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LETTERS TO THE EDITOR

A NOTE ON THE OCCURRENCE TIMES OF A PÓLYA–LUNDBERG PROCESS

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Recently Albrecht (1983) has pointed out that for the occurrence-time sequence $\{T_n\}$ of an appropriate version of a mixed Poisson process $\bar{N}(t) = N(t\Delta)$ (where $N(t)$ is a Poisson process with unit intensity and Δ is the mixing variable) the relation

$$\frac{T_n}{n} \to \frac{1}{\Delta} \quad \text{a.s.} \quad (n \to \infty) \tag{1}$$

holds. We shall show here by a martingale argument that such a relation is always valid for any version $N^*(t)$ of a Pólya–Lundberg process which is a mixed Poisson process with a gamma mixing variable with mean $\lambda > 0$ and variance $\alpha\lambda^2$, $\alpha > 0$.

Theorem. The sequence $\{S_n\}$ defined by $S_n = n/(1+\alpha\lambda T_n)$, $n \geqq 1$ forms a submartingale with $E(S_n) \leqq 2/\alpha$, $n \geqq 2$.

Proof. By the Markov property of $\{T_n\}$ (Pfeifer (1982)) we need only show

$$E\left(\frac{1}{1+\alpha\lambda T_{n+1}} \,\middle|\, T_n = t\right) \geqq \frac{n}{n+1}\frac{1}{1+\alpha\lambda t} \quad \text{a.s.} \tag{2}$$

which follows easily from the transition probabilities

$$P(T_{n+1} > s \mid T_n = t) = \left(\frac{1+\alpha\lambda t}{1+\alpha\lambda s}\right)^{n+1/\alpha} \quad \text{a.s.,} \quad s \geqq t \geqq 0. \tag{3}$$

Also, $E((n-1)/T_n) = \lambda$ for $n \geqq 2$, hence $E(S_n) \leqq 2/\alpha$.

By the martingale convergence theorem now $S_n \to S$ a.s. $(n \to \infty)$ for some random variable S, from which we also have

$$\frac{T_n}{n} \to \frac{1}{\alpha\lambda S} \quad \text{a.s.} \quad (n \to \infty), \tag{4}$$

i.e. $\alpha\lambda S$ is a canonical representation of the mixing variable.

Note that using estimations given in Albrecht (1983) (4) also implies that for the process $N^*(t)$ itself

$$\frac{N^*(t)}{t} \to \alpha\lambda S \quad \text{a.s.} \quad (t \to \infty). \tag{5}$$

References

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