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EXTREMAL PROCESSES, RECORD TIMES AND STRONG APPROXIMATION

Dietmar Pfeifer
University of North Carolina at Chapel Hill
and
Technical University Aachen

Abstract

Given an i.i.d. sequence of random variables with continuous cumulative distribution function, we present a construction for the jump times of an extremal process on the same probability space which interpolate the given record times. This provides another approach to the strong approximation of extremal processes developed by Deheuvels (1981, 1982, 1983), and allows for a simplified investigation of the relationship between the record times and the jump times of the extremal process. In particular, we show that the surplus number S of extremal jumps in $(1, \infty)$ over the record times is approximately Poisson distributed with exact mean $E(S) = 1 - \gamma$ where γ denotes Euler's constant. Further, we show that an approximation of record times by the sequence of records rather than extremal jumps is a.s. of the same order, with a doubly exponential limiting distribution. The possibility of a joint strong approximation of records, record and inter-record times by Wiener processes derived from this is also pointed out.

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1. Introduction

Let $\{X_n;\ n \in \mathbb{N}\}$ be an i.i.d. sequence of random variables (r.v.'s) with continuous cumulative distribution (cdf) F, and let $X_{(n)} = \max\{X_1,..,X_n\}$, $n \in \mathbb{N}$. Of particular interest are the times U_n when these partial maxima change their values, defined by

$$(1.1) \qquad U_0 = 1, \quad U_{n+1} = \inf\{k;\ X_k > X_{U_n}\},\ n \geq 0.$$

Due to the continuity of F, (1.1) is a.s. well-defined; U_n is called the n^{th} record time, and X_{U_n} the n^{th} record value of the sequence. Several efforts have been made to clarify the asymptotic properties of record times, using different approaches such as canonical representations ([15],[16]), strong approximation techniques ([3],[8]) or embedding into extremal processes ([9],[11]), all of them saying that $\{\log U_n;\ n \in \mathbb{N}\}$ asymptotically behaves like a homogeneous Poisson point process with unit rate. Here the extremal process $\{E(t);\ t > 0\}$ (called extremal-F) is a right continuous non-decreasing pure jump Markov process such that for all selections $0 < t_1 < \ldots < t_k$ of time points we have

$$(1.2) \qquad P\left(\bigcap_{i=1}^{k}\{E(t_i) \leq x_i\}\right) =$$

$$F^{t_1}(\min\{x_1,..,x_n\}) \prod_{i=2}^{k} F^{t_i - t_{i-1}}(\min\{x_i,..,x_k\})$$

where $x_1,..,x_k \in \mathbb{R}$. Especially, from (1.2) it follows that we have $\{X_{(1)},\dots,X_{(n)}\} \overset{\mathcal{L}}{=} \{E(1),\dots,E(n)\}$ for all $n \in \mathbb{N}$, where $\overset{\mathcal{L}}{=}$ means equality in distribution. The structural properties of such extremal processes are well-investigated (cf. [5], [9] - [12],[14]), and their importance is given by the fact that they occur as functional weak limits of the normalized processes $\left\{\frac{1}{b_n}\left(X_{([nt])} - a_n\right);\ t > 0\right\}$ ([.] denoting integer part) where $a_n \in \mathbb{R}$, $b_n > 0$ are constants which make $\frac{1}{b_n}\left(X_{(n)} - a_n\right)$ converge weakly to an extreme value distribution (whenever this is possible) (see e.g. [11] and further references therein). Also, if $\{\tau_n;\ -\infty < n < \infty\}$ denotes the jump times of the extremal-F process, it has been shown that these form a non-homogeneous Poisson point process with intensity $\lambda(t) = \frac{1}{t}$, $t > 0$ (in fact, the extremal process has infinitely many jumps in every neighbourhood of the origin). Correspondingly, the sequence $\{E(\tau_n);\ -\infty < n < \infty\}$ of states visited forms a Markov chain with transititon probabilities

$$(1.3) \qquad P(E(\tau_{n+1}) > y \mid E(\tau_n) = x) = \frac{-\log F(y)}{-\log F(x)},\ y \geq x$$

where x,y are chosen such that $0 < F(x) \leq F(y) < 1$. Since the distribution of $\{\tau_n\}$ is independent of F, $\{E(t)\}$ can be transformed to an extremal-Λ process $\{E^*(t)\}$ by letting $E^*(t) = -\log\{-\log F(E(t))\}$, $t > 0$, where $\Lambda(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$ is the cdf of a doubly exponential distribution. Then $\{E^*(\tau_n)\}$ forms a homogeneous Poisson process on $\mathbb{R}$ with unit rate. It follows that the time-transformed process $\{E^*(e^t);\ t \in \mathbb{R}\}$ now is homogeneous Poisson both in time and space.

In the light of (1.2), one might ask whether extremal processes can also be constructed by some sort of extension of the partial maxima (or records) from the original sequence, on the same space. Such considerations have recently been made by Deheuvels ([1],[2]) who started with a strong approximation of the record times $\{U_n\}$, which he then extended to a strong approximation of the inverse extremal process, and finally to the extremal process itself. We shall show here, that in a certain sense also a direct approach is possible, constructing first the extremal jump times from the given record times. This simplifies the investigation of the relationship between the jump times $\{\tau_n\}$ of the extremal process and the record times $\{U_n\}$, completing results of Resnick ([9],[11]). In particular, if S denotes the surplus of extremal jumps over the record times in the interval $(1,\infty)$, then S is approximately Poisson-distributed, with an exact mean

$$(1.4) \qquad E(S) = 1-\gamma$$

where $\gamma = .577$ denotes Euler's constant. Also,

$$(1.5) \qquad \log U_n = \log \tau_{n+S} + o(1) \quad \text{a.s.}$$

$$= \log \tau_n + O(\log n) \quad \text{a.s.} \qquad (n \to \infty)$$

where $\{\log \tau_n\}$ forms a homogeneous Poisson process with unit rate. On the other hand, if F is the cdf of an exponential distribution with unit mean, it can be shown that

$$(1.6) \qquad X_{U_n} - \log U_n \xrightarrow{\mathcal{L}} \Lambda \qquad (n \to \infty)$$

from which it follows that

$$(1.7) \qquad \log U_n = X_{U_n} + O(\log n) \quad \text{a.s.} \qquad (n \to \infty)$$

where again $\{X_{U_n}\}$ forms a unit-rate Poisson process. This indicates that it is possible to approximate simultaneously records, record and inter-record times by the same Poisson process, or, likewise, by the same Wiener process in the strong sense.

2. Construction of the extremal jumps

In view of what has been said earlier, it is easier to work with the time-transformed process $\{E(e^t);\, t \in \mathbb{R}\}$ since then the corresponding jump times $\{\log \tau_n; -\infty < n < \infty\}$ form a homogeneous Poisson process with unit rate. Further, by the general structure of extremal processes, the jump times $\log \tau_n$ must be a.s. concentrated in the random set $\bigcup_{k=1}^{\infty} (\log(U_k - 1), \log U_k)$. In fact, in our construction, $\log \tau_1 \in (\log(U_1 - 1), \log U_1)$.

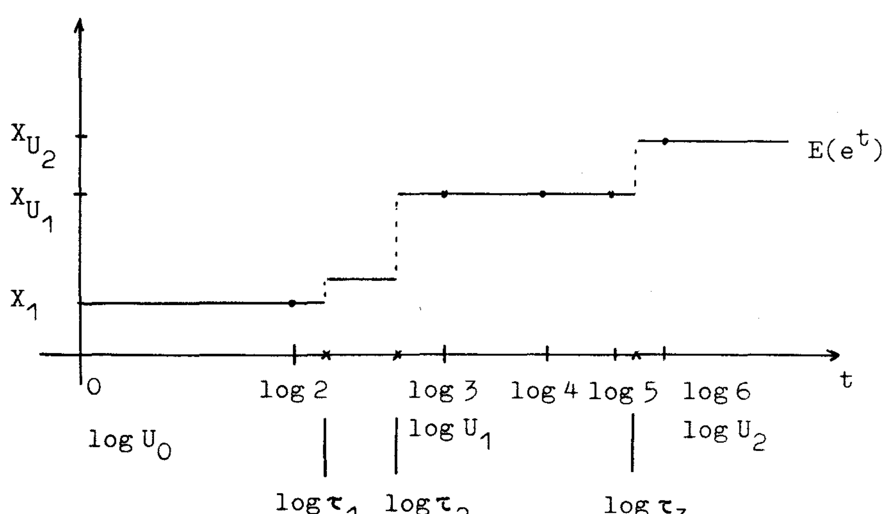
Let, for real numbers $a < b$, N(a,b) denote the number of $\log \tau_n$ - points in the interval (a,b). As a simple consequence of the Poissonian nature of $\{\log \tau_n\}$, in a successful construction, the random variables $A_k = N(\log(U_k-1), \log U_k)$ should be conditionally independent given the σ-field $\mathcal{A} = \sigma(U_1, U_2, \ldots)$, following a below truncated Poisson distribution $Q(\lambda_k)$, say, with parameter

$$(2.1) \qquad \lambda_k = \log\left(\frac{U_k}{U_k - 1}\right)$$

where

$$(2.2) \qquad Q(\lambda, j) = \frac{1}{e^{\lambda}-1} \frac{\lambda^j}{j!}, \quad j \in \mathbb{N} \qquad (\lambda > 0).$$

Further, conditioned on $\mathcal{A}$ and the number A_k, the location of the jumps in $(\log(U_k-1), \log U_k)$ should be distributionally the same as that of an ordered sample of a population distributed uniformly over this interval.



By means of two independent i.i.d. uniformly $\mathcal{U}(0,1)$ - distributed sequences $\{W_n(i);\ n \in \mathbb{N}\}$, $i = 1,2$ (which can independently of $\{X_n\}$ be defined on the same probability space, eventually after enlarging by products) we are thus able to interpolate the given record times by extremal jumps, in the following way.

Step 1. Determination of number of jumps.

Let $F_{Q(\lambda)}$ denote the cdf of the truncated Poisson distribution with parameter $\lambda > 0$. Define

$$(2.3) \qquad A_k = F^{-1}_{Q(\lambda_k)}(W_k(1)),\ k \in \mathbb{N},$$

where $\lambda_k = \log\left(\frac{U_k}{U_k-1}\right)$. A_k denotes the number of jumps to be implanted in the interval $(\log(U_k-1), \log U_k)$.

Step 2. Determination of position of jumps.

Let

$$B_k = \begin{cases} 0, & k = 0 \\ A_1 + \ldots + A_k, & k \geq 1. \end{cases}$$

Define unordered samples

$$(2.4) \qquad D_j^{(k)} = W_{B_{k-1}+j}(2) \log(U_k-1) + (1-W_{B_{k-1}+j}(2)) \log U_k$$

for $1 \leq j \leq A_k$, $k \geq 1$. Let

$$(2.5) \qquad \log \tau_{B_{k-1}+j} = D^{(k)}_{(j)},\ 1 \leq j \leq A_k,\ k \geq 1$$

(ordered samples).

3. Extremal jumps and record times

From the foregoing it is clear that the sequences $\{\log \tau_n;\ n \in \mathbb{N}\}$ and $\{\log U_n;\ n \in \mathbb{N}\}$ are closely related since $\log \tau_n$ only takes values in the set $\bigcup_{k=1}^{\infty} (\log (U_k-1), \log U_k)$; particularly, there exists some a.s. finite r.v. $S \geq 0$ such that

$$(3.1) \qquad \log \tau_{n+S} \in (\log (U_n-1),\ \log U_n) \quad \text{a.s.}$$

for sufficiently large n (cf. [9],[11]). From here it follows that

$$(3.2) \qquad \log U_n = \log \tau_{n+S} + o(\exp\{-n + n\,H(\tfrac{1}{n})\})$$
$$= \log \tau_n + O(\log n) \quad \text{a.s.} \quad (n \to \infty)$$

where $t\,H(\frac{1}{t})$, $t > 0$ belongs to the upper class of a Wiener process (see [8]) since $\log U_n - \log (U_n-1) \sim \frac{1}{U_n}$ $(n \to \infty)$.

Relation (3.2) does not provide the best possible strong approximation of $\{\log U_n\}$ by a homogeneous Poisson process with unit rate. In fact, in [8] it was proved that there exists a Poisson arrival process $\{T_n;\ n \in \mathbb{N}\}$ with unit rate, defined on the same probability space as the original sequence, and a r.v. $Z \geq 0$ which is asymptotically independent of this process such that

$$(3.3) \qquad \log U_n = T_n + Z + o(\exp\{-n + n\,H(\tfrac{1}{n})\}) \quad \text{a.s.} \ (n \to \infty)$$

which gives an a.s. $O(1)$ rate result. It was also shown that Z can be represented as

$$(3.4) \qquad Z = \sum_{k=1}^{\infty} \log\left(1 + \frac{W_k}{U_k - 1}\right)$$

where $\{W_k\}$ is an i.i.d. sequence of $\mathcal{U}(0,1)$-distributed r.v.'s, independent of $\{U_k\}$, and that $E(Z) = 1 - \gamma$ (cf. also [7]). Although $\{T_n\}$ and $\{\log \tau_n\}$ are not directly comparable, there is however an interesting conditional relationship between Z and S, given the σ-field $\mathcal{A}$ generated by the record times.

Theorem 1. We have

$$(3.5) \qquad E(S \mid \mathcal{A}) = E(Z \mid \mathcal{A}) \quad \text{a.s.},$$

hence $E(S) = 1 - \gamma$.

Proof. According to what has been said in Section 2, we have

$$E(A_k \mid U_k) = U_k \log\left(\frac{U_k}{U_k - 1}\right) \quad \text{a.s.}$$

which is the (conditional) mean of the truncated Poisson distribution $Q(\lambda_k)$ with $\lambda_k = \log\left(\frac{U_k}{U_k - 1}\right)$. A little analysis shows that also

$$(3.6) \qquad U_k \log\left(\frac{U_k}{U_k - 1}\right) = E\left\{\log\left(1 + \frac{W_k}{U_k - 1}\right) \,\Big|\, U_k\right\} + 1 \quad \text{a.s.}$$

from which the result follows by the observation that

$S = \sum_{k=1}^{\infty} (A_k - 1)$; hence

$$(3.7) \qquad E(S \mid \mathcal{A}) = \sum_{k=1}^{\infty} E\left\{\log\left(1 + \frac{W_k}{U_{k-1}}\right) \mid \mathcal{A}\right\} = E(Z \mid \mathcal{A}) \text{ a.s.}$$

In the light of (3.1), S is the surplus number of extremal jumps over the record times counted in $(1,\infty)$.

It should be pointed out that since $\{\log \tau_n\}$ has i.i.d. increments following an exponential distribution with unit mean, the limiting distribution of $\log U_n - \log \tau_n$ is that of $Z^* = \sum_{k=1}^{S} Y_k$, where $\{Y_k\}$ are i.i.d. exponential r.v.'s with unit mean, independent of S, such that again $E(Z^*) = 1 - \gamma$. However, Z^* and Z are not identical in distribution since $P(S = 0) \geq \gamma > 0$; hence Z^* has an atom at zero, while Z has no atoms.

Let now I_n, $n \in \mathbb{N}$ denote the indicator r.v. for the event that a record occurs at time n. It is well-known that $\{I_n\}$ is an independent sequence with success probabilities

$$(3.8) \qquad P(I_n = 1) = \frac{1}{n} \quad (= p_n, \text{ say}).$$

Let ζ denote the point process of superpositions of the Bernoulli processes induced by I_n, $n \geq 2$, and

$$(3.9) \qquad \mu_n = -\log(1 - p_n) = \log\left(\frac{n}{n-1}\right), \quad n \geq 2.$$

Let ξ denote a Poisson point process with mean measure $E(\xi(B)) = \sum_{k \in B} \mu_k$, $B \subseteq \mathbb{N}\setminus\{1\}$ which is coupled with ζ

in the sense of Serfling [13] and Karr [6], Chapter 1.6. Then, from Section 2, we see that

$$(3.10) \qquad S \overset{\mathcal{L}}{=} \lim_{n\to\infty}\left\{\xi(B_n) - \zeta(B_n)\right\}$$

with $B_n = \{2,3,..,n\}$ which from another point of view shows that

$$(3.11) \qquad E(S) = \lim_{n\to\infty}\left\{E(\xi(B_n)) - E(\zeta(B_n))\right\}$$
$$= \lim_{n\to\infty}\left\{\log n - \sum_{k=2}^{n}\frac{1}{k}\right\} = 1-\gamma.$$

Alternatively, a distributionally equivalent representation for the r.h.s. of (3.10) is

$$(3.11) \qquad S \overset{\mathcal{L}}{=} \sum_{n=2}^{\infty} I_n(Y_n-1) = \sum_{n=2}^{\infty} W_n, \text{ say}$$

where $\{Y_n\}$ is an independent sequence of below truncated Poisson r.v.'s with distribution $Q(\mu_n)$, independent of $\{I_n\}$. Let now $\nu_n = \mu_n - p_n$, and Z_n be Poisson r.v.'s with mean ν_n. Then the following estimation holds.

<u>Theorem 2</u>.

$$(3.12) \qquad d(\mathcal{L}(W_n),\mathcal{L}(Z_n)) = \sup_A \left| P(W_n \in A) - P(Z_n \in A)\right|$$
$$= P(Z_n = 1) - P(W_n = 1)$$
$$= e^{-\mu_n}\left\{ e^{p_n}(\mu_n - p_n) - \frac{\mu_n^2}{2}\right\} \leq \frac{\mu_n^3}{3} e^{-\mu_n}.$$

Proof. We have

$$P(W_n = 0) = 1 - p_n + \frac{\mu_n}{e^{\mu_n}-1} = (1-p_n)(1+\mu_n)$$

$$P(W_n = k) = p_n \frac{\mu_n^{k+1}}{(k+1)!(e^{\mu_n}-1)} = (1-p_n) \frac{\mu_n^{k+1}}{(k+1)!}, \quad k \geq 1$$

$$P(Z_n = k) = e^{-\nu_n} \frac{\nu_n^k}{k!}, \qquad k \geq 0$$

from which it follows that

$$\frac{P(Z_n = 0)}{P(W_n = 0)} = \frac{e^{p_n}}{1+\mu_n} < 1 \qquad \text{(by series expansion)}$$

$$\frac{P(Z_n = k)}{P(W_n = k)} = (k+1) \frac{e^{p_n}\nu_n^k}{\mu_n^{k+1}} \sim \frac{k+1}{2^k} e^{p_n}\mu_n^{k-1} \to 0 \quad (n \to \infty, k \geq 2)$$

(actually, in the cases considered here this ratio is always strictly less than 1)

$$\frac{P(Z_n = 1)}{P(W_n = 1)} = 2 \frac{e^{p_n}\{e^{-\mu_n} - (1-\mu_n)\}}{\mu_n^2}$$

which is strictly greater than 1 in the range considered here.

It follows that the sup in the total variation distance between Z_n and W_n is achieved for the set $\{1\}$, for all n, hence the result.

Summing up in (3.12), we get the following estimation, according to Serfling [13].

Corollary. The surplus number S of extremal jumps over the record times in $(1, \infty)$ is approximately Poisson distributed with mean $1-\gamma$; the distance in total variation is bounded by

$$(3.13) \qquad d(\mathcal{L}(S), \mathcal{P}(1-\gamma)) \leq \sum_{n=2}^{\infty} \{P(Z_n=1) - P(W_n=1)\} \leq .065.$$

Here $\mathcal{P}(\mu)$ denotes a Poisson distribution with mean μ.

4. Records and record times

In this section we shall show that record times can likewise be strongly approximated by the record values, rather than extremal jumps, with the same a.s. rate. Before doing so, we shall briefly pass over to lower records and lower record times $\{L_n;\ n \geq 0\}$ of the original sequence $\{X_n\}$, defined by

$$(4.1) \qquad L_0 = 1, \quad L_{n+1} = \inf\{k;\ X_k < X_{L_n}\}, \quad n \geq 0.$$

Obviously, $\{L_n\}$ again is a Markov chain of the same type as $\{U_n\}$. Suppose now that all X_n are exponentially distributed with unit mean. It is clear that $X_{(1)} = \min\{X_1, \ldots, X_n\}$ then also is exponentially distributed, and $n X_{(1)}$ is exponentially distributed with unit mean. The following surprising result shows that the latter statement remains valid if n is replaced by the Markov time L_n.

Theorem 3.

i) L_n and $L_n X_{L_n}$ are independent for all $n \geq 0$.

ii) $L_n X_{L_n}$ is exponentially distributed with unit mean.

Proof. The statement is trivial for $n = 0$.
According to the upper record case, $\{(L_n, X_{L_n})\}$ also forms a Markov chain with transition probabilities

$$(4.2) \qquad P(L_{n+1} = k,\ X_{L_{n+1}} > x \mid L_n = m,\ X_{L_n} = y) =$$

$$P(X_2, \ldots, X_{k-m} \geq y > X_{k-m+1} > x) =$$

$$e^{-(k-m-1)y}(e^{-x} - e^{-y}), \qquad k > m,\ x \leq y.$$

For $n = 0$ (i.e. $m = 1$) this yields

$$P(L_1 = k,\ X_{L_1} > x) = \int_x^\infty e^{-(k-2)y}(e^{-x} - e^{-y})e^{-y}dy = \frac{e^{-kx}}{k(k-1)},$$

hence

$$P(L_1 = k,\ L_1 X_{L_1} > x) = P(L_1 = k,\ X_{L_1} > \tfrac{x}{k}) = \frac{e^{-x}}{k(k-1)}$$

$$= P(L_1 = k)\ e^{-x}$$

which proves i) and ii) for $n = 1$.
Assume now that the statement is true for some $n \geq 0$.
Then

$$P(L_n = k,\ X_{L_n} > x) = P(L_n = k)\ e^{-kx},$$

hence

$$f_n(k,y) = k\,P(L_n = k)\ e^{-ky} \text{ is a } \#\otimes\tau\text{-density of } (L_n, X_{L_n})$$

where $\#$ denotes counting measure and τ Lebesgue measure on $\mathbb{R}^+$. Now

$$P(L_{n+1} = k,\ X_{L_{n+1}} > x) =$$

$$\sum_{m<k} \int_x^\infty e^{-(k-m-1)y}(e^{-x} - e^{-y}) f_n(m,y)\ dy =$$

$$\sum_{m<k} m\,P(L_n = m) \int_x^\infty e^{-(k-1)y}(e^{-x} - e^{-y})\ dy =$$

$$\sum_{m<k} \frac{m}{k(k-1)}\ P(L_n = m)\ e^{-kx} =$$

$$e^{-kx} \sum_{m<k} P(L_{n+1} = k \mid L_n = m)\ P(L_n = m) =$$

$$P(L_{n+1} = k)\ e^{-kx}$$

which proves the statement also for n+1, and hence the theorem by induction.

<u>Corollary</u>.

i) $X_{U_n} - \log U_n$ is asymptotically Λ-distributed $(n \to \infty)$.

ii) $\log U_n = X_{U_n} + O(\log n)$ a.s. $(n \to \infty)$.

<u>Proof</u>. Define $V_n = -\log(1 - e^{-X_n})$, $n \in \mathbb{N}$. Then $\{V_n\}$ is also i.i.d. exponentially distributed with unit mean,

and, if $\{L_n\}$ denote the lower record times w.r.t. $\{V_n\}$, $\{U_n\}$ the upper record times w.r.t. $\{X_n\}$, then $L_n = U_n$ for all n. By Theorem 3,

$$(4.3) \quad -\log L_n - \log V_{L_n} = -\log U_n - \log V_{U_n} \quad \text{is}$$
(exactly) Λ-distributed for all n.

But $-\log V_{U_n} = -\log(-\log(1-\exp(-X_{U_n}))) = X_{U_n} + o(1)$ a.s. for $n \to \infty$, which proves i).
ii) is a simple consequence of the Borel-Cantelli Lemma since every Λ-distributed sequence of r.v.'s (independent or not) is at most $O(\log n)$ a.s. for $n \to \infty$.

Note that under the condition of exponential distribution for $\{X_n\}$ the sequence of records $\{X_{U_n}\}$ is a Poisson arrival process with unit rate. This means that in any case, $\{\log U_n\}$ can - up to $O(\log n)$ - be strongly approximated by such a Poisson process; we simply have to choose the sequence $\{-\log(1-F(X_{U_n}))\}$.

Let $\Delta_n = U_n - U_{n-1}$, $n \in \mathbb{N}$ denote the corresponding inter-record times. Theorem 3 then gives immediately rise to the following strong approximation result.

Theorem 4.

i) There exists a Poisson arrival process $\{T_n\}$ with unit rate, defined on the same probability space where $\{X_n\}$ is defined, such that

$$(4.4) \qquad \log U_n = T_n + O(\log n) \quad \text{a.s.} \qquad (n \to \infty)$$
$$\log \Delta_n = T_n + O(\log n) \quad \text{a.s.} \qquad (n \to \infty)$$

ii) W.l.o.g., there exists a Wiener process $\{W(t); t > 0\}$ on the same probability space where the $\{X_n\}$ are defined, with $E(W(t)) = 0$, $\sigma^2(W(t)) = t$, such that

$$(4.5) \qquad \log U_n = n + W(n) + O(\log n) \quad \text{a.s.} \quad (n \to \infty)$$
$$\log \Delta_n = n + W(n) + O(\log n) \quad \text{a.s.} \quad (n \to \infty)$$
$$-\log(1-F(X_{U_n})) = n + W(n) + O(\log n) \quad \text{a.s.} \quad (n \to \infty).$$

Proof. i) is obvious from Theorem 3 and [8], choosing $T_n = -\log(1 - F(X_{U_n}))$. ii) follows from i) by the usual strong approximation procedure à la Kómlos-Major-Tusnady for $\{T_n\}$.

It should be pointed out that the last result has been extended also to the case of k^{th}-records by Deheuvels [4] recently, using different techniques.

Note that the Corollary to Theorem 3, i) also reflects the fact that $E(\log U_n) = n + 1 - \gamma + o(1) \quad (n \to \infty)$ as shown in [7] since γ is the mean of a Λ-distributed r.v. Likewise, this can also be concluded from (3.2) since $\log U_n = \log \tau_n + Z_n^* + o(1)$ a.s. $(n \to \infty)$, where $E(\log \tau_n) = n$, $E(Z_n^*) = 1 - \gamma$.

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Institut für Statistik

RWTH Aachen

Wüllnerstr. 3

D-5100 Aachen

Germany