Martingale characteristics of mixed Poisson processes*)

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1. Introduction

Structural properties of mixed Poisson processes have always been of specific importance in the field on insurance mathematics since they are widely used in the modelling of claim number processes, especially in non-life insurance (see Albrecht (1985) for a general survey). Recently, Heller and Pfeifer (1987) have shown that a mixed Poisson process can be characterized among the class of pure birth processes by martingale properties of certain transformations of the occurrence times of the process. These transformations are connected with the intensities of the birth process, and give rise to strong limit theorems due to the Martingale Convergence Theorem. The aim of the present paper is to investigate which kind of transformations besides intensities render the occurrence times of a mixed Poisson process a martingale, and give characterizations of mixed Poisson processes in the class of birth processes by martingale properties of such transformations. Among other results, it is shown that such transformations are essentially ratios of derivatives of completely monotonic functions (see Bernstein (1928)), and thus natural generalizations of the intensities of mixed Poisson processes (see Lundberg (1940)).

2. Basic notions of the model

We consider the class of pure birth processes $\{N(t);\ t \geqslant 0\}$ with standard transition probabilities

$$p_{nm}(s,t) = P(N(t) = m \mid N(s) = n), \quad 0 \leqslant n \leqslant m, \quad 0 \leqslant s \leqslant t, \tag{1}$$

possessing right-continuous paths and positive and finite intensities

$$\lambda_n(t) = \lim_{h \downarrow 0} \frac{1}{h}\, p_{n,n+1}(t, t+h), \quad n, t \geqslant 0 \tag{2}$$

such that all finite-dimensional marginal distributions of the corresponding occurrence times $\{T_n;\ n \geqslant 0\}$ are absolutely continuous with respect to Lebesgue measure. It was shown in Pfeifer (1982) that under these assumptions $\{T_n;\ n \geqslant 0\}$ forms a Markov chain with transition probabilities

$$P(T_n > t \mid T_{n-1} = s) = \frac{1 - F_n(t)}{1 - F_n(s)}, \quad 0 \leqslant s \leqslant t, \quad n \geqslant 1 \tag{3}$$

and initial distribution function F_0 where

$$F_n(x) = 1 - \exp\left(-\int_0^x \lambda_n(u)\, du\right), \quad x \geqslant 0, \quad n \geqslant 0\,. \tag{4}$$

*) Gefördert durch die Deutsche Forschungsgemeinschaft im Rahmen eines Heisenberg-Stipendiums.

Especially, all F_n are cumulative distribution functions which are absolutely continuous, with densities f_n, say. The intensities can hence likewise be expressed as

$$\lambda_n(t) = \frac{f_n(t)}{1 - F_n(t)}, \quad n, t \geqslant 0, \tag{5}$$

i.e. the intensities are the hazard rates of the distributions given by F_n.
Actually, Lundberg (1940) has proved that (in our terminology) if $(-1)^n f_n$ is the $(n+1)$-th derivative of a completely monotonic function (corresponding to $1 - F_0$ here), then $\{N(t); t \geqslant 0\}$ must be a mixed Poisson process, and vice versa. This result was also used in Heller and Pfeifer (1987) to show that the martingale property of the post-jump intensities $\{\lambda_n(T_{n-1}); n \geqslant 1\}$, together with a suitable initial condition, also characterizes a mixed Poisson process in the class considered. The transformations H_n $(n \geqslant 0)$ which render the occurrence times a martingale are here of the form

$$H_n = \lambda_{n+1} = -\frac{(1-F_0)^{(n+2)}}{(1-F_0)^{(n+1)}} = \frac{h^{(n)}}{g^{(n)}}, \quad n \geqslant 0 \tag{6}$$

where $g = f_0$, $h = -f_0'$ are themselves completely monotonic functions. In this paper we want to show that more generally, for mixed Poisson processes, transformations H_n of the type (6) (with h an essentially arbitrary completely monotonic function) result in martingales $\{H_n(T_n); n \geqslant 0\}$, and essentially only these. Further, it will be shown that the martingale property of the latter sequence, together with a suitable initial condition, with H_n of the general form (6), again characterizes the mixed Poisson process.

3. Characterization theorems

Theorem 1. Let $\{N(t); t \geqslant 0\}$ be a mixed Poisson birth process as considered above, and H_n, $n \geqslant 0$ be positive absolutely continuous functions. Then $\{H_n(T_n); n \geqslant 0\}$ is a martingale with respect to the σ-fields $\mathscr{A}_n$ generated by $T_0, \ldots, T_{n-1}$ if and only if H_n is of the form

$$H_n = \frac{h^{(n)}}{g^{(n)}}, \quad n \geqslant 0 \tag{7}$$

where h is a completely monotonic function and g as in (6).

Proof. The martingale property is equivalent to the almost sure validity of the relation

$$E(H_n(T_n) \mid T_{n-1} = t) = \int_t^\infty H_n(s) \frac{f_n(s)}{1 - F_n(t)} ds = H_{n-1}(t), \quad n \geqslant 1. \tag{8}$$

By the absolute continuity of the H_n we are free to choose versions of the densities f_n such that equality holds on the right hand side in (8) after differentiation, giving

$$H_{n-1}'(t) = H_{n-1}(t)\,\lambda_n(t) - H_n(t)\,\lambda_n(t), \quad t \geqslant 0, \quad n \geqslant 1. \tag{9}$$

Standard arguments from calculus show that the solution of this differential equation is given by

$$H_{n-1}(t) = \frac{c_n(t)}{1-F_n(t)}, \quad t \geqslant 0, \quad n \geqslant 1 \tag{10}$$

where $c_n(t)$ is a suitable absolutely continuous function which must fulfill the relation

$$c_n'(t) = -H_n(t)\,f_n(t) \quad \text{by (9)}\,. \tag{11}$$

Using (10) repeatedly we see that we have

$$c_n'(t) = -c_{n+1}(t)\frac{f_n(t)}{1-F_{n+1}(t)} = -\frac{1}{c_n}c_{n+1}(t) \quad \text{or} \tag{12}$$

$$c_{n+1}(t) = -c_n\,c_n'(t), \quad t \geqslant 0, \quad n \geqslant 1 \tag{13}$$

where c_n are the same constants as in relation (20) in Heller and Pfeifer (1987) (note that we have used a specific property of mixed Poisson processes in this step).
We thus obtain by iteration

$$c_n(t) = (-1)^{n-1}\prod_{k=1}^{n-1} c_k\,c_1^{(n-1)}(t), \quad t \geqslant 0, \quad n \geqslant 1 \tag{14}$$

saying that $c_1(\cdot)$ must be a completely monotonic function by the positivity of $H_1 = c_1(\cdot)/(1-F_1)$. Again with relation (20) in Heller and Pfeifer (1987) and (10) above, we obtain

$$H_n(t) = \frac{c_{n+1}(t)}{1-F_{n+1}(t)} = \frac{\prod\limits_{k=1}^{n} c_k\,c_1^{(n)}(t)}{\prod\limits_{k=0}^{n} c_k\,f_0^{(n)}(t)} = \frac{1}{c_0}\,\frac{c_1^{(n)}(t)}{f_0^{(n)}(t)}, \quad n \geqslant 0. \tag{15}$$

This proves the necessity of (7) with $h = \frac{1}{c_0}c_1(\cdot)$.
Conversely, it is easy to see that if (7) is satisfied, relations (9) to (14) also hold (in reverse order), and hence also (8).
A remarkable consequence of Theorem 1 is the fact that there are no transformations H, independent of n, such that $\{H(T_n); n \geqslant 0\}$ is a martingale, except for constants (when $\frac{h}{g} = \text{const}$), presumed H is positive and absolutely continuous.
Obviously, the case considered in Heller and Pfeifer (1987) is reobtained for the choice $h = -g'$ since by Lundberg (1940), intensities of mixed Poisson processes are precisely of the form

$$\lambda_{n+1}(t) = -\frac{g^{(n+1)}(t)}{g^{(n)}(t)}, \quad t \geqslant 0, \quad n \geqslant 0 \tag{16}$$

with g completely monotonic.
The following result shows that indeed the martingale property of $\{H_n(T_n); n \geqslant 0\}$ with H_n as in (7) characterizes mixed Poisson processes again.

Theorem 2. Let $\{N(t); t \geqslant 0\}$ be a pure birth process of the type considered in §2, and h and g be completely monotonic functions such that h/g is not constant. Let further denote

$H_n = h^{(n)}/g^{(n)}$, $n \geq 0$. Then if $\{H_n(T_n); n \geq 0\}$ is a martingale with respect to the σ-fields $\mathscr{A}_n$ of Theorem 1, we have necessarily

$$\lambda_n(t) = -\frac{g^{(n)}(t)}{g^{(n-1)}(t)}, \quad t \geq 0, \quad n \geq 1. \tag{17}$$

If additionally g is a positive multiple of a density f_0 and $\lambda_0(t) = \frac{f_0(t)}{1-F_0(t)}$, $t \geq 0$ holds as initial condition, where F_0 is the cumulative distribution function corresponding to f_0, then $\{N(t); t \geq 0\}$ is necessarily a mixed Poisson process.

Proof. Analogously to relations (8) and (9) we see that we must have

$$\lambda_n(t) = \frac{H'_{n-1}(t)}{H_{n-1}(t) - H_n(t)}, \quad t \geq 0, \quad n \geq 1 \tag{18}$$

which is well-defined under the conditions above. By definition of H_n, we thus obtain

$$\lambda_n = \frac{h^{(n)}g^{(n-1)} - h^{(n-1)}g^{(n)}}{\{g^{(n-1)}\}^2} \; \frac{1}{\frac{h^{n-1)}}{g^{(n-1)}} - \frac{h^{(n)}}{g^{(n)}}} = -\frac{g^{(n)}}{g^{(n-1)}} \tag{19}$$

as desired. From the initial condition, we see that $1-F_0$ then also is completely monotonic, and

$$\lambda_n = -\frac{(1-F_0)^{(n+1)}}{(1-F_0)^{(n)}}, \quad n \geq 0 \tag{20}$$

which means that the process is mixed Poisson according to Lundberg (1940).

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Zusammenfassung

In der vorliegenden Arbeit wird untersucht, welche Transformationen der Eintrittszeitpunkte gemischter Poisson-Prozesse zu Martingalen führen, und wie solche Transformationen zur Charakterisierung gemischter Poisson-Prozesse in der Klasse der Geburtsprozesse durch Martingaleigenschaften herangezogen werden können.

Summary

We investigate which kind of transformations of the occurrence times of mixed Poisson processes give rise to martingales, and how such transformations can be used to characterize mixed Poisson processes among the class of pure birth processes by martingale properties.