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A NOTE ON STABILITY OF MAXIMA AND RECORDS OF AN IID SEQUENCE

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ABSTRACT

An iid sequence $\{X_n; n \in \mathbb{N}\}$ is called-max-stable if there is a sequence of constants $\{A_n; n \in \mathbb{N}\}$ sucht that $X_{n:n} - A_n \xrightarrow{P} 0$, $n \to \infty$ ($X_{i:n}$ denotes the i-th smallest value among $X_1, \ldots, X_n$). We call $\{X_n; n \in \mathbb{N}\}$ D-max-stable if $X_{n:n} - X_{n-1:n} \xrightarrow{P} 0$, $n \to \infty$. Correspondingly, record-stability and D-record-stability can be defined by passing over to the strictly increasing subsequence of the successive maxima. While a well-known theorem of Geffroy (1958) states that max-stability and D-max-stability are equivalent, this is no longer true for record-stability as was implicitly pointed out by Goldie (1981). In this paper, the relationship between these four stability concepts is investigated further. Especially, it is shown that D-max-stability and D-record-stability are different concepts in that it is possible to maintain the latter property by stretching the observations, while the property of D-max-stability may be lost.

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1. INTRODUCTION

Let $\{X_n;\ n\in \mathbb{N}\}$ be an iid sequence of random variables with cdf F such $F(x)<1$ for alle $x\in \mathbb{R}$. To avoid complications, we also assume that F is continuous. (The problem of max-stability in the discrete case is e.g. discussed in Anderson (1970) and Gather and Mathar (1983).) Let further denote $X_{1:n}\leq\ldots\leq X_{n:n}$ the order statistics of $X_1,\ldots,X_n$, $n\in\mathbb{N}$, and

$$(1.1)\qquad U_0 = 1,\quad U_{n+1} = \inf\left\{k;\ X_k > X_{U_n}\right\},\ n\geq 0\ ,$$

the sequence of record times (which is a.s. well-defined under the assumptions above; cf. Shorrock (1972)). The record sequence $\{X_{U_n};\ n\geq 0\}$ then is precisely the strictly increasing subsequence of $\{X_{n:n};\ n\in\mathbb{N}\}$. Note that we always have

$$(1.2)\qquad X_{U_n:U_n} - X_{U_n-1:U_n} = X_{U_n}-X_{U_{n-1}},\ n\in\mathbb{N}\ ,$$

but usually the sequence $\{X_{n:n}-X_{n-1:n};\ n\geq 2\}$ contains more values than the sequence $\left\{X_{U_n}-X_{U_{n-1}};\ n\in\mathbb{N}\right\}$. For instance, if $X_{U_{n-1}}<X_{U_n+1}<X_{U_n}$, then the difference $X_{U_n+1:U_n+1} - X_{U_n:U_n+1}$ is not contained in the record increments sequence.

Moreover, as will be shown in this paper, the asymptotic behaviour of maxima, records and their increments are not necessarily the same. We call $\{X_n;\ n\in\mathbb{N}\}$ max-stable (in probability) iff there exists a sequence $\{A_n;\ n\in\mathbb{N}\}$ of constants such that $X_{n:n}-A_n\overset{P}{\to}0$, $n\to\infty$, and D-max-stable (in probability) iff $X_{n:n}-X_{n-1:n}\overset{P}{\to}0$, $n\to\infty$. Correspondingly, $\{X_n;\ n\in\mathbb{N}\}$ is called record-stable (in probability) iff for a suitable sequence $\{B_n;\ n\in\mathbb{N}\}$ of constants we have $X_{U_n}-B_n\overset{P}{\to}0$, $n\to\infty$, and D-record-stable (in probability) iff $X_{U_n}-X_{U_{n-1}}\overset{P}{\to}0$.

Gnedenko (1943) and Geffroy (1958) have shown that in case of max-stability, we have

$$(1.3) \qquad A_n \sim F^{-1}(1-\tfrac{1}{n}), \; n\to\infty ,$$

where $F^{-1}(y) = \inf\{x; F(x) \geq y\}$, $0<y<1$, while in case of record-stability,

$$(1.4) \qquad B_n \sim F^{-1}(1-e^{-n}), \; n\to\infty ,$$

(Resnick (1973)). Resnick also proved that an equivalent condition for record-stability is

$$(\mathrm{I}) \qquad \lim_{x\to\infty} \frac{G(x+\varepsilon)-G(x)}{\sqrt{G(x+\varepsilon)}} = \infty \text{ for all } \varepsilon>0 \text{ where } G = -\log(1-F).$$

On the other hand, Geffroy showed that max- and D-max-stability are equivalent, a necessary and sufficient condition for both being

$$(\mathrm{II}) \qquad \lim_{x\to\infty} \frac{1-F(x+\varepsilon)}{1-F(x)} = 0 \text{ for all } \varepsilon>0$$

(even without the assumption of continuity for F).

It is easy to see that (I) implies (II); however, in general, record-stability and D-record-stability are no longer equivalent. A necessary and sufficient condition for D-record-stability (Goldie (1981)) is

$$(\mathrm{III}) \qquad \lim_{x\to\infty} \frac{1}{\sqrt{x}} \int_{G^{-1}(x)}^{G^{-1}(x+\alpha\sqrt{x})} \frac{1-F(u+\varepsilon)}{1-F(u)}\, dG(u) = 0 \text{ for all } \varepsilon,\alpha>0$$

which shows that (II) is (only) a sufficient condition for (III). Of course, (III) implies that

$$(\mathrm{IV}) \qquad \liminf_{x\to\infty} \frac{1-F(x+\varepsilon)}{1-F(x)} = 0 \text{ for all } \varepsilon>0 .$$

It is the aim of this paper to show that D-max- and D-record-stability generally are different concepts, more precisely, that condition (III) is generally a weaker condition than (II).

2. D-MAX-STABILITY AND D-RECORD-STABILITY

Summarizing the implications from the introduction we obtain

(I) $\Leftrightarrow$ record-stability $\Rightarrow$ max-stability $\Leftrightarrow$ (II) $\Leftrightarrow$

D-max-stability $\Rightarrow$ D-record-stability $\Leftrightarrow$ (III) $\Rightarrow$ (IV) .

In general, conditions (I) and (II) are not equivalent as can be seen from the choice

$$(2.1) \qquad F(x) = \begin{cases} 1 - \exp(-x^{\alpha}), & x > 0 \\ 0 & , x < 0 \end{cases} \qquad \text{for some } \alpha > 1 \ .$$

Here $\{X_n; n \in \mathbb{N}\}$ is (D-)max-stable for all $\alpha > 1$, whereas it is record-stable only for $\alpha > 2$ (Resnick (1973)). Note that for $\alpha = 1$, condition (IV) is violated, reflecting the fact that in this case the record sequence has iid exponentially distributed increments, such that D-record-stability is impossible.

However, if we assume that 1-F is log-concave (i.e. G is convex) in some infinite interval $[a,\infty)$, then the ratio $[1-F(x+\varepsilon)]/[1-F(x)]$ is decreasing in x in $[a,\infty)$ for every $\varepsilon > 0$ (Mathar (1981)). Hence (IV) implies (II) such that in this case, (D-)max-stability and D-record-stability are indeed equivalent (actually, in (2.1), we have $G(x) = x^{\alpha}$ which is convex for every $\alpha > 1$ in $[0,\infty)$).

In spite of this observation, we shall show in the sequel that a suitable stretching of observations may affect the (D-)max-stability of $\{X_n; n \in \mathbb{N}\}$, without affecting the D-record-stability, thus implying that D-max- and D-record-stability are in fact different concepts.

We shall consider the following class of cdf's F:

(V) F has, for x large enough, a derivative f(x), and

$$\lim_{x\to\infty} \frac{f(x)}{1-F(x)} = \infty \ .$$

For such a cdf F, the corresponding integrated hazard-rate G is also differentiable for x large enough, and (V) is equivalent to

$$\lim_{x\to\infty} G'(x) = \infty \quad . \tag{2.2}$$

Distributions with (V) have been considered before by von Mises (1936) and Geffroy (1958); the latter showed that (V) implies (II), i.e. (D-)max-stability. Moreover, he pointed out that every F with (II) is 'associated' to a cdf F_1 satisfying (V) in the sense that for all F with (II) there exists F_1 with (V) and a strictly increasing sequence $\{x_n; n \in \mathbb{N}\}$ with $x_{n+1} - x_n \to 0$, $n \to \infty$ such that

$$F(x_n) = F_1(x_n) \quad \text{for all} \quad n \in \mathbb{N} \ . \tag{2.3}$$

Hence, the cdf's satisfying (V) are the 'smoothly max-stable' cdf's satisfying (II).

The following result will be the key for a more detailed investigation of the relationship between (D-)max- and D-record-stability.

<u>Lemma 2.1</u>: If a cdf F satisfies (V), then

$$\lim_{x\to\infty} \frac{1}{\sqrt{x}} \left\{ G^{-1}(x+\alpha\sqrt{x}) - G^{-1}(x) \right\} = 0 \quad \text{for all} \quad \alpha > 0 \ . \tag{2.4}$$

Proof: Since F is a cdf, G as well as G^{-1} must be (weakly) increasing. Moreover, by the mean-value theorem, there exists for all x large enough a real number y(x) with $x \leq y(x) \leq x + \alpha\sqrt{x}$ and

(2.5) $$\frac{G^{-1}(x+\alpha\sqrt{x}) - G^{-1}(x)}{\alpha\sqrt{x}} = (G^{-1})'(y(x)) = \frac{1}{G'(G^{-1}(y(x)))} .$$

Since with $x \to \infty$ we have $y(x) \to \infty$, thus $G^{-1}(y(x)) \to \infty$, hence the result follows by (2.2). ∇

Lemma 2.2: Let $\{X_n; n \in \mathbb{N}\}$ be a 'smoothly max-stable' (and hence (D-) max- and D-record-stable) sequence, i.e. F fulfills (V). Define

(2.6) $$Y_n = X_n + [\![X_n]\!] , \; n \in \mathbb{N}$$

where $[\![\cdot]\!]$ denotes the integer part. Then $\{Y_n; n \in \mathbb{N}\}$ is no longer (D-) max-stable but still D-record-stable.

Proof: Let $F^* = 1-\exp(-G^*)$ denote the cdf of the Y-sequence. Then

(2.7) $$G^*(x) = \begin{cases} G(x-k), & 2k \leq x \leq 2k+1 \\ G(k+1), & 2k+1 \leq x \leq 2(k+1) \end{cases} , \; k \in \mathbb{Z}^+ , \; x \geq 0 .$$

From here we see that

(2.8) $$\limsup_{x\to\infty} \frac{1-F^*(x+\varepsilon)}{1-F^*(x)} = 1, \; \liminf_{x\to\infty} \frac{1-F^*(x+\varepsilon)}{1-F^*(x)} = 0, \text{ all } \varepsilon > 0 ,$$

hence $\{Y_n; n \in \mathbb{N}\}$ cannot be (D-)max-stable.

To prove D-record-stability for the Y-sequence, observe that for sufficiently large y we have

(2.9) $$(G^*)^{-1}(y) = G^{-1}(y) + [\![G^{-1}(y)]\!] , \qquad \text{hence}$$

(2.10) $$2G^{-1}(y) - 1 \leq (G^*)^{-1}(y) \leq 2G^{-1}(y) .$$

Without loss of generality, let $0<\varepsilon<1$, and

$$(2.11)\qquad r_x(\varepsilon) = \sup\left\{\frac{1 - F(u+\varepsilon)}{1 - F(u)};\ u > G^{-1}(x/2)\right\} \quad .$$

For abbreviation, put

$$(2.12)\qquad u_x = G^{-1}(x) \qquad u_x^* = G^{*-1}(x)$$

$$o_x = G^{-1}(x+\alpha\sqrt{x}) \qquad o_x^* = G^{*-1}(x+\alpha\sqrt{x}) \ .$$

Then, for sufficiently large x, we have

$$(2.13)\qquad \frac{1}{\sqrt{x}} \int_{u_x^*}^{o_x^*} \frac{1 - F^*(u+\varepsilon)}{1 - F^*(u)}\, dG^*(u) =$$

$$\sum_{k=0}^{\infty} \frac{1}{\sqrt{x}} \int_{[2k,2k+1-\varepsilon]\cap[u_x^*,o_x^*]} \frac{1-F^*(u+\varepsilon)}{1-F^*(u)}\, dG^*(u) +$$

$$\sum_{k=0}^{\infty} \frac{1}{\sqrt{x}} \int_{[2k+1-\varepsilon,2k+1]\cap[u_x^*,o_x^*]} \frac{1-F^*(u+\varepsilon)}{1-F^*(u)}\, dG^*(u) =: I + J \ ,$$

say. But by assumption, for sufficiently large x,

$$(2.14)\qquad I < r_x(\varepsilon)\, \frac{1}{\sqrt{x}} \int_{u_x^*}^{o_x^*} dG^*(u) < \alpha\ r_x(\varepsilon)\ \to 0,\ x\to\infty$$

for all $\alpha>0$. Also, for all $k\in\mathbb{N}$,

$$(2.15)\qquad \int_{2k+1-\varepsilon}^{2k+1} \frac{1 - F^*(u+\varepsilon)}{1 - F^*(u)}\, dG^*(u) = \{1-F(k+1)\} \int_{k+1-\varepsilon}^{k+1} e^{G(u)}\, dG(u) =$$

$$1 - \frac{1 - F(k+1)}{1 - F(k+1-\varepsilon)} < 1 \ ,$$

hence, for x sufficiently large, we have

$$(2.16)\qquad J < \frac{1}{\sqrt{x}} \left\{1 + \frac{1}{2}(o_x^* - u_x^*)\right\} < \frac{1}{\sqrt{x}} \left\{\frac{3}{2} + o_x - u_x\right\} =$$

$$\frac{3}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \left\{G^{-1}(x+\alpha\sqrt{x}) - G^{-1}(x)\right\} \to 0,\ x\to\infty \quad \text{for all} \quad \alpha>0$$

by Lemma 2.1. ∇

REMARK

a) The foregoing result shows that D-record-stability is not equivalent to a mere tail-property of the underlying distribution as are conditions (I), (II) or (IV) unless the cdf F is e.g. log-concave. Without further conditions on F, D-record-stability is in general a weaker property than (D-)max-stability.

b) Corresponding to the weaker notions of relative max- and relative D-max-stability defined by $X_{n:n}/C_n \xrightarrow{P} 1$, $n \to \infty$ for a suitable sequence of constants $\{C_n; n \in \mathbb{N}\}$, and $X_{n:n}/X_{n-1:n} \xrightarrow{P} 1$, $n \to \infty$, resp. (cf. Gnedenko (1943), Geffroy (1958)), we can treat relative record- and D-record-stability analogously by considering the random variables

$$Y_n = \begin{cases} \log X_n, & \text{if } X_n > 1 \\ 0, & \text{otherwise} \end{cases} .$$

In this case, the ratios $\frac{1-F(x+\varepsilon)}{1-F(x)}$ have simply to be replaced by the ratios $\frac{1-F(\gamma x)}{1-F(x)}$ with $\gamma > 1$.

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