ESTIMATES FOR THE SYRACUSE PROBLEM VIA A PROBABILISTIC MODEL

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Abstract

In the paper we employ a simple stochastic model for the 'Syracuse problem' (aka '$3x + 1$ problem') to get estimates for the 'average behaviour' of the trajectories of the original deterministic dynamical system. The use of the model is supported not only by certain similarities between the governing rules in the systems, but also by a qualitative estimate of the rate of approximation (Theorem 2). From the model, we derive explicit formulae for the asymptotic densities of some sets of interest for the original sequence. We also approximate the asymptotic distributions for the 'stopping times' (times till absorption in the only known cycle $\{1, 2\}$) of the original system and give numerical illustrations to our results.

Keywords: Syracuse problem; dynamical system; random walk.

1 Introduction

The Syracuse problem, also known as $3x + 1$ *problem, Collatz' problem, Kakutani's problem, Ulam's problem,* and *Hasse's algorithm*, has been drawing attention of many mathematicians for more than 40 years. The problem belongs to the class of questions which can be explained to anybody, but which are very difficult to answer. For its history, details, references and prizes for its solution see Lagarias (1985) and Lagarias and Weiss (1992).

The Syracuse problem is concerned with the behaviour of a simply defined discrete time dynamical system. In general, such a system is given by the relation

$$x_{n+1} = f(x_n), \quad n \geq 0, \ f : \mathcal{X} \mapsto \mathcal{X}, \tag{1}$$

which defines the sequence $\{x_n\}_{n\geq 0}$ in the phase space $\mathcal{X}$ of the system, starting with some initial point $x_0 \in \mathcal{X}$. Such systems arise in a very natural way in many fields

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of mathematics and its applications. Unfortunately, even for the simplest spaces $\mathcal{X}$ and simple functions f, the description of the behaviour of $\{x_n\}_{n\geq 0}$ often turns out to be a difficult problem.

Our problem is on the sequence $\{x_n\}_{n\geq 0}$ in $\mathcal{X} = \mathbf{N}$, which is the trajectory of the system (1) with

$$f(x) = \begin{cases} (3x+1)/2 & \text{if } x \text{ is odd,} \\ x/2 & \text{if } x \text{ is even.} \end{cases} \tag{2}$$

It is easy to see that this system has a simple cycle $\{1, 2\}$. Call the least positive k (if it exists) for which $x_k = 1$ the *total stopping time* $t(x_0)$ of x_0:

$$t(x_0) = \inf\{k \geq 1 : \; x_k = 1\}. \tag{3}$$

The Syracuse problem consists in verifying the following

CONJECTURE. *Each $x_0 \in \mathbf{N}$ has a finite total stopping time.*

In other words,

$$\mathbf{T} \equiv \{x_0 \in \mathbf{N} : \; t(x_0) < \infty\} = \mathbf{N}.$$

So far this problem has not been solved. However, the conjecture has been numerically checked for a large range of values of n; it turned out to be true for all $x_0 < 2^{40}$ (N.Yoneda's result cited in Lagarias (1985)). There are also some results on the asymptotic density of the sets of x_0 having certain properties concerned with the conjecture.

For a set $D \subset \mathbf{N}$, denote its asymptotic density by

$$\rho(D) = \lim_{n\to\infty} \frac{1}{n} |\{x \in D : \; x \leq n\}|,$$

where $|A|$ is the cardinality of A. Crandall (1978) showed that

$$|\{x \in \mathbf{T} : \; x \leq n\}| > n^c$$

for some $c \geq 0.05$. More is known about the so–called *stopping times*

$$t_0(x_0) = \inf\{k \geq 1 : \; x_k < x_0\}.$$

Clearly, the conjecture above can also be restated as

$$\mathbf{T}_0 \equiv \{x_0 : \; t_0(x_0) < \infty\} = \mathbf{N}$$

(note also that $\mathbf{T} \subseteq \mathbf{T}_0$). It was shown by Terras (1976, 1979) and Everett (1977) that $\rho(\mathbf{T}_0) = 1$. Moreover, it was also proved there that, for any $k > 0$, there exists the density

$$F(k) = \rho(\mathbf{T}_0(k)), \tag{4}$$

where $\mathbf{T}_0(k) = \{x_0 \in \mathbf{N} : \; t_0(x_0) \leq k\}$.

In the present note, we give further results on the asymptotic densities of sets related to the behaviour of $\{x_n\}$, using a simple probabilistic model for the system (1)–(2), and give an estimate for the rate of approximation of this system by the model. We also derive an explicit formula for the density (4). The results will be illustrated by experimental data for the system. Our interest will be concerned mainly with the behaviour of the total stopping time $t(x_0)$.

In conclusion of this section, we would like to compare the behaviour of the Syracuse system (1)–(2) with that of an arbitrary 'continuous extension' of the system to the phase state $\mathcal{X} = \mathbf{R}$. While for the former, the existence of non-trivial periodic points is a hard open problem, for the latter, one has the following result.

Proposition. *For any continuous extension $\tilde{f}$ of the function f given by (2) to $\mathbf{R}_+$, the dynamical system*

$$x_{n+1} = \tilde{f}(x_n) \tag{5}$$

has periodic points of all periods.

Proof. To prove the proposition, it suffices, by Lemma 2.3 from Block and Coppel (1992) to show that the system (5) is turbulent, i.e. there exist disjoint compact intervals A_1 and A_2 such that $A_1 \cup A_2 \subseteq f(A_1) \cap f(A_2)$. Taking $A_1 = [2k+\varepsilon, 2k+1-\varepsilon]$ and $A_2 = [2k+1+\varepsilon, 2k+2-\varepsilon]$ for a $k \in \mathbf{N}$, it is easy to verify that, due to continuity of $\tilde{f}$, the intervals A_j will have the desired property for all small enough $\varepsilon > 0$.

2 A probabilistic model and estimates

Note that (2) can be re-written in the form

$$f(x) = \left[\frac{1}{2} + J(x)\left(1 + \frac{1}{2x}\right)\right] x, \tag{6}$$

where $J(x) = 1$ if x is odd and 0 if x is even. This suggests that, if we consider the 'transition rule' on a sufficiently large 'regular' set $A \subset \mathbf{N}$, it should look like $x \to (1/2+I)x$, where I is a Bernoulli random variable taking the values 1 and 0 with probabilities 1/2 each. Moreover, heuristic considerations make it likely that these I's could be (almost) independent for consequent steps. The following observation made in Terras (1976) and Everett (1977) shows that, to some extent, this is true. For the brevity's sake put $J_k(x_0) = (J(x_0), J(x_1), \ldots, J(x_{k-1})) \in \{0,1\}^k$.

Theorem 1 *If $A = [a_0, a_1) \subset \mathbf{N}$, $|A| = a_1 - a_0 = 2^k$, $k \in \mathbf{N}$, then the mapping $J_k : A \mapsto \{0,1\}^k$ is one-to-one. The function $J_k(x_0)$, $x_0 \in \mathbf{N}$, is 2^k-periodic.*

For completeness of the exposition, we give a simple short proof by induction of this assertion.

Proof. 1. $k = 1$. Obvious from the definition.

2. $k-1 \mapsto k$. Clearly, for any A of the type specified, one has $A = A_0 + A_1$, where $A_0 = [a_0, a_0 + 2^{k-1})$, $A_1 = A_0 + 2^{k-1} \equiv \{x_0' = x_0 + 2^{k-1} : x_0 \in A_1\}$. Now note that

$$x_1' = f(x_0') = f(x_0 + 2^{k-1}) = \begin{cases} (3x_0' + 1)/2 = x_1 + 3 \cdot 2^{k-2}, & \text{if } x_0 \text{ is odd,} \\ x_0'/2 = x_1 + 2^{k-2}, & \text{if } x_0 \text{ is even.} \end{cases}$$

Continuing this chain, we see that

$$x_j' = f(x_{j-1}') = x_j + 3^{\alpha_j} \cdot 2^{k-1-j}, \quad j = 1, \dots, k-1,$$

where $\alpha_j = J(x_0) + \cdots + J(x_{j-1})$. From this we infer that

$$J_{k-1}(x_0) = J_{k-1}(x_0'), \quad J(x_{k-1}) + J(x_{k-1}') = 1. \tag{7}$$

Since by the induction hypothesis, J_{k-1} provides a one-to-one mapping of A_0 onto $\{0,1\}^{k-1}$, (7) completes the proof of the theorem.

Recall that the uniform distribution on $\{0,1\}^k$ is just the product of k Bernoulli distributions with success probabilities 1/2. Hence Theorem 1 implies the following

Corollary 1 *If x_0 is uniformly distributed over the set $A = [a, a + 2^k)$, $k \geq 1$, then $J(x_0), J(x_1), \dots, J(x_{k-1})$ are i.i.d. Bernoulli random variables with success probabilities* 1/2.

This suggests the use of the following stochastic sequence (which resembles the first stochastic model in Lagarias and Weiss (1992); about the difference between them see Remark 3 below):

$$X_{n+1} = \left(\frac{1}{2} + I_n\right) X_n = X_0 \prod_{m=0}^{n} \left(\frac{1}{2} + I_m\right), \qquad X_0 \in \mathbf{N}, \tag{8}$$

where I_n are i.i.d. Bernoulli r.v.'s, $\mathbf{P}(I_m = 1) = 1/2$. The appropriateness of this model follows not only from the similarity of the relations (6) and (8) above but also from the following quantitative result.

Theorem 2 *For any $k \geq 1$, if x_0 is uniformly distributed over the set $A = [a, a + 2^k) \subset \mathbf{N}$, $X_0 = a$, and $I_m = J(x_m)$, $m = 0, 1, \dots, k-1$, then always*

$$X_m \leq x_m \leq X_m e^{R_m}, \quad m = 1, 2, \dots, k,$$

where $R_m \geq R_{m-1} \geq 0$ and $\mathbf{E}\, R_m \leq 2^{k-1}(1 + 2^{2m-k+1} 3^{-m})/a$, $m = 0, 1, \dots, k$.

Thus, this approximation "on the average" of $\{x_n\}_{n \leq k}$ by $\{X_n\}_{n \leq k}$ is good whenever $k \ll \log a$. However, as it often happens with various approximations, it seems to work in a larger range of k (e.g. for $k \sim \log a$, as we shall see below when considering the behaviour of $t(x_0)$).

Proof. Clearly, for $m = 0, 1, \ldots, k$,

$$(9)\qquad \begin{aligned} y_m &\equiv \ln x_m = \ln x_0 - \sum_{j=0}^{m-1} \left(-\ln(1/2 + I_j + I_j/2x_j)\right), \\ Y_m &\equiv \ln X_m = \ln X_0 - \sum_{j=0}^{m-1} \xi_j, \end{aligned}$$

where $\xi_j = \ln 2 - I_j \ln 3$ are i.i.d.r.v.'s with

$$(10)\qquad \mathbf{E}\,\xi_j = \frac{1}{2}\ln\frac{4}{3} = \mu > 0, \quad \mathrm{Var}\,(\xi_j) = (\ln 3)^2 \mathrm{Var}\,(I_j) = \left(\frac{1}{2}\ln 3\right)^2 = \sigma^2,$$

$$\varphi = \mathbf{E}\, e^{\xi_j} = \mathbf{E}\,[(1/2 + I_j)^{-1}] = 4/3.$$

Since $x_0 \geq X_0 = a$, we have $y_m \geq Y_m$, $m = 0, 1, \ldots, k$. Clearly, this implies that

$$\ln\left(\frac{1}{2} + I_j + \frac{I_j}{2x_j}\right) \leq \ln\left(\frac{1}{2} + I_j\right) + \frac{I_j}{3x_j} \leq - - \xi_j + \frac{1}{3X_j}, \quad j = 0, 1, \ldots, k-1.$$

Hence

$$y_m \leq Y_m + \frac{1}{3}\sum_{j=0}^{m-1} \frac{1}{X_j} + \ln\left(1 + \frac{x_0 - a}{a}\right), \quad j = 0, 1, \ldots, k,$$

so that

$$y_m - Y_m \leq \frac{x_0 - a}{a} + \frac{1}{3}\sum_{j=0}^{m-1}\frac{1}{X_j} = R_m.$$

But $\mathbf{E}\,(x_0 - a) = 2^{k-1}$ and $\mathbf{E}\,(1/X_j) = a^{-1}\mathbf{E}\,\exp\left(\sum_{i=0}^{j-1}\xi_i\right) = a^{-1}\varphi^{j-1}$. Therefore

$$\mathbf{E}\,R_m \leq \frac{2^{k-1}}{a} + \frac{1}{3a}\sum_{j=0}^{m-1}\varphi^{j-1} = (2^{k-1} + \varphi^m - 1)a^{-1} \leq 2^{k-1}(1 + 2^{2m-k+1}3^{-m})a^{-1},$$

which completes the proof.

Now turn back to the conjecture above: does the sequence $\{x_n\}$ fall eventually in the absorbing set $\{1, 2\}$ for any initial value x_0? Relations (9) and (10) show that our stochastic sequence $\{X_n\}$ tends to 0 a.s., so in view of Theorem 2 we could expect that, at least for "almost all x_0", the answer is positive. However, so far we can only prove the following results concerning the "rate of decrease" of $\{x_n\}$, which is an improvement of the results on the asymptotic density of $\mathbf{T}_0$ mentioned in the Introduction.

THEOREM 3 *For any $k \geq 1$ and $C_m \geq 0$, $m = 1, \ldots, k$, the set*

$$D^k(C_1, \ldots, C_k) = \left\{x_0 \in \mathbf{N} : \frac{x_1}{x_0} \geq C_1, \ldots, \frac{x_k}{x_0} \geq C_k\right\}$$

has the density

$$\rho(D^k(C_1,\dots,C_k)) = \mathbf{P}\left(\bigcap_{m=1}^{k}\left\{U_m \ge \frac{m+\log_2 C_m}{\log_2 3}\right\}\right),$$

where $U_m = I_1 + \dots + I_m$ *are cumulative sums of i.i.d. Bernoulli r.v.'s with success probabilities* $1/2$.

Putting $C_1 = \dots = C_{k-1} = 0$, we immediately get the following

Corollary 2 *For any* $k \ge 1$ *and* $C > 0$*, the set* $D_k(C) = \{x_0 \in \mathbf{N} : \, x_k/x_0 < C\}$ *has the density*

$$\rho(D_k(C)) = B_{k,1/2}((k+\log_2 C)/\log_2 3), \tag{11}$$

where $B_{k,p}$ *is the binomial distribution function with parameters* k *and* p.

This means, for example, that, for a fixed k, $x_k/x_0 < 2^{-k}3^{k/2}$ for "half" of the x_0's.

Remark 1 For large k, the right hand side of (11) is close to

$$\Phi(((2-\log_2 3)k + \log_2 C)/\log_2 3),$$

where Φ is the standard normal distribution function.

It is easy to see that

$$D^k(1,\dots,1) = \{x_0 \in \mathbf{N} : \, t_0(x_0) > k\} = \mathbf{N}\backslash\mathbf{T}_0(k).$$

We thus have the following explicit form for the density (4) of $\mathbf{T}_0(k)$.

Corollary 3

$$\rho(\mathbf{T}_0(k)) = 1 - \mathbf{P}\left(\bigcap_{m=1}^{k}\{U_m \ge m/\log_2 3\}\right).$$

The standard estimates for large deviations probabilities for random walks imply now estimates for this density (which is a function of k) like those from Theorem D in Lagarias (1985).

Proof of Theorem 3. For the sake of simplicity we shall only prove here Corollary 2, for the proof of Theorem 3 goes essentially along the same lines.

For any given $\delta > 0$, only for finitely many x_0 (and definitely not for $x_0 \ge 2^k/\delta$) can one get $x_j < \delta^{-1}$ for some $j = 0, 1, \dots, k-1$. Hence

$$\left(\frac{3}{2}\right)^{\alpha_k}\left(\frac{1}{2}\right)^{k-\alpha_k} \le \frac{x_k}{x_0} \le \left(\frac{3+\delta}{2}\right)^{\alpha_k}\left(\frac{1}{2}\right)^{k-\alpha_k}, \quad x_0 \ge 2^k/\delta \tag{12}$$

(the left inequality is true for all $x_0 \geq 1$), where $\alpha_k = J(x_0) + \cdots + J(x_{k-1})$. Now let $m = m(n) = \lfloor n2^{-k} \rfloor$ ($\lfloor z \rfloor$ denoting the integer part of z); clearly, for any set D,

$$0 \leq |\{x \in D : x \leq n\}| - -|\{x \in D : x \leq m2^k\}| \leq 2^k.$$

To complete the proof it remains to note that, if x_0 is uniformly distributed over $\{x : x \leq m2^k\}$, it follows from Corollary C1 that in that case α_k has the distribution $B_{k,1/2}$. Therefore

$$\frac{1}{n}|\{x \in D_k(C) : x \leq n\}| \sim \frac{1}{m2^k}|\{x \in D_k(C) : x \leq m2^k\}| = \mathbf{P}\,(x_k/x_0 < C),$$

and since we have from (12) that

$$\begin{aligned} \mathbf{P}\,(U_k < (k + \log_2 C)/\log_2 3) &\geq \mathbf{P}\,(x_k/x_0 < C) \geq \\ &\geq \mathbf{P}\,(U_k < (k + \log_2 C)/\log_2(3 + \delta)) - -1/m\delta, \end{aligned}$$

it remains to let $n \to \infty$; our assertion follows since $\delta > 0$ is arbitrary small.

The next result gives a somewhat more precise characterization of the "average rate" of decrease of x_j.

THEOREM 4 *For any sequence*

$$r(m) = \frac{1}{2}(1 + \beta_m m^{-1/2}) \log_2 3$$

with $\beta_m \to \infty$ *as* $m \to \infty$, *one has* $\rho(D) = 1$ *for the set*

$$D = \{x_0 : x_m \leq x_0^{r(m)},\ m = \lfloor \log_2 x_0 \rfloor\}.$$

Remark 2 Note that this result is exact in the following sense. If

$$r(m) = \frac{1}{2}(1 + O(m^{-1/2})) \log_2 3,$$

then the corresponding set D cannot have the unit density.

Proof. Set $A_k = [2^k, 2^{k+1}) \subset \mathbf{N}$, and let $x_0(k)$ be a random variable uniformly distributed over A_k, $k \geq 1$. It is easy to see that it suffices to prove that

$$\mathbf{P}\,(x_k(k) > x_0(k)^{r(k)}) \to 0 \quad \text{as} \quad k \to \infty, \tag{13}$$

for $\lfloor \log_2 x_0(k) \rfloor = k$ a.s.

This probability does not exceed

$$\mathbf{P}\left(\min_{j \leq k} x_j(k) < N\right) + \mathbf{P}\,(x_k(k) > x_0(k)^{r(k)};\ \min_{j \leq k} x_j(k) \geq N). \tag{14}$$

Put $N = 3^{k/4}$. Since $x_j(k) \geq 3^{\alpha_j} 2^{-j} x_0(k) \geq 3^{\alpha_j} 2^{k-j}$ by (12), the first probability in (14) is estimated by

$$\begin{aligned}\mathbf{P}\left(\min_{j\leq k} 3^{\alpha_j}2^{-j} < 3^{k/4}2^{-k}\right) &\leq \mathbf{P}\left(\min_{j\leq k}(\alpha_j - j/\log_2 3) < -k/\log_2 3 + k/4\right)\\ &\leq \mathbf{P}\left(\min_{j\leq k}(\alpha_j - j/2) < - - k/4\right) \to 0\end{aligned}$$

as $k \to \infty$ by the Donsker–Prokhorov invariance principle (recall that α_j, $j = 1, \dots, k$, are sequential sums of the i.i.d.r.v.'s $J(x_j(k))$, $j = 0, \dots, k-1$, and hence the distribution of $2k^{-1/2} \min_{j\leq k}(\alpha_j - j/2)$ converges weakly to that of $\min_{0\leq t\leq 1} w(t)$, where w is the standard Wiener process).

Now let us turn to the second term in (14). It does not exceed

$$\begin{aligned}\mathbf{P}\left((3+3^{-k/4})^{\alpha_k}2^{-k} > (2^k)^{r(k)-1}\right) &\leq \mathbf{P}\left(\alpha_k > \frac{kr(k)}{\log_2 3(1+3^{-k/4-1})}\right)\\ &= \mathbf{P}\left(\alpha_k > k/2 + \beta_k' k^{1/2}\right) \to 0 \quad \text{as} \quad k \to \infty\end{aligned}$$

by the central limit theorem, for $\beta_k' \to \infty$ as β_k does (cf. (12); we again make use of the fact that $x_0(k) \geq 2^k$). Therefore, (13) is proved, and hence the proof of Theorem 4 is complete.

3 Absorption times

In the previous section, we estimated how the sequence $\{x_n\}$ decreases "on the average". Now we turn to the absorption times themselves. We shall consider the total stopping time $t(x_0)$ for original sequence (see (3; in fact, $\{x_n\}$ hits the absorbing set $\{1, 2\}$ at time $t(x_0) - 1$), and its analogue for the stochastic sequence, the hitting time

$$T(a) = \min\{n > 0 : \ X_n \leq 1\}, \quad X_0 = a.$$

From Corollary 1 and Theorem 2 one could expect that

$$|A|^{-1}|\{x_0 \in A : \ t(x_0) < u\}| \sim \mathbf{P}\,(T(a) < u), \tag{15}$$

say, for $A = [a, a + 2^k) \subset \mathbf{N}$. Below we shall see what the numerical experiments say about (15), but first we shall estimate the right-hand side of that relation.

Remark 3 Some results on the almost sure behaviour of $T(a)$ as $a \to \infty$ are given in Lagarias and Weiss (1992), but for their own stochastic model which looks like ours with the essential difference that they consider, roughly speaking, a new random walk for each level; the total stopping time is defined there as

$$\sigma_\infty(\omega_n) = \min\{k : \ Z(n, k) \geq \ln n\},$$

where $Z(n,k) = \sum_{i=1}^{k} X(n,i)$, and $\{X(n,i)\}$ is a set of independent random variables with common distribution coinciding with that of our ξ_1. For this modified total stopping time they establish the almost sure behaviour, obtaining the following result: with probability 1,

$$\limsup_{n\to\infty} \frac{\sigma_\infty(\omega_n)}{\ln n} = \gamma_{RW} \approx 41.678, \tag{16}$$

while in our model, by the law of large numbers in extended renewal theory (see e.g. Gut (1988)), we have just that almost surely

$$\lim_{n\to\infty} \frac{T(n)}{\ln n} = \frac{1}{\mu} = \left(\frac{1}{2}\ln\frac{4}{3}\right)^{-1} \approx 6.952.$$

(Further comments on this subject see below). After that, results for the stochastic model are compared in the cited paper with the real behaviour of the original sequence, when the initial value is taken from a sequence of sets like our A_k from Theorem 4 with a resulting relative error of about 25%.

We should note here the following two aspects of such modelling. Firstly, the assumption about independence of all $X(n,k)$ might have been inspired by a representation of the form $x_k = 2^{-k}3^{J(x_0)+\cdots+J(x_{k-1})}x_0 + \delta_k(x_0)$ established in Theorem 1.1 in Terras (1976). Nevertheless, it looks a bit strange (since, say, at time one, the jumps in the original system have the same sign for all initial values of the same parity etc.), whereas our choice is justified at least by Theorems 1 and 2. Moreover, the large value obtained for the upper limit is apparently due to the fact that there are "too many" different $X(n,k)$'s. By the way, using the model from Lagarias and Weiss (1992), one could establish also the central limit theorem and even the law of the iterated logarithm for the quantity

$$|\{x_0 : \ x_1/x_0 \ge 3/2\}|,$$

which reflects *no real property* of the original sequence whatsoever. Secondly, the extreme values for the original sequence given in the tables in Lagarias and Weiss (1992) could (quite surely) correspond, in a sense, to a set of probability null in the stochastic model, of which the a.s. behaviour has been described possibly just on the complement of that set. For a more accurate use of the model, one should also calculate the number of these "extreme" points giving large total stopping time values, and compare it with n.

In other words, the use of the a.s. behaviour of such stochastic models to make predictions for the original sequence looks quite unreliable and unjustified. On the other hand, as we have seen above, the distributional properties of the probabilistic model could help in describing the "average" properties of the original one (including its behaviour on the set of initial points having asymptotic density 1).

Return now to the problem of estimating the terms in (15). Clearly,

$$\ln X_n = b - S_n, \quad b = \ln a, \quad S_n = \sum_{k=1}^{n} \xi_k,$$

where $\xi_k = \ln 2 - I_k \ln 3$ are i.i.d. r.v.'s with the positive mean μ and variance σ^2 given in (10), and

$$T(a) = \min\{n > 0 : \ S_n \geq b\}.$$

Hence, by the central limit theorem in the extended renewal theory (see e.g. Theorem III.5.1 in Gut (1988)), we see that $T(a)$ is, under proper normalization, asymptotically normal. We shall give now yet another argument proving that fact, because it provides some motivation for our next step.

If we put $\eta_k = 1 - 2I_k$ and $V_n = \sum_{k\leq n} \eta_k$, then $\mathbf{E}\,\eta_1 = 0$, $\mathrm{Var}(\eta_1) = 1$, and

$$T(a) = \min\{n \geq 1 : \ V_n \geq f(n)\}, \quad f(t) = \sigma^{-1}(b - \mu t).$$

Now define the normalized process $s_h(t) = h^{-1/2}V_{ht}$, $t \geq 0$, and put $h = h(a) = b/\mu = (2\ln a)/\ln(4/3)$. Then we have

$$T(a) = h \min\{t > 0 : \ s_h(t) \geq (1-t)g\}, \quad g = (b\mu)^{1/2}\sigma^{-1}. \tag{17}$$

Put

$$\tau(g) = \inf\{t : \ w(t) \geq (1-t)g\} \tag{18}$$

for the standard Wiener process $w(t)$, $t \geq 0$. Since w is continuous, and $\tau(g) \to 1$ a.s. as $g \to \infty$,

$$w(\tau(g)) \to w(1) \quad \text{a.s. as} \ \ g \to \infty. \tag{19}$$

Now by the Donsker–Prokhorov invariance principle the process s_h converges in distribution to w in the Skorokhod space $D[0,\infty)$ as $h \to \infty$. By Skorokhod's theorem, we may assume without loss of generality that $s_h \to w$ a.s. uniformly on any finite interval $[0, C]$, $h \to \infty$. But then it follows from (17) to (19) that

$$s_h(h^{-1}T(a)) \to w(1) \quad \text{a.s. as} \ \ a \to \infty. \tag{20}$$

Further, since the jumps in the random walk V_n are bounded, it is not difficult to see from (17) that

$$gh^{-1}(T(a) - h) = -s_h(h^{-1}T(a)) + O(h^{-1/2}), \tag{21}$$

and hence it follows from (20) that the distribution of $\mu\sigma^{-1}h^{-1/2}(T(a)-h)$ converges to that of $-w(1)$. We shall state this result as a theorem.

Theorem 5 *For any real x,*

$$\mathbf{P}\left(\frac{T(a) - \mu^{-1}\ln a}{\mu^{-3/2}\sigma(\ln a)^{1/2}} < x\right) \to \Phi(x) \quad \textit{as} \ \ a \to \infty.$$

Moreover, relation (21) suggests that

$$gh^{-1}(T(a) - h) \overset{d}{\approx} - - w(\tau(g)) = g(\tau(g) - 1) \tag{22}$$

(the last equality holds due to the continuity of w), and therefore makes the following "second order approximation" plausible:

$$T(a)/\ln a \overset{d}{\approx} \mu^{-1}\tau(g), \tag{23}$$

which would be of great value since the rate of convergence in Theorem 5 is very slow: it is only $O((\ln a)^{-1/2})$. Unfortunately, we could not prove so far such a relation in the general case (a direct proof of a similar result, that is, the proof of the closeness of the distributions of the first hitting times for random walks and for Wiener processes, is possible for integer-valued $\xi_i \le 1$; it is based on an analogue of the formula (24) below for skip-free random walks). However, simulations show that in fact we have here such an approximation.

How is that $\tau(g)$ distributed? Clearly, it is the crossing time of the level g by the continuous process $v(t) = w(t) + gt$. Since $v(t)$ has the density

$$p_{v(t)}(x) = \frac{1}{(2\pi t)^{1/2}} \exp\left(-\frac{(x-gt)^2}{2t}\right),$$

we have by a well-known formula for such processes (see e.g. Borovkov (1976)), that $\tau(g)$ has a density $p_{\tau(g)}(t)$ and

$$g^{-1}p_{\tau(g)}(t) = t^{-1}p_{v(t)}(g), \quad g, t > 0, \tag{24}$$

so that

$$p_{\tau(g)}(t) = \frac{g}{(2\pi)^{1/2}t^{3/2}} \exp\left(-\frac{g^2(1-t)^2}{2t}\right), \quad t > 0$$

(the standard approach to deriving this formula for $\tau(g)$ is to combine the reflection principle with a change of measure, cf. Siegmund (1985); the above derivation is much shorter). Therefore, the density of the right-hand side of (23) is

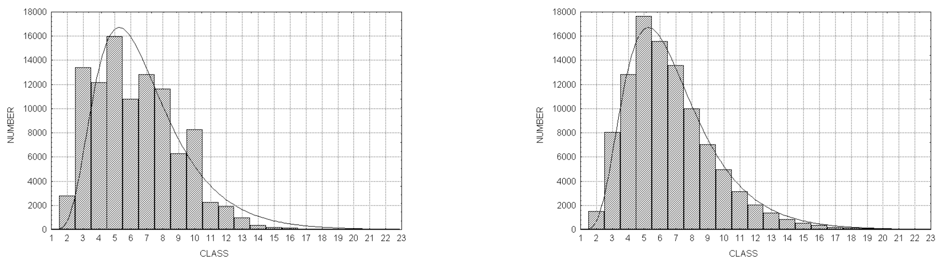
$$\frac{(\ln a)^{1/2}}{(2\pi x^3)^{1/2}\sigma} \exp\left\{-\frac{(\mu x - 1)^2 \ln a}{2\sigma^2 x}\right\}, \quad x > 0. \tag{25}$$

The above argument makes it plausible that this density provides a refinement of the normal approximation for the distribution of $T(a)$ from Theorem 5. The numerical data from the next section show that this is really the case, and moreover, that expression (25) can serve as a reasonable approximation to the distribution of the stopping times in the original system (1)–(2) as well.

4 Numerical results

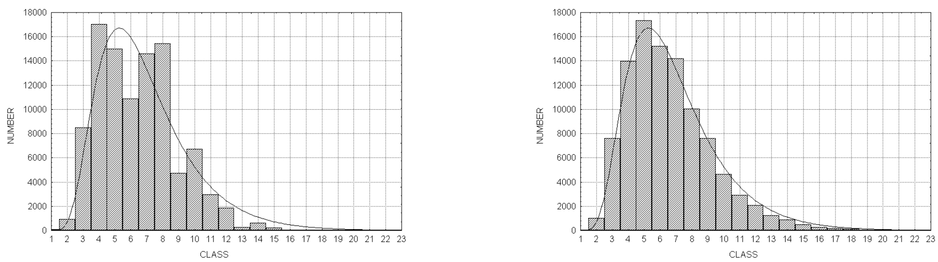
The first four figures below show the histograms of the normalized total stopping times $t(x_0)/\ln x_0$ of the original model for all 10^5 initial values x_0 in the sets $[0.95 \times 10^6, 1.05 \times 10^6)$ and $[1.95 \times 10^6, 2.05 \times 10^6)$, resp., and the histograms for samples of 10^5 simulated values of $T(a)/\ln a$ each with $a = 10^6$ and $a = 2 \times 10^6$, resp.

We also plot the densities (25) on the same graphs. The histograms for simulated samples are given in order to compare the error of approximation to the original model with what one gets for truly stochastic systems. The empirical averages for $t(x_0)$ are 6.86 and 6.88, resp., which is *very close* to the renewal theoretic value 6.95 above. Also, the histograms for $T(a)$ follow quite precisely the density (25). The fifth figure shows the histogram of the normalized stopping time $t(x_0)/\ln x_0$ in the range $x_0 \in [3.5\times 10^6,\, 4.5\times 10^6)$. Due to the larger interval, this histogram is closer in shape to the histograms of the simulated values of $T(a)$, giving an empirical average of 6.86.



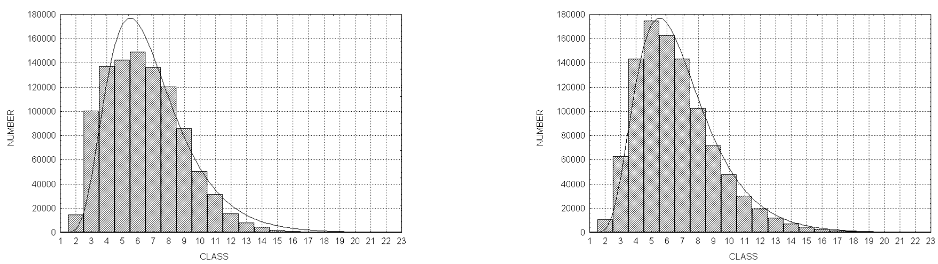
$x_0 \in [0.95 \times 10^6,\, 1.05 \times 10^6)$ $a = 10^6$

Histograms for $t(x_0)/\ln x_0$ and $T(a)/\ln a$ with fitted density



$x_0 \in [1.95 \times 10^6,\, 2.05 \times 10^6)$ $a = 2 \times 10^6$

Histograms for $t(x_0)/\ln x_0$ and $T(a)/\ln a$ with fitted density



$x_0 \in [3.5 \times 10^6,\, 4.5 \times 10^6)$ $a = 4 \times 10^6$

Histograms for $t(x_0)/\ln x_0$ and $T(a)/\ln a$ with fitted density

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