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An Alternative Proof of a Limit Theorem for the Pólya–Lundberg Process

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Abstract

For the jump time sequence $\{X_n; n \geq 0\}$ of a Pólya–Lundberg process it is shown that n/X_n is asymptotically gamma-distributed, the limiting distribution being related to the unconditional risk distribution of the process. A statistical inference problem arising from this fact is also discussed.

1. Introduction

We consider a Pólya–Lundberg process $\{N(t); t \geq 0\}$ with intensities

$$\lambda_n(t) = \lambda \frac{1+\alpha n}{1+\alpha\lambda t}, \quad n, t \geq 0 \ (\alpha, \lambda > 0). \tag{1}$$

This kind of process has in detail been studied by Lundberg (1940) who also investigated applications to insurance problems.

As has recently been shown by the author, the process $\{N(t); t \geq 0\}$ can equivalently be described by the jump times

$$X_n = \sup\{t \geq 0; N(t) = n\}, \quad n \geq 0 \tag{2}$$

which actually form a Markov chain (MC) with transition probabilities

$$P(X_n > t | X_{n-1} = s) = \frac{1-F_n(t)}{1-F_n(s)}, \quad 0 \leq s \leq t, \ n \geq 1 \tag{3}$$

and initial distribution F_0 where

$$F_n(t) = 1 - \exp\left\{\int_0^t \lambda_n(s)\, ds\right\} = 1-(1+\alpha\lambda t)^{-(n+1/\alpha)}, \quad n, t \geq 0. \tag{4}$$

It is the purpose of this paper to show that $T_n := n/X_n$ is asymptotically gamma-distributed for $n \to \infty$ with a limiting density of the form

$$f(t) = \frac{1}{(\alpha\lambda)^{1/\alpha}\Gamma\left(\frac{1}{\alpha}\right)} t^{1/\alpha - 1} e^{-(t/\alpha\lambda)}, \quad t > 0. \tag{5}$$

Note that for $\lambda = 1$, this coincides with the unconditional risk distribution of $\{N(t); t \geq 0\}$. This result is also implicit in the work of Lundberg (1940), but has not been stated in the above form.

2. The Limit Theorem

Let $f_n(t)=F_n'(t)=\lambda(\alpha n+1)(1+\alpha\lambda t)^{-(n+1+1/\alpha)}$, $t>0$, $n\geq 0$, denote the density function corresponding to F_n. By the MC-property of $\{X_n;\ n\geq 0\}$, a density h_n for $(X_0, \ldots, X_n)$ is given by

$$h_n(t_0, \ldots, t_n)=\prod_{k=0}^{n-1}\frac{f_k(t_k)}{1-F_{k+1}(t_k)}f_n(t_n),\ t_0<\ldots<t_n. \tag{6}$$

In our case,

$$h_n(t_0, \ldots, t_n)=\lambda^{n+1}\prod_{k=1}^{n}(\alpha k+1)(1+\alpha\lambda t_n)^{-(n+1+1/\alpha)},\quad 0<t_0<\ldots<t_n. \tag{7}$$

The density g_n of X_n can now easily be obtained by integration, giving

$$g_n(t)=\frac{t^n}{n!}\lambda^{n+1}\prod_{k=1}^{n}(\alpha k+1)(1+\alpha\lambda t)^{-(n+1+1/\alpha)},\quad t>0. \tag{8}$$

Note that (8) is similar to the frequency function of a Pólya-distribution (Lundberg, 1940, (13)).

The density of T_n now is

$$\frac{n}{t^2}g_n\left(\frac{n}{t}\right)=\frac{1}{(\alpha\lambda)^{1/\alpha}\Gamma\left(\frac{1}{\alpha}\right)}t^{1/\alpha-1}\frac{\Gamma(n+1+1/\alpha)}{n^{1/\alpha}\Gamma(n+1)}\left(1+\frac{t}{\alpha\lambda n}\right)^{-(n+1+1/\alpha)},\quad t>0. \tag{9}$$

A passage to the limit shows that

$$\lim_{n\to\infty}\frac{n}{t^2}g_n\left(\frac{n}{t}\right)=f(t),\quad t>0, \tag{10}$$

hence we have proved the following limit law:

Theorem. *Under the assumptions made above, $T_n\xrightarrow{\mathcal{D}}T$ for $n\to\infty$ where T is following a gamma-distribution given by* (5).

As can easily be seen by the densities (7) and (8), $(1+1/n)T_n$ is the maximum-likelihood-estimate for λ, and T_n is a minimal sufficient and complete unbiased estimate for λ, hence among all unbiased estimates for λ depending on $X_0, \ldots, X_n$, T_n has minimum variance $V(T_n)=\lambda^2(1+\alpha n)/(n-1)$. However, by the Theorem, T_n is not consistent for $n\to\infty$, hence there is no suitable estimation of λ from the observation of a single process.

For a convenient estimation of λ from several independent processes see Lundberg (1940), p. 137.

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References

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