J. Appl. Prob. **19**, 127–135 (1982)
Printed in Israel
0021-9002/82/010127-09 $01.15

CHARACTERIZATIONS OF EXPONENTIAL DISTRIBUTIONS BY INDEPENDENT NON-STATIONARY RECORD INCREMENTS

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Abstract

A non-homogeneous version of the classical record process is presented which allows two different characterizations of exponential distributions by independent non-stationary record increments. A connection with the interarrival times of the corresponding record counting process (which is pure birth) is also pointed out.

RECORD VALUES; MARKOV ADDITIVE CHAIN; EXPONENTIAL DISTRIBUTION; INDEPENDENT NON-STATIONARY INCREMENTS; COUNTING PROCESS; PURE BIRTH PROCESS

1. Introduction

In recent years a fruitful theory has been developed concerning record values from i.i.d. random variables (see Glick (1978) for a list of references), but only a relatively small number of authors investigated more general record models (Biondini and Siddiqui (1975), Guthrie and Holmes (1975), Yang (1975), Gaver (1976), Westcott (1977), Deken (1978)). Unlike these we consider a non-homogeneous version of the classical record model which arises from possible changes of the underlying distributions after every record event. To be more precise, let $\{X_{00}, X_{nk}\,;\, n, k \geqq 1\}$ be a family of independent random variables on a probability space $(\Omega, \mathcal{A}, P)$ with P_n being the distribution of the X_{nk} and F_n being the corresponding cumulative distribution function (c.d.f.), $n \geqq 0$. The sequence $\{\Delta_n\,;\, n \geqq 0\}$ of *interrecord times* is recursively defined by

$$\Delta_0 = 0, \quad \Delta_{n+1} = \min\{k\,;\, X_{n+1,k} > X_{n,\Delta_n}\}. \tag{1.1}$$

To be well-defined, let $\min(\emptyset) = X_{n\infty} = \infty$. The sequence $\{U_n\,;\, n \geqq 0\}$ of *record times* is defined by

$$U_n = 1 + \sum_{k=0}^{n} \Delta_k. \tag{1.2}$$

Received 30 September 1980; revision received 16 June 1981.

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The sequence $\{R_n; n \geqq 0\}$ of *record values* is defined by

$$R_n = X_{n,\Delta_n}. \tag{1.3}$$

Let $\sigma(X)$ denote the σ-algebra generated by the random variable X. As can easily be seen by induction, Δ_n and U_n, $n \geqq 1$, are stopping times with respect to $\{\mathcal{A}_{nk}; 1 \leqq k \leqq \infty\}$ where

$$\mathcal{A}_{nk} = \sigma(X_{00}, X_1, \cdots, X_{n-1}, X_{n1}, \cdots, X_{nk})$$

and

$$X_m = (X_{m1}, X_{m2}, \cdots), \qquad m \geqq 1;$$

also, R_n is measurable with respect to $\mathcal{A}_{n\infty}$.

To give an example, suppose the $\{X_{nk}; k \geqq 1\}$ correspond to random shocks attacking a component which works without failure unless a shock greater than R_{n-1} occurs. Let for safety reasons a modified component then be used which endures shocks up to a magnitude of the last shock, R_n; also, let safety factors be built in which influence the distribution of the subsequent shocks, $\{X_{n+1,k}; k \geqq 1\}$. Then U_n denotes the time of the nth accident, and obviously time periods Δ_n between successive accidents will be stochastically increasing, if the underlying distributions P_n are stochastically decreasing. Hence the latter case corresponds to a shock model with increasing safety. (A similar shock model in reliability theory has been described by Gaver (1976).)

2. Non-homogeneous records and Markov additive chains

Throughout this paper, we shall only be concerned with the non-degenerate case, i.e. $\Delta_n < \infty$ a.s. for all n. Let ξ_n denote the right end of F_n. The following proposition gives necessary and sufficient conditions for the record process to be non-degenerate.

Proposition 2.1. For all $n \geqq 1$, $\Delta_n < \infty$ a.s. iff $\Delta_{n-1} < \infty$ a.s. and

$$\xi_{n-1} \leqq \xi_n \quad \text{with } \xi_{n-1} < \xi_n \quad \text{if } \xi_{n-1} \text{ is an atom of } F_{n-1}. \tag{2.1}$$

Proof. Let $\Delta_n < \infty$ a.s., then $\Delta_{n-1} < \infty$ a.s. by definition. Since $R_{n-1} \geqq X_{n-1,1}$ (let $X_{01} = X_{00}$) we have

$$0 = P(\Delta_n = \infty) = P\left(\bigcap_{k=1}^{\infty} \{X_{nk} \leqq R_{n-1}\}\right) \geqq P(R_{n-1} \geqq \xi_n)$$
$$\geqq P(X_{n-1,1} \geqq \xi_n)$$

from which (2.1) immediately follows.

For the converse part note that by the independence assumption

$$P\left(\bigcap_{j=1}^{\infty}\{X_{nj}\leqq X_{n-1,k}\}\,\Big|\,X_{n-1,k}\right)=\prod_{j=1}^{\infty}F_n(X_{n-1,k})=0 \quad \text{a.s.},$$

hence

$$P(\Delta_n=\infty)=P\left(\bigcup_{k=1}^{\infty}\{\Delta_{n-1}=k\}\cap\bigcap_{j=1}^{\infty}\{X_{nj}\leqq X_{n-1,k}\}\right)$$
$$\leqq\sum_{k=1}^{\infty}P\left(\bigcap_{j=1}^{\infty}\{X_{nj}\leqq X_{n-1,k}\}\right)=0.$$

In the i.i.d. case Proposition 2.1 reduces to a result of Shorrock (1972).

In what follows we shall always assume that (2.1) is valid for all $n\geqq 1$.

Theorem 2.2. $\{(\Delta_n, R_n); n\geqq 0\}$ is a Markov chain with transition probabilities

$$P_{n-1,n}(k,x\mid A\times B)=P_n((x,\infty)\cap B)\sum_{j\in A}F_n^{j-1}(x), \tag{2.2}$$

$k\in\mathbb{N}$, $x<\xi_n$, $A\subseteq\mathbb{N}$, $B\in\mathcal{B}$, $n\geqq 1$, where $\mathcal{B}$ denotes the collection of all Borel sets $B\subseteq\mathbb{R}$. This can be proved by methods similar to those of the homogeneous case.

As can easily be seen from (2.2), $\{(U_n, R_n); n\geqq 0\}$ also is a Markov chain with transition probabilities

$$Q_{n-1,n}(k,x\mid A\times B)=P_n((x,\infty)\cap B)\sum_{j\in A\cap(k,\infty)}F_n^{j-k-1}(x), \tag{2.3}$$

$k\in\mathbb{N}$, $x<\xi_n$, $A\subseteq\mathbb{N}$, $B\in\mathcal{B}$, which are translation invariant with respect to k.

As immediate consequences from (2.2) and (2.3) we have the following results, paralleling those known from the i.i.d. case.

Corollary 2.3.

(a) $\{(U_n, R_n); n\geqq 0\}$ is a Markov additive chain (cf. Çinlar (1972)), and $\{R_n; n\geqq 0\}$ is a Markov chain with transition probabilities

$$P_{n-1,n}(x\mid B)=P_n(B\mid(x,\infty)),\qquad x<\xi_n,\ B\in\mathcal{B}. \tag{2.4}$$

(b) $\Delta_1,\cdots,\Delta_n$ are conditionally independent given $R_0,\cdots,R_{n-1}$ with

$$P\left(\bigcap_{i=1}^{n}\{\Delta_i=k_i\}\,\Big|\,R_0,\cdots,R_{n-1}\right)=\prod_{i=1}^{n}P(\Delta_i=k_i\mid R_{i-1})$$
$$=\prod_{i=1}^{n}\{1-F_i(R_{i-1})\}F_i^{k_i-1}(R_{i-1}) \tag{2.5}$$

a.s., $k_1,\cdots,k_n\in\mathbb{N}$.

Note that unlike the i.i.d. case the sequence $\{U_n; n\geqq 0\}$ of record times will *not* be a Markov chain in general. Let for instance $F_n = F^{c_n}$ where F is a continuous c.d.f. and $c_n > 0$. Then

$$P(U_n = k_n \mid U_1 = k_1, \cdots, U_{n-1} = k_{n-1})$$

$$= c_n \frac{\sum_{i=1}^{n-1} (c_i - c_{i-1})k_{i-1} + c_{n-1}k_{n-1}}{\left\{\sum_{i=1}^{n} (c_i - c_{i-1})k_{i-1} + c_n(k_n - 1)\right\}\left\{\sum_{i=1}^{n} (c_i - c_{i-1})k_{i-1} + c_n k_n\right\}}, \tag{2.6}$$

$$1 < k_1 < \cdots < k_n.$$

3. Characterizations of exponential distributions by independent record increments

In this section we shall for simplicity assume that all P_n are concentrated on the non-negative real axis $\mathbb{R}^+$ with right end ∞ and $F_n(0) = 0$; this implies that all record distributions are also concentrated on that set.

Theorem 3.1. Let $n \geqq 1$. If P_n is an exponential distribution, then R_{n-1} and $R_n - R_{n-1}$ are independent, and the distribution of $R_n - R_{n-1}$ is the same as P_n. Conversely, if R_{n-1} and $R_n - R_{n-1}$ are independent and F_{n-1} is strictly increasing on $\mathbb{R}^+$, then P_n is an exponential distribution.

Proof. We first prove the converse part. From relation (2.4) we see that

$$P(R_n - R_{n-1} \leqq s \mid R_{n-1} = t) = \frac{F_n(s+t) - F_n(t)}{1 - F_n(t)} \quad \text{a.s. } P^{R_{n-1}} \tag{3.1}$$

for $s \geqq 0$, the exceptional set possibly depending on s. Also,

$$g_m(x) = \int_{(-\infty,x)} \frac{1}{1 - F_m(y)} P^{R_{m-1}}(dy), \qquad x \in \mathbb{R},\ m \geqq 1 \tag{3.2}$$

is a P_m-density of R_m, non-decreasing with x, hence by our assumptions G_{n-1}, the c.d.f. of R_{n-1}, must be strictly increasing on $\mathbb{R}^+$. This implies that for every $s \geqq 0$ the set of points $t \geqq 0$ for which equality holds in (3.1) is a dense subset of $\mathbb{R}^+$. By the right continuity of c.d.f.'s and the independence assumption we then have

$$\frac{F_n(s+t) - F_n(t)}{1 - F_n(t)} = 1 - H_n(s) \quad \text{for } s, t \geqq 0, \tag{3.3}$$

where $1 - H_n$ is the c.d.f. of $R_n - R_{n-1}$. This implies

$$\begin{aligned} H_n(s+t) &= \frac{1 - F_n(s+t+u)}{1 - F_n(u)} = \frac{1 - F_n(s+u)}{1 - F_n(u)} \cdot \frac{1 - F_n(t + (s+u))}{1 - F_n(s+u)} \\ &= H_n(s)H_n(t) \quad \text{for } s, t \geqq 0 \text{ with } H_n(0) = 1. \end{aligned} \tag{3.4}$$

The only (bounded) solution of (3.4) is thus given by

$$(3.5) \qquad H_n(s) = e^{-\lambda_n s} \quad \text{for some real } \lambda_n > 0,$$

i.e. P_n is an exponential distribution.

Conversely, if P_n is an exponential distribution, the right-hand side of (3.1) is a.s. $P^{R_{n-1}}$ independent of t, hence R_{n-1} and $R_n - R_{n-1}$ are independent with the distribution of $R_n - R_{n-1}$ being the same as P_n.

Note that the assumption of the strict increasingness of F_{n-1} in Theorem 3.1 is not redundant as can be seen by the following counterexample: let $[\![x]\!]$ denote the greatest integer not exceeding $x, x \in \mathbb{R}$ and take

$$F_1(x) = \begin{cases} 1 - \exp\{-([\![x]\!] + x)\}, & x \geqq 0 \\ 0, & x < 0. \end{cases}$$

Let P_0 be any distribution concentrated on the integers $\mathbb{N}$. Then R_0 and $R_1 - R_0$ are independent although F_1 is *not* an exponential distribution. This is true since for all $s \geqq 0$, $k \in \mathbb{N}$ we have $[\![s + k]\!] = [\![s]\!] + k$, hence by (3.1),

$$P(R_1 - R_0 \leqq s \mid R_0 = k) = \frac{F_1(s+k) - F_1(k)}{1 - F_1(k)} = F_1(s),$$

independent of k.

Proceeding inductively, Theorem 3.1 leads to the following generalization of Tata's (1969) characterization theorem.

Corollary 3.2. If all P_n, $n \geqq 1$ are exponential distributions, $\{R_n ; n \geqq 0\}$ is an independent increments process, the distribution of $R_n - R_{n-1}$ being the same as P_n. Conversely, if $\{R_n ; n \geqq 0\}$ is an independent increments process and F_0 is strictly increasing on $\mathbb{R}^+$, then all P_n, $n \geqq 1$, are exponential distributions.

Another explanation of the first part of Corollary 3.2 could be given as follows:

Suppose all F_n possess right-continuous densities f_n with respect to Lebesgue measure. Define a counting process $\{N(t); t \geqq 0\}$ for record values by $N(t) = \#\{n; R_n \leqq t\}$.

Then $\{N(t); t \geqq 0\}$ is a (Markovian) pure birth process with intensies $\lambda_n(t)$ given by the hazard rates

$$(3.6) \qquad \lambda_n(t) = \frac{f_n(t)}{1 - F_n(t)}, \qquad n, t \geqq 0.$$

This is true since for $0 < t_1 < \cdots < t_k$ and non-negative integers $n_1, \cdots, n_k$, $k \geqq 1$ we have $(R_{-1} = 0)$

$$(3.7) \qquad P(N(t_1) = n_1, \cdots, N(t_k) = n_k) = P\left(\bigcap_{i=1}^{k} \{R_{n_i - 1} \leqq t_i < R_{n_i}\} \right).$$

By the Markov chain property of $\{R_n ; n \geqq 0\}$ and repeated integration we thus get

$$P(N(t_k) = n_k \mid N(t_1) = n_1, \cdots, N(t_{k-1}) = n_{k-1})$$

$$(3.8) \qquad = \begin{cases} \dfrac{1-F_{n_k}(t_k)}{1-F_{n_{k-1}}(t_{k-1})} \displaystyle\int \cdots \int \dfrac{P_{n_k-1}(du_{n_k-1}) \cdots P_{n_{k-1}}(du_{n_{k-1}})}{\{1-F_{n_k}(u_{n_k-1})\} \cdots \{1-F_{n_{k-1}+1}(u_{n_{k-1}})\}}, \\ \qquad t_{k-1} < u_{n_{k-1}} < \cdots < u_{n_k-1} \leqq t_k \\ \qquad \text{if } n_k \geqq n_{k-1} \\ 0 \qquad \text{if } n_k < n_{k-1} \end{cases}$$

which says that $\{N(t); t \geqq 0\}$ is a (Markovian) pure birth process with standard transition matrix and intensities

$$\lambda_n(t) = \lim_{s \downarrow t} \frac{1}{1-F_n(t)} \frac{F_n(s) - F_n(t)}{s-t} = \frac{f_n(t)}{1-F_n(t)}.$$

But in the exponential case, $\lambda_n(t) \equiv \lambda_n$, independent of t, hence $\{N(t); t \geqq 0\}$ is time-homogeneous. Also, the waiting time for the nth jump, $n \geqq 1$, is exactly the nth record R_{n-1}, hence the interarrival times for the counting process are just the successive record increments. From the general theory we thus can conclude that $\{R_n ; n \geqq 0\}$ is an independent increments process, the increments $R_n - R_{n-1}$, $n \geqq 1$ being exponentially distributed with parameter λ_n (see Breiman (1968), Chapter 15.6).

In the remainder of this section we want to investigate the question whether the characterization of exponential distributions given by Corollary 3.2 also holds under the weaker condition that only *successive* record increments are independent. The answer is positive if we additionally assume that P_0 is an exponential distribution; otherwise a counterexample can be constructed which shows that a corresponding characterization of exponential distributions does not necessarily hold, not even if F_0 is strictly increasing on $\mathbb{R}^+$. For the proof of the main theorem the following two lemmas are needed.

Lemma 3.3. Let $\{Y_k ; k \geqq 0\}$ be independent exponentially distributed random variables with mean $1/\lambda_k > 0$ and let f_n, $n \geqq 0$ denote the density of $\Sigma_{k=0}^n Y_k$, continuous on $\mathbb{R}^+$. Then f_{n+1} is differentiable with

$$(3.9) \qquad f'_{n+1}(x) = \lambda_{n+1}(f_n(x) - f_{n+1}(x)), \qquad x > 0.$$

Proof. By the convolution formula we have

$$f_{n+1}(x) = \int_0^x f_n(y) \lambda_{n+1} \exp(-\lambda_{n+1}(x-y)) dy.$$

The desired result now follows by differentiation.

Lemma 3.4. Let $\mathscr{C}_r^b(\mathbb{R}^+)$ denote the set of all real-valued measurable functions defined on $\mathbb{R}^+$ which are right continuous and bounded and let f_n, $n \geqq 0$ be as in Lemma 3.3. Define a linear operator $\mathscr{I}_n$ from $\mathscr{C}_r^b(\mathbb{R}^+)$ into $\mathscr{C}_r^b(\mathbb{R}^+)$ by

$$\mathscr{I}_n(g; x) = \int_0^\infty g(x+y) f_n(y) dy, \qquad g \in \mathscr{C}_r^b(\mathbb{R}^+).$$

Then $\mathscr{I}_n(g) \equiv 0$ only if $g \equiv 0$, i.e. the transformation given by $\mathscr{I}_n$ is unique.

Proof. For $x > 0$ we have

$$\mathscr{I}_{n+1}(g; x) = \int_x^\infty g(y) f_{n+1}(y - x) dy,$$

hence

$$\begin{aligned} \mathscr{I}'_{n+1}(g; x) &= -\int_x^\infty g(y) f'_{n+1}(y-x) dy - g(x) f_{n+1}(0) \\ &= -\lambda_{n+1}(\mathscr{I}_n(g; x) - \mathscr{I}_{n+1}(g; x)) \quad \text{a.e. by (3.9)} \end{aligned} \tag{3.10}$$

since $f_{n+1}(0) = 0$. Therefore, by the right continuity,

$$\mathscr{I}_{n+1}(g) \equiv 0 \quad \text{implies} \quad \mathscr{I}_n(g) \equiv \cdots \equiv \mathscr{I}_0(g) \equiv 0$$

with $0 = \mathscr{I}'_0(g; x) = -\lambda_0(g(x) - \mathscr{I}_0(g; x))$ a.e., hence $g \equiv 0$ a.e. But g is right continuous, hence $g \equiv 0$ everywhere.

Theorem 3.5. Suppose that P_0 is an exponential distribution. Then if for the record sequence $\{R_n \,;\, n \geqq 0\}$ successive increments are independent all P_n, $n \geqq 1$ are necessarily exponential distributions.

Proof. From Theorem 3.1 we see that P_1 must be an exponential distribution. Suppose now the theorem is proved for $P_1, \cdots, P_n$ with $n \geqq 1$. Then $R_0, R_1 - R_0, \cdots, R_n - R_{n-1}$ are independent exponentially distributed random variables. Hence by the Markov chain property of $\{R_n \,;\, n \geqq 0\}$ given by (2.4) we have for $n \geqq 1$, $u, v \geqq 0$, using the notation of Lemma 3.3,

$$\begin{aligned} &P(R_{n+1} - R_n > u, R_n - R_{n-1} > v) \\ &= P(R_{n-1} + u + v < R_n + u < R_{n+1}) \\ &= \iiint_{s+u+v<t+u<w} \frac{P_{n+1}(dw) P_n(dt)}{\{1 - F_{n+1}(t)\}\{1 - F_n(s)\}} f_{n-1}(s) ds \\ &= e^{-\lambda_n v} \int_0^\infty \frac{1 - F_{n+1}(t+u+v)}{1 - F_{n+1}(t+v)} f_n(t) dt. \end{aligned} \tag{3.11}$$

Let

$$g_u(t) = \frac{1 - F_{n+1}(t+u)}{1 - F_{n+1}(t)}, \qquad u, t \geqq 0.$$

Then by the independence assumption, $\mathcal{I}_n(g_u) = \text{const.}$, hence $g_u = \text{const.}$ (depending on u) by Lemma 3.4. Now for $H(u) = g_u(0)$ we have

$$H(u+v) = H(u)H(v) \quad \text{with } H(0) = 1, \tag{3.12}$$

hence $H(u) = e^{-\lambda_{n+1}u}$ for some real $\lambda_{n+1} > 0$ which implies that P_{n+1} also is an exponential distribution.

For the counterexample indicated above let F_0 have the density $f_0 = \sum_{n=0}^{\infty} a_n 1_{[n,n+1)}$ where $\{a_n; n \geqq 0\}$ is a sequence of positive reals with $\sum_{n=0}^{\infty} a_n = 1$ and let

$$F_n(x) = 1 - e^{-x}, \qquad n = 1, 3, 4, \cdots$$

$$F_2(x) = 1 - \exp\{-(2\pi x + \sin 2\pi x)\}.$$

Then R_0 and $R_1 - R_0$ are independent as well as $R_{n+1} - R_n$ and $R_n - R_{n-1}$ for $n \geqq 3$ by Theorem 3.1. Similar to (3.11), we further have

$$P(R_3 - R_2 > u, R_2 - R_1 > v) = e^{-u} \int_0^{\infty} \frac{1 - F_2(s+v)}{1 - F_2(s)} P^{R_1}(ds) \tag{3.13}$$

and

$$P(R_2 - R_1 > u, R_1 - R_0 > v) = e^{-v} \int_0^{\infty} \int_0^{\infty} e^{-t} \frac{1 - F_2(s+t+u+v)}{1 - F_2(s+t+v)} f_0(s) ds dt$$

$$= e^{-v} \sum_{n=0}^{\infty} a_n \int_0^{\infty} \int_n^{n+1} e^{-t} \frac{1 - F_2(s+t+u+v)}{1 - F_2(s+t+v)} f_0(s) ds dt \tag{3.14}$$

$$= e^{-v} \int_0^1 \frac{1 - F_2(s+u)}{1 - F_2(s)} ds$$

since $(1 - F_2(\cdot + u))/(1 - F_2(\cdot)) = \exp\{-2(\pi u + \sin(\pi u)\cos(2\pi \cdot + \pi u))\}$ is periodic with period 1. But (3.13) and (3.14) immediately imply that $R_{n+1} - R_n$ and $R_n - R_{n-1}$ are also independent for $n = 1, 2$. Hence successive record increments are independent although F_2 is *not* an exponential distribution.

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CORRECTIONS

PFEIFER, D. (1982) Characterizations of exponential distributions by independent non-stationary record increments. *J. Appl. Prob.* **19**, 127–135.

(1) Relation (2.6) should be corrected as follows:

$$P(U_n = k_n \mid U_1 = k_1, \cdots, U_{n-1} = k_{n-1})$$

$$(2.6) \qquad = c_n \frac{\sum_{i=1}^{n-1} (c_{i-1} - c_i) k_{i-1} + c_{n-1} k_{n-1}}{\left\{\sum_{i=1}^{n} (c_{i-1} - c_i) k_{i-1} + c_n (k_n - 1)\right\}\left\{\sum_{i=1}^{n} (c_{i-1} - c_i) k_{i-1} + c_n k_n\right\}},$$

$$1 = k_0 < k_1 < \cdots < k_n.$$

(2) Theorem 3.1 should read as follows:

Theorem 3.1. Let $n \geqq 1$. If P_n is an exponential distribution, then R_{n-1} and $R_n - R_{n-1}$ are independent, and the distribution of $R_n - R_{n-1}$ is the same as P_n. Conversely, let the origin be the left end of $F_0, \cdots, F_{n-2}$, if $n \geqq 2$. Then if R_{n-1} and $R_n - R_{n-1}$ are independent and F_{n-1} is strictly increasing on $\mathbb{R}^+$, P_n is an exponential distribution.

Without the additional assumption in the converse part of the theorem, G_{n-1} might be 0 in some neighbourhood of the origin; in this case, the conditional distribution $P_n(\cdot \mid (\zeta, \infty))$ could be characterized as being exponential only where $\zeta =$ the largest of the left ends of $F_0, \cdots, F_{n-2}$.

(3) In the second line of (3.14), the term $f_0(s)$ should be deleted.