ON ASYMPTOTIC BEHAVIOR OF WEIGHTED SAMPLE QUANTILES

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We study the limit behavior of L-statistics with the 'delta type' weight sequences, of which the O-statistics and usual kernel quantile estimators are special cases. Deviations of these statistics from the sample quantiles are shown to have (after proper normalization) Gaussian limit distributions. We evaluate the main term of the asymptotics of the covariance of this deviation with the corresponding sample quantile. This gives also the asymptotic relative deficiency of sample quantiles w.r.t. our L-statistics.

Key words: sample quantile, quantile estimator, L-statistic, quantile process, relative efficiency, second order properties.

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1. Introduction.

Let $X_1, \ldots, X_n$ be a sample of random variables with a hypothetical continuous distribution function $F = F^{(n)}$ (depending in general case on n, so that we are in the triangular array scheme setup), $X_{(1)} \le X_{(2)} \le \ldots \le X_{(n)}$ be the order statistics of the sample. Last decade a lot of papers were devoted to the study of various versions of L-statistics

$$(1.1) \qquad \sum_{j=1}^{n} v_j X_{(j)}, \qquad v_j = v_j^{(n)},\ j = 1, \ldots n, \qquad \sum_{j=1}^{n} v_j = 1,$$

having the property that, for any $\varepsilon > 0$,

$$\sum_{|j/n - t| > \varepsilon} |v_j| \to 0 \qquad \text{as} \qquad n \to \infty$$

for some fixed $t \in (0, 1)$. Such statistics are intended for estimating the value of the quantile function

$$(1.2) \qquad Q(t) = Q^{(n)}(t) = F^{-1}(t) := \sup\{x : \ F(x) \le t\}, \quad 0 \le t \le 1,$$

at this point t, and in regular case they turn to be superior to the sample quantiles, i.e. the values of the empirical quantile function (e.q.f.)

$$Q_n(t) = F_n^{-1}(t) = X_{(j)} \qquad \text{for} \qquad \frac{j-1}{n} \le t < \frac{j}{n}, \qquad j = 1, \dots, n,$$

F_n being the empirical distribution function (e.d.f.) of the sample $X_1, \dots, X_n$. This observation was supported by both experimental data (see e.g. discussions and simulation data in Kaigh and Cheng [16], Kaigh and Driscoll [17], Sheater and Marron [24], Steinberg and Davis [27], Stewart [28], and Yang [30]) and theoretical considerations for the case of independent identically distributed (i.i.d.) X's (here one could mention in addition the works by Azzalini [1], Driscoll and Kaigh [6], Kaigh [15], Falk [7 – 10], Padgett [20], Parzen [21], Reiss [22], and Zelterman [31]). The first result to be cited here states that in the i.i.d. case the relative deficiency of the sample quantile w.r.t. the linear combination of *finitely many* order statistics goes to infinity (rather quickly) as the sample size $n \to \infty$ (Reiss [22]).

Now note that statistic (1.1) may be viewed (for a special choice of $v_j \ge 0$) as a result of averaging the sample quantiles for a 'subsampling' procedure. Thus, sampling without replacement leads to the 'O-statistic'

$$\sum_{j=r}^{n+r-d} \frac{\binom{j-1}{r-1}\binom{n-j}{d-r}}{\binom{n}{d}} X_{(j)};$$

we draw a subsample of size d from the original sample $X_1, \dots, X_n$, and take the rth order statistic, $r < d$, with $r/d \sim t$ (Kaigh – Lachenbruch, or K-L-statistic; sampling with replacement gives another version called Harrel – Davis, or H-D-statistic). These statistics were shown to be asymptotically normal and further asymptotically equivalent to each other and to the kernel type quantile estimator (1.6) with the normal 'window' function k and 'bandwidth'

$$\alpha_n(t) = n^{-1/2}\big(t(1-t)\big)^{1/2} \tag{1.3}$$

(Zelterman [31]).

Another asymptotically equivalent estimator (which is close to the 'continuous' version of resampling without replacement) is given by the binomial weights:

$$\widetilde{Q}_n(t) = \sum_{j=0}^{n-1} \binom{n-1}{j} t^j (1-t)^{n-1-j} X_{(j+1)} \tag{1.4}$$

(cf. e.g. Muñoz-Pérez and Fernández Palacín [19]). Clearly $\widetilde{Q}_n(t)$ is the Bernstein polynomial of order $n-1$ for the sample quantile function $Q_n(t)$. It follows from the well-known properties of Bernstein polynomials and continuity theorems for empirical processes that if $Q^{(n)}$ converges to a continuous limit Q, then so does Q_n, and hence $\widetilde{Q}_n$ also converges uniformly to Q. The Moivre – Laplace theorem implies that in the i.i.d. case (1.4) is asymptotically equivalent to the kernel estimator (1.6) with the standard normal density k and time dependent bandwidth (1.3). The special case of (1.4) for $t = 1/2$ was studied (under the name of 'Islamic mean') in Dehling et al. [5]. It was shown in particular that the deviation of this version of the 'smoothed' sample median from the usual one has the following limit behavior: if $F(t) = t,\ 0 \le t \le 1$, then the law

$$\mathcal{L}\left(n^{3/4}(\widetilde{Q}_n(1/2) - Q_n(1/2))\right) \Rightarrow N(0, \sigma^2), \tag{1.5}$$

the normal distribution with mean 0 and variance $\sigma^2 = (2^{1/2} - 1)/2\pi^{1/2}$ (the sign $\Rightarrow$ denotes here and in what follows the weak convergence of the corresponding laws).

See also Falk [11, 12] and Falk and Reiss [13, 14] for relevant problems for bootstrap quantile estimators.

Another special class of statistics of the form (1.1) admitting representation in the kernel estimator form

$$\hat{Q}_n(t) = \frac{1}{\alpha_n} \int_0^1 Q_n(x)\, k\left(\frac{t-x}{\alpha_n}\right) dx \tag{1.6}$$

(with a fixed kernel k for all n with some bandwidth parameter $\alpha_n \to 0$ as $n \to \infty$) was extensively studied in the literature of nonparametric estimation (see e.g. Azzalini [1], Falk [8, 9], Padgett [20], Parzen [21], Sheater and Marron [24], Yang [30], Zelterman [31]). We state here the main result of Falk [8] giving the asymptotic behavior of the relative deficiency

$$d(n) = \min\left\{j :\ \mathbf{E}\big(Q_{n+j}(t) - Q(t)\big)^2 \le \mathbf{E}\big(\hat{Q}_n(t) - Q(t)\big)^2\right\} \tag{1.7}$$

of the sample quantile $Q_n(t)$ w.r.t. the estimator (1.6) when X's are i.i.d. and $F = F^{(n)}$ does not depend on n.

Let $\lim_{t\to\infty} t^{\delta}(1 - F(t) + F(t)) = 0$ for some $\delta > 0$, $Q(t)$ be $(m+1)$ times differentiable in a neighbourhood A of $t \in (0, 1),\ m \ge 2$, and the derivative of order $(m+1)$ be bounded in A, $Q'(t) > 0$. Suppose that the kernel k has bounded support $[-c, c]$, and that

$$\int k(x)\, dx = 1, \qquad \int x^j k(x)\, dx = 0, \qquad j = 1, \dots, m.$$

If

$$\alpha_n n^{1/4} \to \infty \qquad \text{and} \qquad \alpha_n n^{1/(2m+1)} \to 0 \qquad \text{as} \qquad n \to \infty, \tag{1.8}$$

then

$$\lim_{n\to\infty} \frac{d(n)}{n\alpha_n} = \frac{\Psi(k)}{t(1-t)}, \tag{1.9}$$

where

$$\Psi(k) = 2 \int x\, k(x)\, K(x)\, dx, \qquad K(x) = \int_{-c}^{x} k(y)\, dy. \tag{1.10}$$

The functional Ψ, which can be interpreted as a measure of performance of the estimator (1.6) was studied in Falk [7, 8].

In the present paper we study more general L-statistics of the form (1.1) and prove for them a result of the type stated in (1.5) for the deviation of our statistics from sample quantiles. Further we evaluate the asymptotics of the covariance of such a deviation with the sample quantile itself. This gives us the asymptotics of the relative deficiency of the latter w.r.t. statistic (1.1) (without restrictions of the type (1.8)).

2. Main results and proofs.

Let $\{V_t^{(n)}\}_{0 \le t \le 1}$, $n = 1, 2, \ldots$, be a sequence of families of distributions on $\mathbf{R}$. In what follows, we assume that the following uniform 'unbiasedness' condition is satisfied:

$$\int_0^1 x\, V_t^{(n)}(dx) = t + \delta_t^{(n)}, \qquad \delta^{(n)} = \sup_t |\delta_t^{(n)}| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.1}$$

We suppose also that the measures $W_t^{(n)}(A) = V_t^{(n)}(t + \alpha_n A)$ converge weakly for a sequence $\alpha_n \to 0$ (we assume $n\alpha_n \to \infty$ to avoid trivial situations) as $n \to \infty$:

$$W_t^{(n)} \Rightarrow W_t, \quad 0 \le t \le 1, \tag{2.2}$$

Convergence of measures

$$|x| W_t^{(n)} \Rightarrow |x| W_t, \quad 0 \le t \le 1. \tag{2.3}$$

will also be supposed in some of the assertions below.

We shall study statistics of the form

$$Q_n^*(t) = \int Q_n(x) V_t^{(n)}(dx). \tag{2.4}$$

Clearly these are L-statistics of the form (1.1) with

$$v_j = v_j(t, n) = V_t^{(n)}\big(((j-1)/n, j/n]\big), \quad j > 1, \quad v_1 = V_t^{(n)}\big([0, 1/n]\big). \tag{2.5}$$

Kernel estimators (1.6) are of course of the form (2.4) with $V_t^{(n)}$ having density $\alpha_n^{-1} k((t - x)\alpha_n^{-1})$ and thus being of the same form for all $t \in (0, 1)$ (in this case $W_t^{(n)}$ does not depend on n and t).

Remark 1. Taking $V_t^{(n)}$ to be the law of the variable $Bi(n-1,t)/(n-1)$, where $Bi(n,p)$ denotes binomial random variable with parameters n,p, we get our statistic $\widetilde{Q}_n(t)$ from (1.4). In this case clearly $\alpha_n = n^{-1/2}$, and the limiting law is the normal $\big(0, t(1-t)\big)$ distribution:

$$W_t(z) \equiv W_t((-\infty, z]) = \Phi\big(z(t(1-t))^{-1/2}\big),$$

Φ is the standard normal distribution.

Now let us make a small digression. The use of the e.q.f. for statistical inference is a common practice (and any use of order statistics can be viewed this way too), although it still looks less natural and convenient than the use of the e.d.f. Of course, these two objects are in duality (1.2) and behavior of one of them determines uniquely the behavior of another. It is well known that (by the Glivenko – Cantelli theorem)

$$\sup_t |F_n(t) - F^{(n)}(t)| = \sup_t |U_n(t) - t| \equiv \sup_t |R_n(t) - t| \to 0$$

where $U_n(t) = F_n(Q^{(n)}(t))$ is the e.d.f. and $R_n(t) = U_n^{-1}(t)$ is the e.q.f. for the sample

$$Y_k = F^{(n)}(X_k) \tag{2.6}$$

from the uniform $[0,1]$ distribution. On the other hand, if the family $\{Q^{(n)}\}^{n\geq 1}$ is equicontinuous, it follows from the representation

$$X_{(k)} = Q^{(n)}(Y_{(k)}), \tag{2.7}$$

where $Y_{(k)}$ are order statistics of the uniform sample (2.6), that

$$\sup_t |Q_n(t) - Q^{(n)}(t)| = \sup_t |Q^{(n)}(R_n(t)) - Q^{(n)}(t)| \to 0 \quad \text{as} \quad n \to \infty.$$

Moreover, behavior of the corresponding empirical process is closely related to that of the empirical quantile process as well. Assume for the sake of simplicity that all

$$F^{(n)}(t) \equiv t, \qquad 0 \leq t \leq 1 \tag{2.8}$$

(which is not much of a restriction of generality in view of (2.6) and (2.7)). Then, as it was proved in Vervaat [29] (see also Borovkov [4]), the empirical process

$$n^{1/2}(F_n(t) - t) \Rightarrow u \qquad \text{as} \qquad n \to \infty, \tag{2.9}$$

with continuous process u in the limit, if and only if

$$n^{1/2}(Q_n(t) - t) \Rightarrow -u \qquad \text{as} \qquad n \to \infty, \tag{2.10}$$

Remark 2. Convergence of the type $u_n \Rightarrow u$ here and in what follows is that of distributions on the Skorokhod space $D[0,1]$ of right continuous functions having left side limits. However, if the limiting processes are continuous, one can understand it as the possibility of constructing of versions of the processes u_n and u on a common probability space so that u_n converges to u in uniform topology almost surely, see Skorokhod [26] and e.g. Billingsley [2].

Now we turn to the 'weighted' e.q.f. Q_n^* from (2.4) and compare its limiting behavior with that of Q_n. First we show that these two are asymptotically first-order equivalent (here we make no assumptions on dependence and distributions of X's). Put, for a sequence $\beta_n > 0$, $n \geq 1$,

$$Y_n(t) = \beta_n(Q_n(t) - t), \quad Y_n^*(t) = \beta_n(Q_n^*(t) - t), \quad t \in [0,1].$$

Theorem 1. *Let conditions* (2.1) *and* (2.2) *be satisfied.*

i) *If, for any $t \in (0,1)$, the e.q.f. $Q_n(t) \to t$ as $n \to \infty$, then also $Q_n^*(t) \to t$, $t \in (0,1)$. In both cases convergence is uniform.*

ii) *If, for some $\beta_n \to \infty$, the processes*

$$Y_n \Rightarrow u, \tag{2.11}$$

$u(t)$ being a stochastic process in $C[0,1]$, and $\beta_n \delta^{(n)} \to 0$ as $n \to \infty$, then, for any fixed $t \in (0,1)$, the sequence of random variables $Y_n^*(t)$ converges in distribution to $u(t)$ as $n \to \infty$. This convergence takes place for the processes:

$$Y_n^* \Rightarrow u,$$

if

$$\text{for any } \varepsilon > 0, \quad V_t^{(n)}(\{x : \ |t - x| > \varepsilon\}) \to 0 \quad \text{as} \quad n \to \infty \tag{2.12}$$

uniformly in t.

Proof. i) follows immediately from (2.12) which in turn follows from condition (2.2). Indeed,

$$Q_n^*(t) - t = \int\limits_{|x-t| \leq \varepsilon} \big(Q_n(x) - t\big) V_t^{(n)}(dx) + o(1),$$

and the integral admits (by monotonicity of Q_n) the lower and upper bounds

$$(Q_n(t \mp \varepsilon) - t)\, V_t^{(n)}\,(\{x : \ |t - x| \leq \varepsilon\})$$

respectively, which converge correspondingly to $-\varepsilon$ and ε as $n \to \infty$. Since $\varepsilon > 0$ is arbitrary, the assertion follows. The limiting function is continuous, and hence the uniformitiy of the convergence can easilly be shown using a standard 'monotonicity and compactness' argument.

ii). By the Skorokhod representation theorem (Skorokhod [26]), we can assume without loss of generality, that our processes Q_n and u are given on a common probability space and

$$Y_n(t) = u(t) + \varepsilon_n(t)$$

where $\theta_n = \sup_t |\varepsilon_n(t)| \to 0$ a.s. as $n \to \infty$. Therefore

$$\begin{aligned} Y_n^*(t) =& \beta_n \int (Q_n(x) - x) V_t^{(n)}(dx) + \beta_n \int (x-t) V_t^{(n)}(dx) = \\ =& \int u(x) V_t^{(n)}(dx) + \beta_n \delta_t^{(n)} + \widetilde{\theta}_n, \qquad |\widetilde{\theta}_n| \le \theta_n. \end{aligned}$$

The first term in the last line here tends to $u(t)$ similarly to i) by continuity of $u(t)$, and this convergence is uniform if (2.12) holds unifromly in t, which implies the convergence of processes $Y_n^* \Rightarrow u$ (see Remark 2 above). ■

Now we turn to the second order properties of $Q_n^*(t)$. This averaged function turns by obvious reasons to be uniformly closer to the theoretical quantile function $Q^{(n)}(t) = t,\ 0 \le t \le 1$. We restrict ourselves to the case when the limiting $Y_n(t)$ process is the Brownian bridge $b(t),\ 0 \le t \le 1$, which is the case e.g. if the sample is i.i.d. To give a 'more quantitative' idea of how large such a reduction can be, recall that the process $b(t)$ has stochastic differential

$$db(t) = -\frac{b(t)}{1-t} dt + dw(t),$$

$w(t)$ being the standard Wiener process. Hence if the deviation $b(t_0)$ of the Brownian bridge from zero is large at some point t_0, the both processes $b(t_0 + s)$ and $b(t_0 - s),\ s \ge 0$, drift towards zero the faster the large $b(t_0)$ is (recall that $b(t)$ is time-reversible). Therefore the 'averaging' (2.4) will be 'most efficient' at the extremum points of the trajectory of $b(t)$.

We shall make use of the following useful characteristic of the closeness of the laws of two processes $u(t)$ and $v(t),\ 0 \le t \le 1$, introduced in Borovkov and Sakhanenko [3]: for $\varepsilon > 0$, we put

$$\lambda(u, v, \varepsilon) = \inf \mathbf{P}(\sup_t |u(t) - v(t)| > \varepsilon),$$

where the infimum is taken over the set of all possible constructions of the two processes on a common probability space. We note here only that

$$\Lambda(u, v) = \inf\{\varepsilon :\ \lambda(u, v, \varepsilon) < \varepsilon\}$$

is the Lévy – Prokhorov distance between the distributions of u and v generated by the uniform topology.

Introduce the deviation process

$$Z_n(t) = \beta_n \alpha_n^{-1/2}\big(Q_n^*(t) - Q_n(t)\big), \qquad 0 \le t \le 1, \tag{2.13}$$

and let

$$\begin{aligned} H_t = \lim_{n\to\infty} \Big[\sum |y| \left(\Delta W_t^{(n)}(y)\right)^2 + \\ + 2\int_0^\infty z\,(1 - W_t^{(n)}(z))\,dW_t^{(n)}(z) - 2\int_{-\infty}^0 z\,W_t^{(n)}(z-0)\,dW_t^{(n)}(z)\Big] \end{aligned} \tag{2.14}$$

if the limit exists (sum is over the set of all jumps $\Delta W_t^{(n)}(y)$ of the function $W_t^{(n)}(y)$ as usual). Note that the behavior of the expression in brackets in (2.14) depends heavily upon how does $W_t^{(n)}$ converge to the limit W_t near the jumps of the latter. If W_t is continuous, conditions (2.1) – (2.3) are easily seen to imply that the limit in (2.14) always exists (we just recall that in this case $W_t^{(n)}(z)$ converges to $W_t(z)$ uniformly in z), and

$$H_t = 2\int_0^\infty z(1 - W_t(z))\,dW_t(z) + 2\int_{-\infty}^0 |z| W_t(z)\,dW_t(z). \tag{2.15}$$

For a symmetric continuous limit law W_t,

$$H_t = 4\int_{-\infty}^0 |z|\,W_t(z)\,dW_t(z). \tag{2.16}$$

Theorem 2. *Let conditions* (2.1) *and* (2.2) *be satisfied, and, for any* $\varepsilon > 0$,

$$\lambda(Y_n, b, \varepsilon\alpha_n^{1/2}) \to 0 \qquad \textit{as} \quad n \to \infty \tag{2.17}$$

and

$$\alpha_n^{-1/2}\beta_n\delta^{(n)} \to 0, \tag{2.18}$$

where $\delta^{(n)}$ *is from condition* (2.1).

Then the finite-dimensional distributions of the process Z_n *converge to those of the Gaussian white noise process with mean* 0 *and variance* H_t.

Remark 3. In the standard i.i.d. case $\beta_n = n^{1/2}$, and hence, for the averaging (1.4) we have $\alpha_n = n^{-1/2}$ and

$$W_t(z) = \Phi\big(z(t(1-t))^{-1/2}\big), \tag{2.19}$$

(see Remark 1), and the normalizing factor in (2.13) is $n^{3/4}$. Further, for the specified law $V_t^{(n)}$, we have

$$\int_0^1 x\, V_t^{(n)}(dx) = \frac{1}{n-1}\mathbf{E}Bi(n-1,t) = t.$$

Hence $\delta^{(n)} = 0$ and condition (2.18) is fulfilled. On the other hand, it was shown in Komlos et al. [18] that, say,

$$\lambda(Y_n, b, n^{-1/2}\log^2 n) \to 0,$$

and therefore condition (2.17) is also satisfied. Now (2.16) and (2.19) give us the limiting variance

$$\widetilde{H}_t = 4\big(t(1-t)\big)^{1/2} \int_{-\infty}^{0} |x|\Phi(x)\, d\Phi(x) = (2^{1/2}-1)\left(\frac{t(1-t)}{\pi}\right)^{1/2},$$

for the last equality see e.g. Section 2 in Dehling et al. [5]. Thus relation (1.5) is the special case of our Theorem 2.

Proof of Theorem 2. We have from our assumptions that, for a representation on a common probability space,

$$Q_n(t) = t + \frac{1}{\beta_n} b(t) + \frac{\varepsilon_n(t)}{\beta_n},$$

where $\mathbf{P}(\Omega_n) \to 0$ for the event $\Omega_n = \{\sup_t |\varepsilon_n(t)| > \gamma_n\}$ for a sequence $\gamma_n = o(\alpha_n^{1/2})$. One has

$$\begin{aligned} Z_n(t) =& \alpha_n^{-1/2} \int \beta_n\big(Q_n(x) - Q_n(t)\big)\, V_t^{(n)}(dx) = \\ =& \alpha_n^{-1/2} \int \big(b(x) - b(t)\big)\, V_t^{(n)}(dx) + \\ &+ \alpha_n^{-1/2}\beta_n \left(\int x\, V_t^{(n)}(dx) - t\right) + \alpha_n^{-1/2}\varepsilon_n^*(t) = I_1 + I_2 + \alpha_n^{-1/2}\varepsilon_n^*(t), \end{aligned}$$

where $I_j = I_j(t)$, and $\varepsilon_n^*(t) \le 2\gamma_n$ on the complement event Ω_n^c and hence the last term tends to zero in probability. Now we have also that

$$|I_2| \le \alpha_n^{-1/2}\beta_n\delta^{(n)} \to 0$$

by assumption (2.18) of the theorem. The first integral I_1 is clearly a Gaussian random variable with zero mean. Since the process $b(t)$ has covariance function

$$r(x,t) = \mathbf{E}(b(x)b(t)) = x \wedge t - xt \qquad \text{for} \qquad x,t \in [0,1],$$

and

$$r(t+\alpha_n y, t+\alpha_n z) = t + \alpha_n (y \wedge z) - t^2 - t\alpha_n (y+z) - \alpha_n^2 yz,$$

we have from the Fubini theorem that

$$\begin{aligned}
\mathbf{E}\left(I_1^2(t)\right) &= \frac{1}{\alpha_n}\mathbf{E}\left(\int_0^1\int_0^1 \big(b(x)-b(t)\big)\big(b(s)-b(t)\big)V_t^{(n)}(dx)V_t^{(n)}(ds)\right) = \\
&= \frac{1}{\alpha_n}\int W_t^{(n)}(dy)\int W_t^{(n)}(dz)\big(r(t+\alpha_n y, t+\alpha_n z) - \\
&\qquad - r(t+\alpha_n y, t) - r(t, t+\alpha_n z) + r(t,t)\big) = \\
&= \int W_t^{(n)}(dy)\int W_t^{(n)}(dz)\big(y\wedge z - 0\wedge y - 0\wedge z - -\alpha_n yz\big) \\
&= \int\int \varphi(y,z) W_t^{(n)}(dy) W_t^{(n)}(dz) - \alpha_n\left(\int y W_t^{(n)}(dy)\right)^2,
\end{aligned} \tag{2.20}$$

where

$$\varphi(y,z) = \begin{cases} |y|\wedge|z| & \text{for } yz>0, \\ 0 & \text{otherwise,} \end{cases} \tag{2.21}$$

and the last term in (2.20) is $O(\alpha_n^{-1}{\delta^{(n)}}^2) = o(1)$ by the assumption of the theorem.

The first integral in the last line of (2.20) is easily seen from (2.21) to be equal to

$$\begin{aligned}
&\int_0^\infty\int_0^\infty y\wedge z\, W_t^{(n)}(dy)W_t^{(n)}(dz) + \int_{-\infty}^0\int_{-\infty}^0 |y|\wedge|z|\, W_t^{(n)}(dy)W_t^{(n)}(dz) = \\
&= \sum |y|\left(\Delta W_t^{(n)}(y)\right)^2 + 2\int_0^\infty dW_t^{(n)}(z)\, z\cdot\int_z^\infty dW_t^{(n)}(y) + \\
&\qquad + 2\int_{-\infty}^0 W_t^{(n)}(dz)\, |z| \int_{-\infty}^z dW_t^{(n)}(y) = \\
&= \sum |y|\left(\Delta W_t^{(n)}(y)\right)^2 + 2\int_0^\infty \big(1 - W_t^{(n)}(z)\big)\, z\, dW_t^{(n)}(z) + \\
&\qquad + 2\int_{-\infty}^0 W_t^{(n)}(z-0)\, |z|\, dW_t^{(n)}(z) \to H_t,
\end{aligned}$$

the sum here being the result of integrating along the diagonal.

To complete the proof of Theorem 2, it remains to show that $I_1(t_1)$ and $I_1(t_2)$ are asymptotically uncorrelated for $t_1 \neq t_2$. Similarly to (2.20), we have for $a = t_2 - t_1 > 0$

$$\mathbf{E}\big(I_1(t_1)I_1(t_2)\big) =$$
$$= \frac{1}{\alpha_n}\mathbf{E}\left(\int_0^1\int_0^1 \big(b(x_1)-b(t_1)\big)\big(b(x_2)-b(t_2)\big)V_{t_1}^{(n)}(dx_1)V_{t_2}^{(n)}(dx_2)\right) =$$
$$= \alpha_n \int\int y_1 y_2 W_{t_1}^{(n)}(dy_1)W_{t_2}^{(n)}(dy_2) + o(1) \to 0,$$

where the integration domain is

$$D = \left\{(y_1,y_2):\ y_2 - y_1 > -\frac{a}{\alpha_n},\ y_2 > -\frac{a}{\alpha_n},\ y_1 < \frac{a}{\alpha_n}\right\},$$

and the integral over the complement D^c is easily seen from our conditions (2.1) – (2.3) to be $o(1)$. ■

Now we shall show that the deviation $Z_n(t)$ is negatively correlated with the quantile process $Y_n(t)$ and evaluate the asymptotic deficiency of sample quantiles w.r.t. their averaged versions (2.4). In the remaining part of the paper, we consider only the i.i.d. case with X's uniformly distributed over $[0,1]$ (so that (2.11) holds with $u(t) = b(t)$ and $\beta_n = n^{1/2}$, while α_n can be arbitrary).

Theorem 3. *Let conditions* (2.1) − (2.3) *be satisfied. If* $X_1, X_2, \dots$ *are i.i.d with the uniform distribution on* $[0,1]$, *then for* $n \to \infty$

$$\mathbf{E}\big(Z_n(t)Y_n(t)\big) = \alpha_n^{1/2}J_t + o(\alpha_n^{1/2}), \tag{2.22}$$

where $J_t = \int_{-\infty}^{0} yW_t(dy) < 0$.

Proof. Put $k = [nt]+1,\ t \in (0,1)$, and recall notation v_j from (2.5). Then

$$\begin{aligned}\mathbf{E}\big(Z_n(t)Y_n(t)\big) &= \mathbf{E}\Big\{n^{1/2}\alpha_n^{-1/2}(Q_n^*(t) - X_{(k)})\times \\ &\times\left(n^{1/2}\left(X_{(k)} - \frac{k}{n+1}\right) + n^{1/2}\left(\frac{k}{n+1} - t\right)\right)\Big\} = \\ &= n\alpha_n^{-1/2}\sum_{j=1}^{n} v_j \mathbf{E}\big(X_{(j)} - X_{(k)}\big)\left(X_{(k)} - \frac{k}{n+1}\right) + O(n^{-1/2})\end{aligned} \tag{2.23}$$

To estimate the last sum, recall that

$$\mathbf{E}X_{(j)} = \frac{j}{n+1},\quad \mathbf{E}\left(X_{(j)} - \frac{j}{n+1}\right)\left(X_{(k)} - \frac{k}{n+1}\right) = \frac{j(n+1-k)}{(n+1)^2(n+2)} \tag{2.24}$$

for $j \le k$ (see e.g. Lemma 1.7.1 in Reiss [23]).

Hence the sum is equal to

$$
\begin{aligned}
S = & \sum_{j=1}^{n} v_j \mathbf{E}\Big(X_{(j)} - X_{(k)}\Big)\Big(X_{(k)} - \frac{k}{n+1}\Big) = \\
= & \sum_{j=1}^{n} v_j \mathbf{E}\Big(X_{(j)} - \frac{j}{n+1}\Big)\Big(X_{(k)} - \frac{k}{n+1}\Big) + \\
& + \sum_{j=1}^{n} v_j \frac{j-k}{n+1} \mathbf{E}\Big(X_{(k)} - \frac{k}{n+1}\Big) - \sum_{j=1}^{n} v_j \mathbf{E}\Big(X_{(k)} - \frac{k}{n+1}\Big)^2 = \\
= & \sum_{j=1}^{n} v_j \mathbf{E}\Big(X_{(j)} - \frac{j}{n+1}\Big)\Big(X_{(k)} - \frac{k}{n+1}\Big) - \frac{k(n+1-k)}{(n+1)^2(n+2)}.
\end{aligned}
\tag{2.25}
$$

The sum of covariances here equals, after multiplication by $(n+1)^2(n+2)$, to

$$
\begin{aligned}
& \sum_{j \le k} v_j j(n+1-k) + \sum_{j>k} v_j k(n+1-j) = \\
& \quad = -k \sum_{j=1}^{n} j v_j + (n+1) \sum_{j \le k} j v_j + k(n+1) \sum_{j>k} v_j.
\end{aligned}
\tag{2.26}
$$

Now (2.25) and (2.26) give

$$
\begin{aligned}
S = & - \frac{k}{(n+1)^2(n+2)} \sum_{j=1}^{n} j v_j + \frac{1}{(n+1)(n+2)} \sum_{j \le k} j v_j + \\
& + \frac{k}{(n+1)(n+2)} \sum_{j>k} v_j + \frac{k^2}{(n+1)^2(n+2)} - - \frac{k}{(n+1)(n+2)} = \\
& = - \frac{k}{n^3} \left(n \int_0^1 x\, V_t^{(n)}(dx) + O(1) \right) + \frac{1}{n^2} \Big(n \int_0^{k/n} x\, V_t^{(n)}(dx) + \\
& + O(1) + k \int_{k/n+0}^{1} V_t^{(n)}(dx) \Big) + \frac{k^2}{n^3} - - \frac{k}{n^2} = \\
& = - \frac{k}{n^2} \int_0^1 x\, V_t^{(n)}(dx) + \frac{1}{n} \int_0^{k/n} x\, V_t^{(n)}(dx) + \\
& + \frac{t}{n} \int_{k/n+0}^{1} V_t^{(n)}(dx) + \frac{k^2}{n^3} - \frac{k}{n^2} + O\big(n^{-2}\big) = \\
& = - \frac{k}{n^2} \left(t + \delta_t^{(n)} \right) + \frac{1}{n} \int_0^{k/n} (x-t) V_t^{(n)}(dx) + \frac{t}{n} + \frac{k^2}{n^3} - \frac{k}{n^2} + O\big(n^{-2}\big) =
\end{aligned}
$$

$$=\frac{1}{n}\int_0^t (x-t)\,V_t^{(n)}(dx) + o\big(n^{-3/2}\alpha_n^{1/2}\big) + O\big(n^{-2}\big) =$$

$$=\frac{\alpha_n}{n}\int_{-\infty}^0 y\,W_t^{(n)}(dy) + o\big(n^{-3/2}\alpha_n^{1/2}\big) + O\big(n^{-2}\big).$$

Conditions (2.1) – (2.3) imply that

$$\int_{-\infty}^0 y\,W_t^{(n)}(dy) \to \int_{-\infty}^0 y\,W_t(dy),$$

and hence we have from (2.23) that

$$\mathbf{E}\big(Z_n(t)Y_n(t)\big) = n\alpha_n^{-1/2}\Bigg(\frac{\alpha_n}{n}\int_{-\infty}^0 y\,W_t(dy) + o\big(\alpha_n n^{-1} + \alpha_n^{1/2} n^{-3/2}\big) + \\ + O(n^{-2})\Bigg) + O\big(n^{-1/2}\big) = \alpha_n^{1/2} J_t + o\big(\alpha_n^{1/2}\big)$$

which completes the proof of the theorem. ■

Knowing the asymptotics of the covariance (2.22), we can evaluate that of the relative deficiency $d^*(n)$ of $Q_n(t)$ w.r.t. $Q^*(t)$ (defined according to (1.7) but with $\hat{Q}_n$ substituted by Q_n^*).

Corollary 2. *Under assumptions of Theorem 3,*

$$\lim_{n\to\infty} \frac{d^*(n)}{n\alpha_n} = \frac{G_t}{t(1-t)}$$

with $G_t = -2J_t - H_t$.

Remark 4. Note that if the limiting law W_t is continuous, we have, from (2.15) and conditions (2.1) – (2.3), the relation

$$G_t = -2\int_{-\infty}^0 y\,W_t(dy) - 2\int_0^\infty y\,(1-W_t(y))\,W_t(dy) - \\ --2\int_{-\infty}^0 |y|\,W_t(y)\,W_t(dy) = 2\int_{-\infty}^\infty y\,W_t(y)\,W_t(dy) - -2\int_{-\infty}^\infty y\,W_t(dy),$$

which coincides for absolutely continuous W_t with $\Psi(k)$, $k = dW_t(x)/dx$, from (1.10) (when $\int x\,k(x)\,dx = 0$).

Proof of Corollary 2. We have from (2.24) that

$$\mathbf{E}\big(Q_n(t) - t\big)^2 = n^{-1}t(1-t) + O\big(n^{-2}\big).$$

On the other hand,

$$
\begin{aligned}
&\mathbf{E}\big(Q_n^*(t)-t\big)^2 = \mathbf{E}\big(Q_n^*(t)-Q_n(t)+Q_n(t)-t\big)^2 = \\
&= \mathbf{E}\big(Q_n(t)-t\big)^2 + 2\mathbf{E}\big(Q_n^*(t)-Q_n(t)\big)\big(Q_n(t)-t\big) + \mathbf{E}\big(Q_n^*(t)-Q_n(t)\big)^2 \\
&= n^{-1}t(1-t) + O\big(n^{-2}\big) + \alpha_n n^{-1}\big(2J_t + H_t\big) + o\big(\alpha_n n^{-1}\big) = \\
&= n^{-1}t(1-t) - \alpha_n n^{-1}G_t + o\big(\alpha_n n^{-1}\big).
\end{aligned}
$$

Taking into account the main terms only, we get the following asymptotic relation for $m = n + d^*(n)$:

$$
m^{-1}t(1-t) \approx n^{-1}t(1-t) - \alpha_n n^{-1}G_t,
$$

and hence

$$
1 - \frac{d^*(n)}{n} \approx \left(1 + \frac{d^*(n)}{n}\right)^{-1} = \frac{n}{m} \approx 1 - \alpha_n \frac{G(t)}{t(1-t)}.
$$

Corollary 2 is proved. ■

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