e-stat: Basic Stochastic Finance at School Level

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Abstract. This text shows possibilities how basic facts of mathematics of finance can be introduced at school level by using e-stat. The basic facts ground on the discrete option pricing theory in the one-period binomial-tree-model. A simple numerical example is used to explain the term of "arbitrage" and its importance to financial mathematics. The example leads into two interactive Java-applets which give an impression of the ability of computers to bring complex contents to a better understanding by interactivity.

This text will also give an idea of how to teach basic elements of mathematical stochastics by using mathematics of finance such as probability and expectation.

Keywords. Stochastic education at school, options, option pricing theory, discrete mathematics of finance, financial markets, binomial-tree-model, Java-applets, combination of options, arbitrage, e-stat, e-learning

1 Introduction

1.1 The importance of teaching financial mathematics at school

In general financial mathematics is supposed to be quite a difficult subject to examine, which is in general only dealt with at university level. At least since we have publications like EBERLEIN (1998) based on the Cox-Ross-Rubinstein-model this conjecture is no longer true. It is even possible to introduce basic aspects of probability theory by demonstrating simple ideas of financial mathematics.

In e-stat we combine this fact with the possibilities of interactivity using a computer. This does not only enable teaching financial mathematics at school level but gives also motivations and excites pupils in order to get better teaching results.

1.2 Teaching strategy

The course we produced for e-stat is called "Basic facts of financial mathematics". It starts with simple explanations of common assets used at the stock exchange and leads afterwards to an easy pricing problem in the binomial-tree-model. Basic facts of probability and expectation can be explained during this part. Essential is a look from the economic point of view with the terms of "arbitrage", "leverage effect" and "combination of options". The course follows in part the monographs of PLISKA (1997), UHLIR AND STEINER (2001) and BINGHAM AND KIESEL (1998).

2 Options and Arbitrage

An *option* is the right to trade a certain amount of an asset for a fixed price, called "*strike price*" or "*exercise price*", at a future date, called "*time of maturity*". If it is

possible to use the right within a certain time period the option is called *American* option, if it can only be exercised at the time of maturity it is called *European* option. In the following text we will focus on European options. A *call* option gives the owner the right to buy an asset, a *put* option confers the right to sell.

Of course the exertion of an option depends on the asset price. In case of a call option it only makes sense to exercise if the price of the underlying asset is higher than the strike price. If the price is lower it is cheaper to buy the asset directly at the stock exchange than to purchase it using the option.

There are a few other types of options called "exotic options" which differ in some modalities of the owners right.

2.1 Notations

In this text we will not refer to any transaction costs resulting from the stock exchange. The following designations are significant: *T:* the **T**ime of maturity, *X:* e**X**ercise price, strike price, S_t: **S**tock price/asset price at time $t \in [0,T]$, *i:* riskless **I**nterest rate. As said above the exertion of an option depends on the ratio of the asset price at time *T* and the strike price *X*:

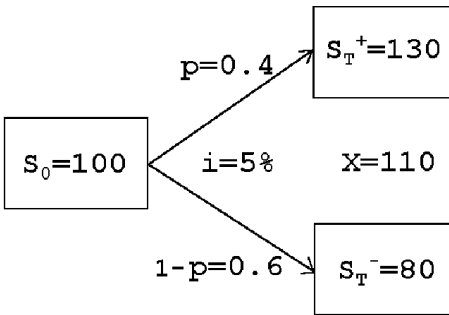|	call	put
exercise option	$S_T > X$	$S_T < X$
do not exercise option	$S_T \leq X$	$S_T \geq X$

The main goal is now to find the „correct" price C_0 of a call option (respectively P_0 of a put option) at time 0, which is the time of buying the option. We therefore look first at the price of an option at its time of maturity *T*, where we have:

$$C_T = \max\{S_T - X; 0\} = (S_T - X)^+ \quad \text{(positive-part)}$$

$$P_T = \max\{X - S_T; 0\} = (X - S_T)^+.$$

2.2 Modelling a financial market

We will now consider the so-called *Two-Step-One-Period-Model*, i.e. we assume that the asset price in the beginning of the trading-period S_0 increases till the time of maturity *T* up to $S_T^+ > S_0$ or decreases down to $S_T^- < S_0$, and we require $S_T^- < X < S_T^+$. Let $p \in (0,1)$ the probability of increase; Thus $1-p$ is the probability of decrease. The following example can be found in an article of EBERLEIN (1998). Our asset is a stock with a start price $S_0 = 100\,€$; with probability $p = 0.4$ it increases to the value of $S_T^+ = 130\,€$, respectively decreases with probability $1-p = 0.6$ to the

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                              ┌─────────────┐
                    p=0.4  ↗  │ S_T⁺=130    │
                              └─────────────┘
┌──────────┐
│ S₀=100   │ ◄    i=5%        X=110
└──────────┘
                              ┌─────────────┐
                 1-p=0.6  ↘   │ S_T⁻=80     │
                              └─────────────┘
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value of $S_T^- = 80 \, €$. The strike price is set to $X = 110 \, €$, the interest rate is $i = 5\%$. We call the interest rate factor $r = (1+i)$ and the discount factor $v = \dfrac{1}{r}$. Which price C_0 has the writer of a call option (i.e. the buyer of the option) on this stock to pay?

2.3 Finding correct option prices

To answer this question we will first look at the discounted option prices, giving $v \cdot C_T^+ = 19.05 \, €$ and $v \cdot C_T^- = 0 \, €$. We will now consider two possibilities of finding the option price C_0 :

2.3.1 First possibility: Fair game "writer vs. market"

We interpret the model as a fair game. The writer buys an option at time 0 and gains a certain amount of money in case of an increasing stock price whereas he loses his money in case of a decreasing stock price. He expects an average gain of zero. Thus, a formula to calculate the "fair" option price uses the expectation:

$$C_0 = E\left(v \cdot C_T\right) = p \cdot v \cdot C_T^+ = \textbf{7.62}$$

The problem of this strategy is the restriction to the writer to trade with options exclusively. If he uses both options and stocks for trading he is able to achieve a result shown in table 2.1.

On the left side we can see the action the writer does. He starts with zero money. At time 0 he sells short two stocks and buys five call options and lends the rest of his money for credit. At time T he has to pay for his stock, gets back his credit and exercises his options in case of an increasing stock price. At the end he earns 10€ independent of the behaviour of the stock price.

$t = 0$		$t = T$	
action	account movements	$S_T^+ > X$	$S_T^- \leq X$
short sell 2 stocks	+200.00	−260.00	−160.00
buy 5 call options	−38.10	+100.00	0.00
give credit	−161.90	+170.00	+170.00
balance / sum	0.00	+10.00	+10.00

Table 2.1. Example of arbitrage

Situations like the one in table 2.1 we call "arbitrage". Of course we want to prevent such situations. We will therefore look at a second possibility to find a correct option price.

2.3.2 Second possibility: The arbitrage-free market

We demand $S_T^+ > r \, S_0$ and take a look at the following table 2.2.

$t = 0$		$t = T$	
action	account movements	$S_T^+ > X$	$S_T^- \le X$
short sell n stocks	$+n\,S_0$	$-n\,S_T^+$	$-n\,S_T^-$
buy m call options	$-m\,C_0$	$m\left(S_T^+ - X\right)$	0.00
give credit	$-\left(n\,S_0 - m\,C_0\right)$	$r\left(n\,S_0 - m\,C_0\right)$	$r\left(n\,S_0 - m\,C_0\right)$
balance / sum	0.00	0.00	0.00

Table 2.2. Arbitrary formulas to calculate the option price

From the two columns below $t = T$ we obtain $-n\,S_T^+ + m\left(S_T^+ - X\right) = -n\,S_T^-$, leading to the ratio $h = n/m$ (so-called *hedge ratio*) given by

$$h = \frac{S_T^+ - X}{S_T^+ - S_T^-};$$

for the corresponding call price we obtain the unique solution $C_0 = h\left(S_0 - v\,S_T^-\right)$.

In the example above we have $h = (130 - 110)/(130 - 80) = 0.4$ and $C_0 = \mathbf{9.52}$. The proof of C_0 being the arbitrage-free price is not very difficult.

2.4 Illustration by interactivity

The question is now which way we take to motivate these ideas at school. Within e-stat, a standard tool is the use of Java applets. In the following Java applet (fig. 2.1) an arbitrage-table as seen above is realized. In the lower left part of the applet the results of all the transfers of stocks and options can be found, exactly as in table 2.1. In the upper part it is possible to edit all the parameters of the *Two-Step-One-Period-Model.* That enables pupils to examine the idea of arbitrage with their own numbers, which is crucial to memorize problems and ways of acting in financial markets. For instance does the applet give the opportunity to make exercises where some parameters are given, e.g. stock prices and type of options, and the others are to find out by the pupils and to test their solution. The applet works as follows:

In the first line of the applet (starting with "Aktien:" [stocks]) can be chosen between buying or selling stocks. Then it is filled in, how many stocks are to buy or sell. On the right side of the first line the stock prices at time $t = 0$ and $t = T$ can be entered.

In the second line (starting with "Optionen:" [options]) almost same choices as in the first line are made but with the option instead of the stock. Additionally it can be chosen whether to trade with put- or call-options and of course there is no need to enter on option price at time $t = T$.
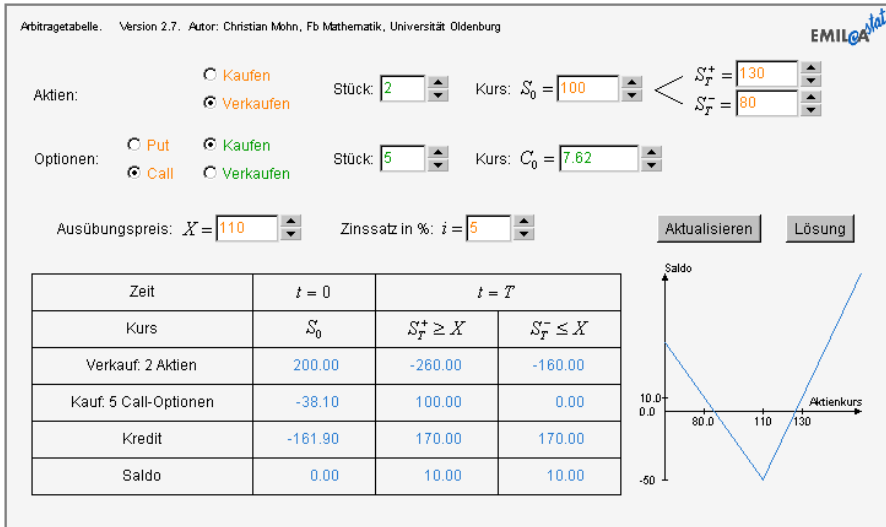
Arbitragetabelle. Version 2.7. Autor: Christian Mohn, Fb Mathematik, Universität Oldenburg   EMIL@stat

Aktien: ○ Kaufen ● Verkaufen   Stück: 2   Kurs: $S_0 = 100$ < $S_T^+ = 130$ / $S_T^- = 80$

Optionen: ○ Put ● Call   ● Kaufen ○ Verkaufen   Stück: 5   Kurs: $C_0 = 7.62$

Ausübungspreis: $X = 110$ Zinssatz in %: $i = 5$ [Aktualisieren] [Lösung]

Zeit	$t = 0$	$t = T$	
Kurs	S_0	$S_T^+ \geq X$	$S_T^- \leq X$
Verkauf: 2 Aktien	200.00	-260.00	-160.00
Kauf: 5 Call-Optionen	-38.10	100.00	0.00
Kredit	-161.90	170.00	170.00
Saldo	0.00	10.00	10.00

Saldo ... Aktienkurs ... 80.0 110 130 ... 10.0 0.0 ... -50

Fig 2.1. Java-applet of an Arbitrage-table with graphical output

In the third line (starting with "Ausübungspreis:" [strike price]) the strike price of the option and the interest is entered. On the right side there are two buttons. The left one, called "Aktualisieren" [actualize] calculates the new arbitrage table in the lower left and also the graphic in the lower right corner with all changes that have been made in the first three lines. The right button called "Lösung" [solution] calculates the correct hedge ratio and option price in an arbitrage-free market, dependent of the stock prices and the choice of buying or selling stocks and of trading with puts or calls. Therefore the upper part of the applet is different coloured: All data that are needed to calculate the correct hedge ratio and option price are orange and all data that are changed by this calculation are green, that is, number of stocks and options, option prices and buying or selling options.

The arbitrage table in the lower left corner is a fixed output, directly controlled by the values of the first three lines of the applet as described with table 2.1.

The graphical output visualizes the total gain or loss in dependence of the stock price. Thus the behaviour aside from the two stock prices S_T^+ and S_T^- is visible.

3 Combination of options

Another aspect of the concept of arbitrage is the investigation of combining different options. Actually this is only an enhancement to the concept above: there is no longer only one kind of option to trade with but several. Still we need to calculate the correct prices for all options we use. This is possible with a little bit more complicated theory of financial mathematics, which shall not be explained in this place (however it is done in another course of e-stat). What we want to show here is again the power of interactivity and the increase of motivation of pupils by enabling them to find out things by themselves. In the Java applet (fig. 3.1) created for this matter, it is possible to combine up to five European put or call options with parameters of one's own choice. The result is a graphical output of the payoff with all chosen options (thin

line) and the combination of them which is the sum (thick line). Thus, pupils are enabled to find out good combinations such as strangles, straddles, strips, straps or even a butterfly-call-spread by trial and error. The applet works similarly to the one above. In the lower five lines you can edit for up to five options all parameters and calculate the graphical output by clicking on the button "Aktualisieren" [actualize].
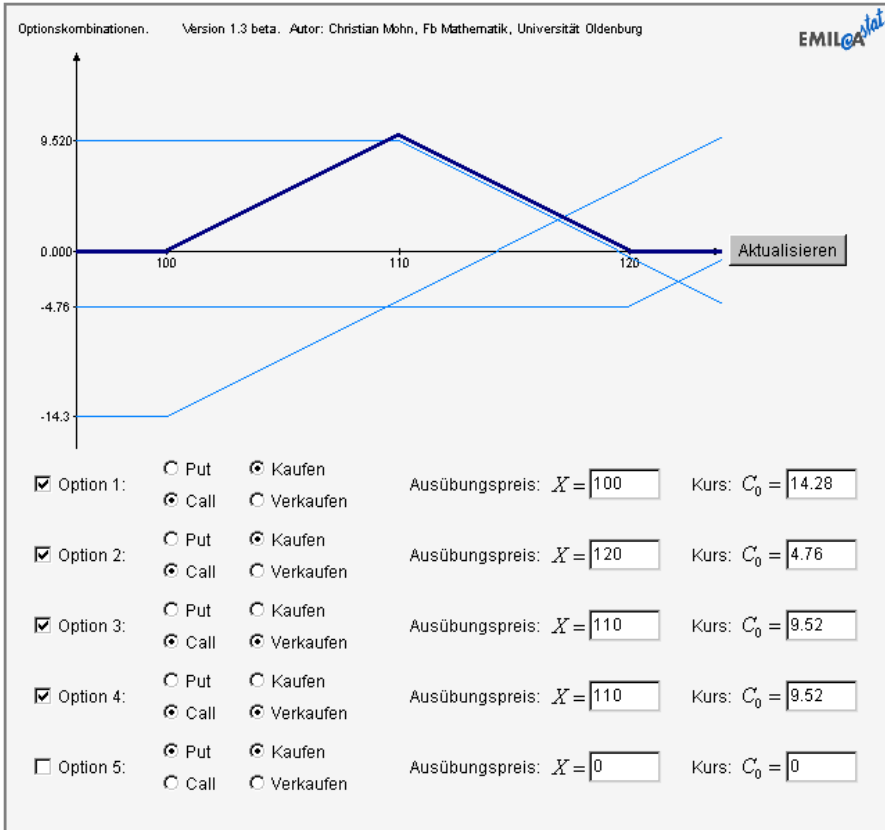

Fig 3.1. Beta-Version of a Java-applet that calculates the sum of a combination of options

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