PÓLYA–LUNDBERG PROCESS

The Pólya–Lundberg process, frequently abbreviated as *Pólya process*, is a stochastic process* with various applications in nuclear physics and insurance mathematics. Depending on the context in which it is used, the Pólya–Lundberg process can be characterized as a pure birth Markov process*, a mixed Poisson process*, or a limit of contagion-type urn models*. Due to its Markovian character, the first approach is especially appropriate for the modeling and description of physical processes such as electron–photon cascades (see Arley [2]; see also Bharucha-Reid [4] and the references given therein). As a weighted Poisson process, the Pólya–Lundberg process plays an important role in non-life insurance, as was shown by Lundberg [8], who fitted the Pólya process to sickness and accident statistics. Here the urn model approach provides a simple interpretation of the contagion property of the Pólya–Lundberg process (see also Beard et al. [3]; for a more advanced exposition of the corresponding urn process, see also Hill et al. [6]). Recently, some connections between the Pólya–Lundberg process and records* have been pointed out; for instance, the Pólya process can be considered as a counting process of record values coming from independent Pareto*-distributed random variables (see Pfeifer [10]). Conversely, the study of record values paced by a Pólya–Lundberg process gives an interesting insight into the probabilistic behavior of this process from a very different point of view (see Orsingher [9]). Lately, the Pólya–Lundberg process has been employed to illuminate structural properties of infinitely divisible* stochastic point processes with respect to the representation of the probability generating functional (see Waymire and Gupta [13]).

In the Markovian setting, a Pólya–Lundberg process $\{N(t),\ t \geqslant 0\}$ is a nonhomogeneous birth process (*see* BIRTH-AND-DEATH PROCESSES) with birth rates

$$\lambda_n(t) = \lambda \frac{1 + \alpha n}{1 + \alpha \lambda t}, \qquad t \geqslant 0, \quad n = 0, 1, 2, \dots, \tag{1}$$

where $\lambda, \alpha > 0$ are scale and shape parameters, respectively; that is, the probability of a new birth in the time interval $(t, t+h)$ is given by $\lambda_n(t)h + o(h)$, while the probability of two or more births in this interval is $o(h)$ for $h \to 0$. Here $o(h)$ is a remainder term with $o(h)/h \to 0$ for $h \to 0$. As solutions of Kolmogorov's backward differential equations, the marginal distributions of the process are obtained, given by the Pólya distributions

$$\Pr(N(t) = n) = \frac{(\lambda t)^n}{n!}(1 + \alpha\lambda t)^{-(n+1/\alpha)} \times \prod_{k=1}^{n-1}(1 + \alpha k) \tag{2}$$

for $n = 0, 1, 2, \dots$ and $t \geqslant 0$ with mean and variance

$$E(N(t)) = \lambda t, \qquad \operatorname{var}(N(t)) = \lambda t(1 + \alpha\lambda t), \tag{3}$$

which also illustrates the meaning of the parameters λ and α. As suggested by the formulas above, the Pólya–Lundberg process approaches a Poisson process* if α approaches zero.

As a birth process, the Pólya–Lundberg process can also equivalently be described by the sequence T_n, $n = 1, 2, \dots,$ of birth occurrence times which form a Markov chain with transition probabilities

$$\Pr(T_{n+1} > s \mid T_n = t) = \left(\frac{1 + \alpha\lambda t}{1 + \alpha\lambda s}\right)^{n+1/\alpha}, \qquad 0 \leqslant t \leqslant s \tag{4}$$

(see Albrecht [1] and Pfeifer [11]). As a special property of the Pólya–Lundberg process, the sequence $S_n = n/(1 + \alpha\lambda T_n)$ forms a mean-bounded submartingale (*see* MARTINGALES). This provides a simple proof of the fact that for the time averages

$$\frac{N(t)}{t} \to \Lambda, \qquad \frac{T_n}{n} \to \frac{1}{\Lambda} \qquad \text{almost certainly} \quad (t, n \to \infty) \tag{5}$$

(*see* CONVERGENCE OF SEQUENCES OF RANDOM VARIABLES and Pfeifer [12]). Here Λ is a random variable following a gamma distribution* with mean λ and variance $\alpha\lambda^2$.

In the setting of mixed Poisson processes, (5) gives a limit representation of the mixing random variable; that is, a Pólya–Lundberg process can be considered as a weighted

Poisson process whose parameter is chosen at random according to the distribution of Λ. In risk theory*, the distribution function of Λ is also called structure function or unconditioned risk distribution. Characteristically, the Pólya–Lundberg process is the only mixed Poisson process whose birth rates for fixed time t are a linear function of n, as was proven by Lundberg [8].

Finally, the Pólya–Lundberg process can be represented as a limit of urn processes of contagion type introduced by Eggenberger and Pólya [5] (*see also* URN MODELS and Johnson and Kotz [7]). Here a certain number of white and black balls is collected in an urn where p denotes the proportion of white balls and $q = 1 - p$ the proportion of black balls. When a ball is drawn at random, it is replaced along with a fixed proportion β (of the initial total number of balls) of the same color, which causes the contagious effect. If N_m denotes the number of white balls drawn in m trials, the probability distribution of N_m is given by the Pólya–Eggenberger distribution

$$\Pr(N_m = k) = \binom{m}{k} \frac{p^{(k,\beta)} q^{(m-k,\beta)}}{1^{(m,\beta)}} \qquad (6)$$

for $k = 0, 1, \ldots, m$, where $p^{(k,\beta)}$ denotes $p(p + \beta) \cdots (p + (k - 1)\beta)$, etc., with

$$E(N_m) = mp, \qquad \operatorname{var}(N_m) = mpq \frac{1 + m\beta}{1 + \beta}. \qquad (7)$$

Now if for $m \to \infty$, the portions $p = p_m$ and $\beta = \beta_m$ are chosen such that

$$mp_m \to \lambda t, \qquad m\beta_m \to \alpha\lambda t, \qquad (8)$$

then N_m tends to $N(t)$ in distribution [i.e., the probabilities (6) approach the Pólya probabilities (2)]; similarly for the moments (7). A vivid interpretation of this could also be given as follows. Suppose that for a fixed time $t > 0$ a series of m drawings at times $h, 2h, \ldots, mh$ is made, where $h = t/m$ and $p_m = \lambda h$, $\beta_m = \alpha p_m$. Further, let $N^*(s)$, $s \geqslant 0$, denote the number of white balls drawn up to time s. Then $N^*(t)$ approximately behaves like a Pólya–Lundberg process with parameters λ and α at time t in that

$$\Pr(N^*(t + h) = n + 1 \mid N^*(t) = n) = \frac{p_m + n\beta_m}{1 + m\beta_m} = \lambda_n(t)h \qquad (9)$$

with birth rates $\lambda_n(t)$ given by (1).

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