Applicable Analysis, 1986, Vol. 23, pp. 111-118
0003-6811/86/2302-0111 $18.50/0

Some General Probabilistic Estimations for the Rate of Convergence in Operator Semigroup Representations

Communicated by Dr. P. L. Butzer

DIETMAR PFEIFER*
Center for Stochastic Processes, UNC at Chapel Hill, USA, and Institut für Statistik und Wirtschaftsmathematik, Technical University, Aachen, W. Germany
AMS(MOS): 41A36, 20M30, 47D05
Abstract Starting from well-known estimations for the rate of convergence in Hille's, Phillips' and Widder's representation formulas for operator semigroups we show that by a suitable probabilistic approach, these results are easily reobtained, and can immediately be generalized to arbitrary (probabilistic) representation formulas. Some examples are also considered.

(*Received for Publication November 15, 1985*)

INTRODUCTION

As has been worked out in the recent papers [6],[7], probability theory has shown up to be a powerful tool within representation theory of semigroups of linear operators, especially in connection with approximation - theoretic questions in this area. In fact, all relevant estimations for direct approximation theorems and many estimations involving various kinds of moduli of continuity are covered by this approach. However, in [7] some questions concerning certain indirect approximation theorems remained open which we shall answer in this paper. Especially, we shall show that Ditzian's [1],[2],[3] estimations for the rate of convergence for the most important semigroup representations of Hille, Phillips and Widder (see [4]) are not only easily reobtained by the probabilistic approach, but also generalize immediately to

*Research supported in part by US AFOSR Contract No. F49620 82 C 0009.

arbitrary semigroup representations of probabilistic type (see [6]). Throughout the paper, we shall consider a strongly continuous one-parameter operator semigroup $\{T(t);\ t \geq 0\}$ of bounded endomorphisms on a Banach space X with norm $\|\cdot\|$ as in [7]. As usual, A will denote the infinitesimal generator of the semigroup, and $R(\lambda) = (\lambda I - A)^{-1}$ stands for the resolvent of the semigroup which always exists for sufficiently large λ. Further, let $\omega^b(\delta,f)$ denote the rectified modulus of continuity in the interval $[0,b]$ given by

$$\omega^b(\delta,f) = \sup\{\|T(t)f - T(s)f\|;\ 0 \leq s,t \leq b,\ |s-t| < \delta\} \tag{1}$$

for $b,\ \delta > 0,\ f \in X$. In 1960, Hsu [5] gave a first estimation for the rate of convergence in Hille's exponential formula in terms of the rectified modulus of continuity; he proved

$$\|\exp(A_h t)f - T(t)f\| \leq \omega^b(h^{1/3},f) + K\|f\|h^{1/3} \tag{2}$$

for $f \in X$, $h > 0$ (such that $t + h^{1/3} < b$) where K is independent of f,h and t, and A_h is the difference operator given by

$$A_h f = \frac{1}{h}(T(h) - I), \quad h > 0,\ f \in X. \tag{3}$$

In 1969 Ditzian [1] proved that in formula (2), the term $h^{1/3}$ could be replaced by any power h^x with $0 < x < 1/2$, and that x could not be extended to values larger than $1/2$. (For an extension of this relation to arbitrary semigroup representations of probabilistic type, see [7]). He also proved [2] that for $x = 1/2$, a similar estimation holds true, however with a larger factor for the modulus; he obtained

$$\|\exp(A_h t)f - T(t)f\| \leq L\omega^b(h^{1/2},f) \tag{4}$$

for $t < b - \delta$ $(0 < \delta < b$ being fixed), and h small enough, where again L is independent of h and t. In [3], he developed analogous estimations for Phillips' and Widder's representation formulas; he showed that

$$\left\|\left\{\frac{n}{t}R\left(\frac{n}{t}\right)\right\}^{n+1}f - T(t)f\right\| \le \begin{cases} K\omega^b(n^{-1/2},f) & f \in X \\ Ln^{-1/2}\omega^b(n^{-1/2},Af), & f \in D(A) \end{cases} \tag{5}$$

$$\|\exp(-t\lambda I + t\lambda^2 R(\lambda))f - T(t)f\| \le \begin{cases} K\omega^b(\lambda^{-1/2},f) & f \in X \\ L\lambda^{-1/2}\omega^b(\lambda^{-1/2},Af), & f \in D(A) \end{cases} \tag{6}$$

for sufficiently large n and λ, where again $t < b-\delta$, and K and L are independent of n, λ, and t. Here D(A) denotes the domain of the infinitesimal generator A.

It is no surprise that the right hand side of (4),(5) and (6) are of the same type; this is essentially due to the fact that the estimations involved here are closely related to the variances of the underlying random variables when the probabilistic forms of the above representations are considered (see [6] and [7]); this will be worked out in more detail in the following chapter.

The basic tool here will be the concept of the probability generating function Ψ_X of a suitable random variable X, given by

$$\Psi_X(t) = E(t^X), \quad t > 0 \quad \text{and} \quad \Psi_X^*(t) = E(e^{tX}), \quad t \in \mathbb{R} \tag{7}$$

where $E(\cdot)$ means expectation.

MAIN RESULTS

The basic estimation from which all relevant results can immediately be derived is given in the following statement.

THEOREM 1. Let $0 \le t \le b$ and assume that X is a random variable which is concentrated on the interval [0,b] with expectation $E(X) = t$. Then the variance $\sigma^2 = \sigma^2(X)$ is finite, and for all $\varepsilon > 0$, we have

$$\|E[T(X)]f - T(t)f\| \le \left(1+\frac{\sigma}{\varepsilon}\right)\omega^b(\varepsilon,f), \quad f \in X \tag{8}$$

$$\|E[T(X)]f - T(t)f\| \leq \sigma(1+\frac{\sigma}{\varepsilon})\omega^b(\varepsilon,Af), \quad f \in D(A). \tag{9}$$

Proof. Under the conditions above, the moment-generating function Ψ_X^* exists everywhere, hence $E[T(X)]$ is well-defined (see [6]). Further,

$$\begin{aligned}\|E[T(X)]f - T(t)f\| &\leq \int\|T(X)f - T(t)f\|dP \leq \int\omega^b(|X-t|,f)dP \\ &\leq \int(1+\frac{|X-t|}{\varepsilon})\omega^b(\varepsilon,f)dP \leq (1+\frac{\sigma}{\varepsilon})\omega^b(\varepsilon,f)\end{aligned} \tag{10}$$

for $f \in \mathcal{X}$ since by the Jensen inequality (see [7], Theorem 2.1), $\{E(|X-t|)\}^2 \leq E((X-t)^2) = \sigma^2$. Here P denotes the underlying probability measure. This proves relation (8).

Now assume $f \in D(A)$. Then

$$T(X)f - T(t)f = (X-t)T(t)Af + (X-t)\int_0^1[T(t+u(X-t)) - T(t)]Af\,du, \tag{11}$$

hence

$$\begin{aligned}\|E[T(X)]f - T(t)f\| &= \|E((X-t)\int_0^1[T(t+u(X-t)) - T(t)]Af\,du)\| \\ &\leq E(|X-t|\{1+\frac{|X-t|}{\varepsilon}\})\omega^b(\varepsilon,Af) \leq \sigma(1+\frac{\sigma}{\varepsilon})\omega^b(\varepsilon,Af)\end{aligned} \tag{12}$$

which proves relation (9).

Since probabilistic representation theorems for operator semigroups are closely related to the law of large numbers (see [6]), it is interesting to see what kind of estimations can be obtained from the Theorem 1 in this case. For this purpose, X has to be replaced by the arithmetic mean $\overline{X}_n = \frac{1}{n}\Sigma_{k=1}^n X_k$ where $X_1,\ldots, X_n$ are independent copies of X. Then for the variance, $\sigma^2(\overline{X}_n) = \sigma^2/n$, hence the following result holds.

COROLLARY 1. Under the conditions of Theorem 1, and if $X_1,\ldots, X_n$ are independent copies of X, we have

$$\|E[T(\overline{X}_n)]f - T(t)f\| \leq (1+\sigma)\omega^b(n^{-1/2},f), \quad f \in \mathcal{X} \tag{13}$$

$$\|E[T(\bar{X}_n)]f - T(t)f\| \le n^{-1/2}\sigma(1+\sigma)\omega^b(n^{-1/2},Af), \quad f \in D(A). \tag{14}$$

<u>Proof</u>. Obvious by letting $\varepsilon = n^{-1/2}$ in Theorem 1.

It is no problem to extend the above results also to more general situations, i.e. arbitrary distributions for the underlying random variables. One such result is the following.

<u>THEOREM 2</u>. Let X be a non-negative random variable whose moment-generating function Ψ_X^* exists for some positive argument. Then for sufficiently large n, $E[T(\bar{X}_n)]$ exists, and

$$\|E[T(\bar{X}_n)]f - T(t)f\| \le K\omega^b(n^{-1/2},f), \quad f \in X \tag{15}$$

$$\|E[T(\bar{X}_n)]f - T(t)f\| \le Ln^{-1/2}\omega^b(n^{-1/2},Af), \quad f \in D(A), \tag{16}$$

where $t = E(X)$, $t < b - \delta$ $(0 < \delta < b$ being fixed), n being sufficiently large, and K and L are independent of n and t.

<u>Proof</u>. Define

$$Y_n = \begin{cases} \bar{X}_n, & \text{if } |\bar{X}_n - t| \le \delta \\ t, & \text{otherwise.} \end{cases}$$

Then

$$\int_{|\bar{X}_n - t| \le \delta} \|T(\bar{X}_n)f - T(t)f\|\,dP = \int \|T(Y_n)f - T(t)f\|\,dP$$

$$\le \int \omega^b(|Y_n - t|,f)\,dP \le (1+\sigma)\omega^b(n^{-1/2},f)$$

(where σ^2 again denotes the variance of X), and

$$\int_{|\bar{X}_n - t| > \delta} \|T(\bar{X}_n)f - T(t)f\|\,dP \le K^* e^{-\delta\sqrt{n}}\|f\|$$

for some constant $K^* > 0$ (independent of n and t) by the proof of

Theorem 5.1 in [7]. Thus

$$\|E[T(\overline{X}_n)]f - T(t)f\| \le (1+\sigma)\omega^b(n^{-1/2},f) + K^* e^{-\delta\sqrt{n}}\|f\|$$

$$\le K\omega^b(n^{-1/2},f)$$

for some suitable constant $K>0$. This proves (15). For the proof of (16), observe that

$$\|E[T(\overline{X}_n)]f - T(t)f\| \le E(\|T(\overline{X}_n)f - T(Y_n)f\|) + \dots$$

$$\dots + \|E[T(Y_n)]f - T(t)f\|$$

$$\le \int_{|\overline{X}_n - t|>\delta} \|T(\overline{X}_n)f - T(t)f\| dP + E(|\overline{X}_n - Y_n|)\|T(t)\,Af\| + \dots$$

$$\dots + n^{-1/2}\sigma(1+\sigma)\omega^b(n^{-1/2},Af)$$

$$\le K^* e^{-\delta\sqrt{n}}\|f\| + \sqrt{E(\overline{X}_n - t)^2}\sqrt{P(|\overline{X}_n - t| > \delta)}L^*\|Af\| + \dots$$

$$\dots + n^{-1/2}\sigma(1+\sigma)\omega^b(n^{-1/2},Af)$$

$$\le K^* e^{-\delta\sqrt{n}}\|f\| + \sigma L^{**} e^{-\delta\sqrt{n}}\|Af\| + \dots$$

$$\dots + n^{-1/2}\sigma(1+\sigma)\omega^b(n^{-1/2},Af)$$

by (12) and Hölder's inequality for suitable constants $L^*, L^{**} > 0$, which gives the desired result.

An immediate consequence of Theorem 2 to general probabilistic representation theorems for operator semigroups is given in the following statement.

<u>COROLLARY 2</u>. Let N be a non-negative integer-valued random variable and Y be a non-negative real random variable such that $\Psi_N(\delta_1) < \infty$ for some $\delta_1 > 1$ and $\Psi_Y^*(\delta_2) < \infty$ for some $\delta_2 > 0$. Then the expectations $E(N) = \zeta$ and $E(Y) = \gamma$ (say) exist, and for sufficiently large n, $S_n = \{\Psi_N(E[T(\frac{Y}{n})])\}^n$ exists as a bounded linear operator. Then

if $t = \zeta\gamma < b - \delta$ $(0 < \delta < b)$ being fixed), there exist constants K and L which are independent of n and t such that

$$\|S_n f - T(t)f\| \le K\omega^b(n^{-1/2}, f), \quad f \in X \tag{17}$$

$$\|S_n f - T(t)f\| \le Ln^{-1/2}\omega^b(n^{-1/2}, Af), \quad f \in D(A). \tag{18}$$

Proof. Obvious from Theorem 4.4 and 5.2 in [7].

It is interesting to notice that Ditzian's estimations (5) and (6) (for integer values of λ) are covered by Corollary 2, as well as a discrete version of (4) with $h = 1/n$; here additionally a corresponding estimation for $f \in D(A)$ is at once available (see [6] and [7]). Moreover, it is also possible to obtain the general estimations (4) and (6) by an application of the above Theorem to the situation under Theorem 4.2 in [7], using Poisson processes for Hille's and Phillips' formulas. The missing estimation for Hille's formula then is

$$\|\exp(A_h t)f - T(t)f\| \le Lh^{1/2}\omega^b(h^{1/2}, Af), \quad f \in D(A) \tag{19}$$

for $t < b - \delta$ $(0 < \delta < b$ being fixed), and h small enough.

It should be pointed out finally that Corollary 1 can immediately be applied to Kendall's representation formula using binomial distributions which has an interesting application to Bernstein polynomials. Namely, we have (20)

$$\|\{(1-t)I + tT(\tfrac{1}{n})\}^n f - T(t)f\| \le \begin{cases} (1+\sqrt{t(1-t)})\omega^b(n^{-1/2}, f), & f \in X \\ n^{-1/2}\sqrt{t(1-t)}\,(1+\sqrt{t(1-t)})\omega^b(n^{-1/2}, Af), & f \in D(A) \end{cases}$$

for $0 \le t \le 1$, and $b = 1$. Applying this to the semigroup of translations as in [7], chapter 7, and letting ω denote the modulus of continuity in the interval [0,1], we obtain for the Bernstein polynomials

$$B_n(g,x) = \sum_{k=0}^{n} \binom{n}{k} g\left(\frac{k}{n}\right) x^k (1-x)^{n-k}, \quad 0 \le x \le 1,\ n \ge 1,\ g \in C[0,1]:$$

$$|B_n(g,x) - g(x)| \le \begin{cases} (1 + \sqrt{x(1-x)})\,\omega(n^{-1/2}, g), & g \in C[0,1] \\ \sqrt{\dfrac{x(1-x)}{n}}(1 + \sqrt{x(1-x)})\,\omega(n^{-1/2}, g'), & g' \in C[0,1]. \end{cases} \tag{21}$$

It should be pointed out finally that the arguments above can also immediately be applied to more general forms of operator semigroup representations, for instance the product representation theorems developed in [6].

Acknowledgements. A first draft of the paper was written during the author's visit to Dalian Institute of Technology, China, and completed during his visit to the Center for Stochastic Processes, UNC, USA. We are indebted to Professor L.C. Hsu for many fruitful discussions on this topic with him and his colleagues, and an anonymous referee for several constructive remarks.

REFERENCES

1. Z. Ditzian, Proc. Amer. Math. Soc., 22, 351-355 (1969).
2. Z. Ditzian, Proc. Amer. Math. Soc., 24, 351-352 (1970).
3. Z. Ditzian, Israel J. Math., 9, 541-553 (1971).
4. E. Hille and R.S. Phillips, Functional Analysis and Semigroups, (Amer. Math. Soc. Colloq. Publ., 31, Providence, Rhode Island, 1957).
5. L.C. Hsu, Czechoslovak Math. J., 10, 323-328 (1960).
6. D. Pfeifer, Semigroup Forum, 30, 17-34 (1984).
7. D. Pfeifer, J. Approximation Theory, 43, 271-296 (1985).