Semigroup Forum Vol. 46 (1993) 279–285

RESEARCH ARTICLE

A probabilistic variant of Chernoff's product formula

Dietmar Pfeifer

Communicated by J. Goldstein

Abstract In this paper we present a version of Chernoff's product formula based on probabilistic representations for equi–bounded operator semigroups which covers e.g. Hille's first exponential formula as well as the Post–Widder real inversion formula. General rates of convergence are given in terms of the rectified modulus of continuity.

1. Introduction

Let $\mathcal{X}$ denote a real or complex Banach space with norm $\|.\|$ and $\mathcal{E}(\mathcal{X})$ the Banach algebra of bounded endomorphisms on $\mathcal{X}$. Let further denote $\{T(t) \mid t \geq 0\} \subseteq \mathcal{E}(\mathcal{X})$ a one–parameter semigroup of operators of class $(\mathcal{C}_0)$. Then the following well–known properties hold true:

a) For each $f \in \mathcal{X}$, $T(.)f$ is strongly continuous.

b) For the characteristic exponent $\omega \geq -\infty$ one has

$$\omega = \inf_{t>0} \frac{1}{t} \log \|T(t)\| = \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\| < \infty. \tag{1}$$

c) For each $\eta > \omega$, the exists a constant $M \geq 1$ such that

$$\|T(t)\| \leq M e^{\eta t}, \quad t \geq 0. \tag{2}$$

In particular, there exists always a pair $(\eta, M) \in \mathbb{R} \times [1, \infty)$ such that relation (2) holds.

In case that in (2) the choice $\eta = 0$ is possible the semigroup is also said to be *equi–bounded*; if in addition $M = 1$ is admissible then $\{T(t) \mid t \geq 0\}$ is called a *contraction semigroup*. Obviously, every semigroup $\{T(t) \mid t \geq 0\}$ of class $(\mathcal{C}_0)$ can be transformed into some equi–bounded semigroup $\{S(t) \mid t \geq 0\}$ by means of

$$S(t) = e^{-\eta t} T(t), \quad t \geq 0, \ \eta > \omega. \tag{3}$$

If A_h denotes the operator

$$A_h = \frac{1}{h}\big[T(h) - I\big], \quad h > 0, \tag{4}$$

and $\mathcal{D} \subseteq \mathcal{X}$ denotes the set of all elements $f \in \mathcal{X}$ such that the strong limit $\lim_{h\downarrow 0} A_h f$ exists in $\mathcal{X}$, then the (infinitesimal) generator A of $\{T(t) \mid t \geq 0\}$ is — as usual — defined to be the operator on $\mathcal{D} = \mathcal{D}(A)$ given by

$$Af = \lim_{h\downarrow 0} A_h f, \quad f \in \mathcal{D}(A). \tag{5}$$

Recall that $\mathcal{D}(A)$ is also called the domain of A.

The following basic result of semigroup theory can be found in almost every textbook on this subject, see e.g. Butzer & Berens (1967) or Goldstein (1985).

Theorem 1.1. *(Representation of Semigroups) Let A be the generator of a semigroup $\{T(t) \mid t \geq 0\}$ of class $(\mathcal{C}_0)$. Then there holds true:*

a) *If A is a bounded operator, then $\mathcal{D}(A) = \mathcal{X}$, and the semigroup is necessarily of the form*

$$T(t)f = \exp(tA)f := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k f, \quad f \in \mathcal{X}. \tag{6}$$

In this case, $\{T(t) \mid t \geq 0\}$ is also uniformly continuous (i.e. stronly continuous in the operator topology over $\mathcal{E}(\mathcal{X})$). Conversely, if the semigroup is uniformly continuous, then A is bounded, and relation (6) holds true. In particular, for any bounded operator A, (6) always defines a uniformly continuous semigroup $\{T(t) \mid t \geq 0\}$.

b) *In general, with the notation of (4),*

$$T(t)f = \lim_{h \downarrow 0} \exp(tA_h)f, \quad f \in \mathcal{X}. \tag{7}$$

For equi–bounded semigroups, an estimation for the rate of approximation is given by

$$\|T(t)f - \exp(tA_h)f\| \leq \begin{cases} M(1+\sqrt{t})\omega^b(\sqrt{h}, f), & \text{if } f \in \mathcal{X} \\ M\sqrt{t}(1+\sqrt{t})\sqrt{h}\omega^b(\sqrt{h}, Af), & \text{if } f \in \mathcal{D}(A) \end{cases} \tag{8}$$

for $0 \leq t \leq b - \sqrt{h}$, where

$$\omega^b(\tau, f) = \sup\left\{\|T(t)f - T(s)f\| \mid s,t \in [0,b],\ |t-s| \leq \tau\right\}, \quad \tau > 0, \tag{9}$$

denotes the rectified modulus of continuity of $\{T(t) \mid t \geq 0\}$ over the interval $[0,b]$. ■

A proof of relation (8) which sharpenes a corresponding statement in Butzer & Berens (1967) is given in section 2 below.

Part b) of the preceding Theorem is usually referred to as *Hille's first exponential formula.* Note that here $\{\exp(tA_h) \mid t \geq 0\}$ is always a uniformly continuous semigroup with bounded generator A_h for all $h > 0$.

For the general case, rates of approximation can e.g. be obtained via the transformation given in (3) observing that — up to terms of order $\mathcal{O}(h)$ and uniformly in t in compact intervals —

$$T(t)f - e^{tA_h}f \approx e^{\eta t}\left[S(t) - e^{tB_h}\right]f, \quad 0 \leq t \leq b,\ f \in \mathcal{X}, \tag{10}$$

where $B_h = \dfrac{S(h) - I}{h}$, $h > 0$.

Besides classical representation formulas as such above there is another way of obtaining the semigroup by limits of operator products, known as *Chernoff's product formula* (Chernoff (1968, 1974)). We state the corresponding theorem as in Goldstein (1985), Theorem 8.4.

Theorem 1.2. *(Chernoff's Product Formula) Suppose* $\{T(t) \mid t \geq 0\}$ *is a contraction semigroup, and* $\{V(t) \mid t \geq 0\}$ *is a family of contractions on* $\mathcal{X}$ *with* $V'(0) = I$ *such that the derivative* $V'(0)f$ *exists for all* f *in a subset* $\mathcal{D}$ *of* $\mathcal{X}$, *and the closure* A *of* $V'(0)|_{\mathcal{D}}$ *generates* $\{T(t) \mid t \geq 0\}$. *Then for each* $f \in \mathcal{X}$,

$$\lim_{n\to\infty} V\left(\frac{t}{n}\right)^n f = T(t)f, \tag{11}$$

uniformly for t *in compact subsets of* $\mathbb{R}^+$. ■

In this paper we want to show that a version of this theorem can be obtained by probabilistic representations as in Pfeifer (1984 a,b, 1985, 1986), together with rates of convergence similar to (8), even for general (not necessarily contraction or equi–bounded) semigroups.

2. The probabilistic setting

A general approach to representations of semigroups of class $(\mathcal{C}_0)$ going back to ideas of Bernstein as early as 1912 — picked up later by M. Riesz (cf. Hille and Phillips (1957)) and Chung (1962) — has been developed in more detail by Pfeifer (1984 a,b, 1985, 1986), covering in particular Hille's exponential formula as a special case. There is, however, the problem of measurability of semigroups, which is in some sense a necessary condition in order to apply probability theory. For the sake of completeness, we shall recall some of the most important results here, mostly without proof.

Proposition 2.1. *(Measurability of Semigroups) Let* $\{T(t) \mid t \geq 0\}$ *be a semigroup of class* $(\mathcal{C}_0)$ *with generator* A. *Then for any* $f \in \mathcal{X}$, *the map* $T(.)f : t \mapsto T(t)f$ *is continuous, hence (strongly) measurable w.r.t. to the Borel* σ*–field generated by the topology over* $\mathcal{X}$. *The map* $T(.) : t \mapsto T(t)$ *is, in general, only measurable w.r.t. the Borel* σ*–field generated by the operator topology over* $\mathcal{E}(\mathcal{X})$ *if* A *is bounded, i.e. if the semigroup is uniformly continuous. A sufficient condition for* $T(.)$ *to be neither measurable nor separably valued is*

$$\liminf_{t\downarrow t_0} \|T(t) - T(t_0)\| > 0 \tag{12}$$

for some $t_0 > 0$. ■

A simple example of a non–measurable semigroup is given by the semigroup $\{T(t) \mid t \geq 0\}$ of translations, defined by

$$T(t)f(.) = f(. + t), \quad t \geq 0, \quad f \in UCB(\mathbb{R}), \tag{13}$$

where $UCB(\mathbb{R})$ denotes the Banach space of uniformly continuous, bounded functions on $\mathbb{R}$, equipped with the norm $\|f\| = \sup\{|f(x)| \mid x \in \mathbb{R}\}$. Here there holds $\|T(t) - T(s)\| = 2$ whenever $t \neq s$, hence condition (12) is fulfilled. Note that in this case $Af = f'$ (the derivative of f) which is an unbounded operator on the domain $\mathcal{D}(A) = \{f \in UCB(\mathbb{R}) \mid f' \text{ exists on } \mathbb{R} \text{ with } f' \in UCB(\mathbb{R})\}$.

When one attempts to consider "random operators" $T(X)$ for a semigroup $\{T(t) \mid t \geq 0\}$ and a random variable X with values in $\mathbb{R}^+$ Proposition 2.1 essentially says that such an object is, in general, not a random element in $\mathcal{E}(\mathcal{X})$ in the strict sense unless the semigroup is uniformly continuous, i.e. the generator is bounded, or X is countably valued. However, by the strong continuity of the semigroup, $T(X)f$ is always a random element in $\mathcal{X}$ for all $f \in \mathcal{X}$.

Proposition 2.2. *(Expectation of Random Semigroup Operators) Let* $\{T(t) \mid t \geq 0\}$ *be a semigroup of class* $(\mathcal{C}_0)$ *with generator* A *and* X *be a random variable with values in* $\mathbb{R}^+$ *such that the moment generating function* Ψ_X, *given by* $\Psi_X(s) = E(e^{sX})$, $s \geq 0$, *is finite for* $s = \eta$ *with* η *as in (2). Then for all* $f \in \mathcal{X}$, $E[T(X)f] = \int T(X)f\,dP$ *exists in the Bochner sense in* $\mathcal{X}$ *with*

$$\Big\| E\big[T(X)f\big] \Big\| \leq M\Psi_X(\eta)\|f\| \tag{13}$$

where M *is as in (2). Moreover, the map* $E\big[T(X)\big] : f \mapsto E\big[T(X)f\big]$ *on* $\mathcal{X}$ *defines a bounded linear operator* $E\big[T(X)\big] \in \mathcal{E}(\mathcal{X})$ *with*

$$\Big\| E\big[T(X)\big] \Big\| \leq M\Psi_X(\eta). \tag{14}$$

$E\big[T(X)\big]$ *is also called the expectation of* $T(X)$. *If, moreover,* A *is bounded, then*

$$E\big[T(X)\big] = E\big[e^{AX}\big] = \sum_{k=0}^{\infty} \frac{E(X^k)}{k!} A^k = \Psi_X(A), \tag{15}$$

i.e. $E\big[T(X)\big]$ *exists in the Bochner sense in* $\mathcal{E}(\mathcal{X})$. ∎

Proposition 2.3. *(expectation of semigroup operator products) Let* $\{T(t) \mid t \geq 0\}$ *be a semigroup of class* $(\mathcal{C}_0)$ *and* X *and* Y *be independent random variables with values in* $\mathbb{R}^+$ *such that the moment generating functions* $\Psi_X(s)$ *and* $\Psi_Y(s)$, $s \geq 0$, *are both finite for* $s = \eta$ *with* η *as in (2). Then* $E\big[T(X)\big]$, $E\big[T(Y)\big]$ *and* $E\big[T(X+Y)\big]$ *exist in* $\mathcal{E}(\mathcal{X})$, *and there holds*

$$E\big[T(X) \circ T(Y)\big] = E\big[T(X+Y)\big] = E\big[T(X)\big] \circ E\big[T(Y)\big]. \tag{16}$$

where "$\circ$" denotes composition. ∎

Theorem 2.1. *(Probabilistic Approximation of Semigroups) Let* $\{T(t) \mid t \geq 0\}$ *be an equi–bounded semigroup of class* $(\mathcal{C}_0)$ *and* X *a random variable with values in* $\mathbb{R}^+$ *such that* $E(X) = t \in \mathbb{R}$, *and for the variance,* $\mathrm{Var}(X) = \sigma^2 < \infty$. *Let* ω^b *again denote the rectified modulus of continuity of* $\{T(t) \mid t \geq 0\}$ *as in (9). Then for any* $\varepsilon > 0$ *the following holds true:*

$$\big\| E\big[T(X)\big]f - T(t)f \big\| \leq \begin{cases} M\Big(1 + \dfrac{\sigma}{\varepsilon}\Big)\omega^b(\varepsilon, f) & \text{for } f \in \mathcal{X} \\ M\sigma\Big(1 + \dfrac{\sigma}{\varepsilon}\Big)\omega^b(\varepsilon, Af) & \text{for } f \in \mathcal{D}(A) \end{cases} \tag{17}$$

in the range $0 \leq t \leq b - \varepsilon$.

Proof. Let N denote the random variable $N = 1 + \left\lfloor \dfrac{|X-t|}{\varepsilon} \right\rfloor$ with $\lfloor . \rfloor$ denoting the integer part, and $Y = \min\{X, t\}$, $\delta = \dfrac{|X-t|}{N}$. Then $\delta \leq \varepsilon$ and, for

$f \in \mathcal{X}$,

$$\begin{aligned}
\|E[T(X)]f - T(t)f\| &\le E\Big[\|T(X)f - T(t)f\|\Big] \\
&= E\left[\left\|\sum_{k=1}^{N}[T(Y+k\delta)f - T(Y+(k-1)\delta)f]\right\|\right] \\
&\le E\left[\left\|\sum_{k=1}^{N}T((k-1)\delta)\circ[T(Y+\delta)-T(Y)]f\right\|\right] \\
&\le ME\left[N\Big\|[T(Y+\delta)-T(Y)]f\Big\|\right] \le ME(N)\omega^b(\varepsilon,f) \\
&\le M\left(1+\frac{E(|X-t|)}{\varepsilon}\right)\omega^b(\varepsilon,f) \le M\left(1+\frac{\sigma}{\varepsilon}\right)\omega^b(\varepsilon,f).
\end{aligned}$$

This proves the first part of relation (17). The second part follows by a linear substitution (cf. Butzer & Berens (1967), Prop. 1.1.6), in an analogous way: for $f \in \mathcal{D}(A)$,

$$T(X)f - T(t)f = (X-t)Af + (X-t)\int_0^1 \left[T(t+u(X-t)) - T(t)\right]Af\,du,$$

hence

$$\begin{aligned}
\|E[T(X)]f - T(t)f\| &\le ME(|X-t|\cdot N)\omega^b(\varepsilon,Af) \\
&\le ME\Big(|X-t| + \frac{(X-t)^2}{\varepsilon}\Big)\omega^b(\varepsilon,Af) \le M\sigma\Big(1+\frac{\sigma}{\varepsilon}\Big)\omega^b(\varepsilon,Af).
\end{aligned}$$ ∎

Corollary 2.1. *(A Probabilistic Variant of Chernoff's Product Formula) Let* $\{T(t) \mid t \ge 0\}$ *be an equi-bounded semigroup of class* $(\mathcal{C}_0)$ *with generator* A *and* X *be a random variable with values in* $\mathbb{R}^+$ *such that* $E(X) = 1$ *and the variance* $\sigma^2 = \mathrm{Var}(X)$ *is finite. Let, for* $t \ge 0$, *denote* $V(t) = E[T(tX)]$. *Then for all* $t \ge 0$, $V(t)$ *is a bounded operator on* $\mathcal{X}$ *with* $\|V(t)\| \le M$, *and there holds*

$$T(t)f = \lim_{n\to\infty} V^n\left(\frac{t}{n}\right)f, \quad f \in \mathcal{X}, \tag{18}$$

with

$$\|T(t)f - V^n(\tfrac{t}{n})f\| \le \begin{cases} M(1+\sigma t)\omega^b(n^{-1/2},f) & \text{for } f \in \mathcal{X} \\ M\sigma t(1+\sigma t)n^{-1/2}\omega^b(n^{-1/2},Af) & \text{for } f \in \mathcal{D}(A) \end{cases} \tag{19}$$

in the range $0 \le t \le b - n^{-1/2}$, $b > 0$ *(i.e. uniform convergence in compact intervals). Moreover,*

$$Af = \lim_{t\downarrow 0}\frac{1}{t}(V(t)-I)f = V'(0)f, \quad f \in \mathcal{D}(A). \tag{20}$$

Proof. First observe that by Proposition 2.2, $V(t)$ is bounded for all $t \ge 0$. To prove the convergence part (18), it suffices to show relation (19). For this purpose, let $Y_1, \dots, Y_n$, $n \in \mathbb{N}$, be independent copies of X, and choose $Y = \frac{t}{n}\sum_{k=1}^{n} Y_k$. Then $V^n(t/n) = E\big[T(Y)\big]$ by Proposition 2.3. But $E(Y) = t$, $\operatorname{Var}(Y) = \dfrac{\sigma^2 t^2}{n}$, hence (19) follows from (17) with $\varepsilon = n^{-1/2}$. To prove (20) it suffices to show, for all $f \in \mathcal{D}(A)$,

$$\lim_{t \downarrow 0} \left\| \frac{1}{t}\big(V(t) - T(t)\big) f \right\| = 0. \tag{21}$$

But again by (17), chosing $\varepsilon = t < b/2$, we obtain

$$\begin{aligned} \left\| \frac{1}{t}\big(V(t) - T(t)\big) f \right\| &= \frac{1}{t}\big\| E\big[T(tX)\big] f - T(t) f \big\| \\ &\le M\sigma(1+\sigma)\omega^b(t, Af) \longrightarrow 0 \quad (t \downarrow 0) \end{aligned}$$

which proves (21) and hence the Corollary. ■

Note that in case that the generator A is bounded and the moment generating function $\Psi_X(s) = E\big(e^{sX}\big)$ exists in some neighbourhood of the origin, V takes the form

$$V(t) = E\left[e^{tXA}\right] = \Psi_X(tA) \tag{22}$$

for sufficiently small t, hence in this case,

$$T(t)f = \lim_{n \to \infty} \Psi_X^n\Big(\frac{t}{n} A\Big) f, \quad f \in \mathcal{X}. \tag{23}$$

A special version of Theorem 1.1 b) can immediately be obtained from the above Corollary in case that X is Poisson–distributed with mean 1, i.e. $P^X = \mathfrak{P}(1)$, since then

$$\begin{aligned} V(t)f = E\big[T(tX)\big] f &= \frac{1}{e}\sum_{k=0}^{\infty} \frac{1}{k!} T(kt) f = \frac{1}{e}\sum_{k=0}^{\infty} \frac{1}{k!} T(t)^k f \\ &= \frac{1}{e}\exp\big\{T(t)\big\} f = \exp\big\{T(t) - I\big\} f, \quad f \in \mathcal{X}, \end{aligned} \tag{24}$$

and hence

$$\exp\big\{tA_{t/n}\big\} f = V^n\left(\frac{t}{n}\right) f \longrightarrow T(t) f, \quad t \ge 0,\ f \in \mathcal{X}, \tag{25}$$

rates of convergence being given by (19) with $\sigma = 1$. The more general version of Theorem 1.1 b) follows from Theorem 2.1 when X is of the form $X = hZ$, where $P^Z = \mathfrak{P}(t/h)$, $t \ge 0, h > 0$, and $\varepsilon = \sqrt{h}$. Here $E(X) = t$, $\sigma^2 = ht$, hence (8) follows from (17) noting that, for any $f \in \mathcal{X}$,

$$E\big[T(X)\big] f = e^{-t/h} \sum_{k=0}^{\infty} \frac{(t/h)^k}{k!} T(h)^k f = \exp\Big(\frac{t}{h}\big[T(h) - I\big]\Big) f = \exp(tA_h) f. \tag{26}$$

In the particular case that X is exponentially distributed with mean 1 we obtain

$$\begin{aligned} V(t)f = E\big[T(tX)\big] f &= \int_0^{\infty} e^{-s} T(ts) f \, ds \\ &= \frac{1}{t}\int_0^{\infty} e^{-u/t} T(u) f \, du = \frac{1}{t} R\left(\frac{1}{t}\right) f, \quad t > 0,\ f \in \mathcal{X}, \end{aligned} \tag{27}$$

where R denotes the resolvent of the semigroup. Corollary 2.1 thus gives the Post–Widder real inversion formula

$$\left\{\frac{n}{t}R\left(\frac{n}{t}\right)\right\}^n = V^n\left(\frac{t}{n}\right)f \longrightarrow T(t)f, \quad t>0,\ f\in\mathcal{X}, \tag{28}$$

rates of convergence being given by (19) with $\sigma = 1$ again.

Finally we should like to mention that an extension of Corollary 2.1 to arbitrary $(\mathcal{C}_0)$–semigroups $\{T(t) \mid t\geq 0\}$ is possible by means of the transformation (3); namely, if under the conditions of the Corollary, we let $U(t) = E\big[S(tX)\big]$ and $V(t) = e^{\eta t}U(t)$, $t \geq 0$, then (18) remains still valid since in (19), we only have to replace M by $Me^{\eta t}$. Note that the modulus of continuity ω^b here refers to the semigroup S rather than T, which by (3) is, however, of the same order for small values of τ (cf. relation (9)), up to some positive constant.

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Fachbereich Mathematik
Carl von Ossietzky Universität Oldenburg
P.O. Box 25 03
W–2900 Oldenburg
Germany

Received June 18, 1991
and in final form February 8, 1992