Modeling dependence in finance and insurance: the copula approach

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1. Introduction

The concept of "copulas" is not really a new one in the mathematical world. It goes essentially back to problems posed by Hoeffding and Fréchet more than 60 years ago concerning the maximal and minimal possible correlation for bivariate distributions when the marginals are fixed, or, more generally, upper and lower bounds for the joint distribution under this condition. A related question originally posed by the Italian geometrist Pompilj concerns the minimum of the L^{α}-distance of random variables X and Y, i.e. the minimum value of $E|X-Y|^{\alpha}$ when the marginal distributions F_X and F_Y are given. It was actually this approach which lead to the concept of "minimal metrics" in the important field of probabilistic metric spaces and the thorough investigation of copulas, see e.g. Schweizer and Sklar [24] or Zolotarev [27] for an extended exposition. The word "copula" however was seemingly first coined by Sklar in his famous 1959 paper [26]. (For a nice survey over the historic development of copulas we refer to the monograph of Dall'Aglio et al. [5].)

The "rediscovery" of copulas in particular in finance and nowadays also in insurance is perhaps due to the need of a more sophisticated analysis of the joint temporal behaviour of assets and the valuation of portfolios containing them, together with the development of applicable mathematical tools in fields like IRM (Integrated Risk Management) and DFA (Dynamic Financial Analysis). Besides the establishment of regulatory standards on a legislative basis in the European markets, the development of largely extended personal computer capacities in the last decade has in particular enforced such a development. A transfer of real-world-processes into a large-scale computer model e.g. in order to simulate the development of a balance sheet over time becomes more and more "easy" today. With the progress made on the hardware side, the challenges for a proper mathematical modeling of all kind of complex financial and actuarial structures increase likewise. A very thorough review of copulas and their applicability to risk management in particular can be found in Embrechts et al. [9].

For the common user of statistical methods, "correlation" is perhaps the first probabilistic notion to remember, in particular since this kind of "dependence" concept has introduced itself at large to the financial markets, supported by theoretical approaches like the Capital Asset Pricing Model (CAPM) and the Arbitrage Pricing Theory (APT). This is due to the fact that these theories are founded on the "normal" world where in the multivariate setting, the joint distributions are indeed characterized by pairwise correlation (or covariance, resp.). This is not only no longer true outside that world, but also very misleading results can be produced when the concept of correlation is applied without caution, as has been pointed out in several papers in the recent past (see e.g. Embrechts et al. [7], [8], or Blum et al. [3]). It seems therefore necessary to propagate the ideas behind copulas as "the" standard tool of description of all kind of dependence structures to a wider audience, in particular in insurance-linked branches. It is remarkable to see that both in expensive training seminars organized for professional staff as well as in national and international congresses on risk theory copula-based models are gaining increasing interest.

2. Basic facts and a start–up example

This section is devoted to some simple mathematical considerations in order to make the reader a bit more familiar with the underlying mathematical framework. To start with, we present the usual definition of a copula due to Sklar, together with his central characterization theorem.

Definition 2.1. A *copula* is a function C of n variables on the unit n-cube $[0,1]^n$ with the following properties:

1. the range of C is the unit interval $[0,1]$;
2. $C(\mathbf{u})$ is zero for all $\mathbf{u}$ in $[0,1]^n$ for which at least one coordinate equals zero;
3. $C(\mathbf{u}) = u_k$ if all coordinates of $\mathbf{u}$ are 1 except the k-th one;
4. C is n-increasing in the sense that for every $\mathbf{a} \le \mathbf{b}$ in $[0,1]^n$ the measure $\Delta C_{\mathbf{a}}^{\mathbf{b}}$ assigned by C to the n-box $[\mathbf{a},\mathbf{b}] = [a_1,b_1] \times \cdots \times [a_n,b_n]$ is nonnegative, i.e.

$$\Delta C_{\mathbf{a}}^{\mathbf{b}} := \sum_{(\varepsilon_1,\cdots,\varepsilon_n)\in\{0,1\}^n} (-1)^{\sum_{i=1}^{n}\varepsilon_i} C\left(\varepsilon_1 a_1 + (1-\varepsilon_1)b_1, \cdots, \varepsilon_n a_n + (1-\varepsilon_n)b_n\right) \ge 0.$$

As can be easily seen, a copula is in fact a multivariate distribution function with univariate uniform margins restricted to the n-cube. Copulas have many useful properties, such as uniform continuity and (almost everywhere) existence of all partial derivatives, just to mention a few (see e.g. Nelsen [17], Theorem 2.2.4 and Theorem 2.2.7). Moreover, it can be shown that every copula is bounded by the so-called *Fréchet-Hoeffding* bounds, i.e.

$$\max(u_1 + \cdots + u_n - n + 1, 0) \le C(u_1,\cdots,u_n) \le \min(u_1,\cdots,u_n)$$

which are commonly denoted by $\mathcal{W}$ and $\mathcal{M}$ in the literature. In two dimensions, both of the *Fréchet-Hoeffding* bounds are copulas themselves, but as soon as the dimension increases, the *Fréchet-Hoeffding* lower bound $\mathcal{W}$ is no longer n-increasing. However, the inequality on the left-hand side cannot be improved, since for any $\mathbf{u}$ from the unit n-cube, there exists a copula $C_{\mathbf{u}}$ such that $\mathcal{W}(\mathbf{u}) = C_{\mathbf{u}}(\mathbf{u})$ (see Nelsen [17], Theorem 2.10.12).

The most well-known copula is perhaps the so-called *independence copula*, $\Pi(\mathbf{u}) = \prod_{i=1}^{n} u_i$, which has for centuries been the standard copula in statistics and probability for modeling sequences of independent experiments.

The following theorem due to Sklar justifies the role of copulas as dependence functions:

Theorem 2.2 (Sklar). Let H denote a n-dimensional distribution function with margins $F_1,\cdots,F_n$. Then there exists a n-copula C such that for all real $(x_1,\cdots,x_n)$,

$$H(x_1,\cdots,x_n) = C\left(F_1(x_1),\cdots,F_n(x_n)\right).$$

If all the margins are continuous, then the copula is unique, and is in general determined uniquely on the ranges of the marginal distribution functions otherwise. Moreover, the converse of the above statement is also true. If we denote by $F_1^{-1},\cdots,F_n^{-1}$ the generalized inverses of the marginal distribution functions, then for every $(u_1,\cdots,u_n)$ in the unit n-cube,

$$C(u_1,\cdots,u_n) = H\left(F_1^{-1}(u_1),\cdots F_n^{-1}(u_n)\right).$$

For a proof, see Nelsen [17], Theorem 2.10.9 and the references given therein.

Note that in terms of the original distributions, the Fréchet-Hoeffding bounds above translate into

$$\mathcal{W}(F_1(x_1),\cdots,F_n(x_n)) \le H(x_1,\cdots,x_n) \le \mathcal{M}(F_1(x_1),\cdots,F_n(x_n)).$$

Sklar's theorem essentially states that in the multivariate setting, marginal distributions of and dependence between observations can be treated separately. This is of great importance for all practical actuarial work, since at least the analysis of marginal distributions is a standard task for actuaries, e.g. in (re)insurance for estimating the PML (probable maximum loss) of a portfolio as a certain high quantile of the loss distribution (in finance, VaR (value at risk) is a similar concept). After a suitable identification of the marginal distributions by statistical methods, a component-wise transformation of the data via the inverse c.d.f.'s extracts the dependence structure as a copula, which can be analyzed afterwards by further appropriate tools.

As an introductory example, suppose that the random vector (X,Y) has a bivariate *triangular distribution*, i.e. (X,Y) possesses a joint density h of the form

$$h(x,y) = \begin{cases} 2, & x+y \le 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{for } (x,y) \in [0,1]^2.$$

The support of the joint distribution is the triangle formed by the three points (0,0), (1,0) and (0,1). It is easy to see that in this case, both X and Y have univariate triangular distributions themselves, with marginal densities

$$f(x) = 2(1-x),\ x \in [0,1] \quad \text{and} \quad g(y) = 2(1-y),\ y \in [0,1].$$

For the corresponding c.d.f.'s, we obtain by integration:

$$F(x) = x(2-x),\ x \in [0,1],\ \ G(y) = y(2-y),\ y \in [0,1]$$

$$\text{and} \quad H(x,y) = \begin{cases} 2xy, & x+y \le 1 \\ 2xy-(1-x-y)^2, & x+y > 1 \end{cases} \quad \text{for} \quad (x,y) \in [0,1]^2.$$

If we want to express the joint distribution of (X,Y) by their marginals, i.e. we want to find a solution of the equation

$$H(x,y) = C\big(F(x), G(y)\big) \quad \text{for} \quad (x,y) \in [0,1]^2$$

with a suitable copula C, then we can proceed as follows: substituting $u = F(x),\ v = G(y)$, we find as feasible solutions $x = 1-\sqrt{1-u},\ \ y = 1-\sqrt{1-v}$, hence

$$C(u,v) = H\left(F^{-1}(u), G^{-1}(v)\right) = \begin{cases} 2\left(1-\sqrt{1-u}\right)\left(1-\sqrt{1-v}\right), & \sqrt{1-u}+\sqrt{1-v} \geq 1 \\ u+v-1, & \sqrt{1-u}+\sqrt{1-v} < 1 \end{cases}$$

for $(u,v) \in [0,1]^2$. From this, we also obtain a density c of the above copula by two-fold differentiation:

$$c(u,v) = \frac{\partial^2 C(u,v)}{\partial u\, \partial v} = \begin{cases} \dfrac{1}{\sqrt{(1-u)(1-v)}}, & \sqrt{1-u}+\sqrt{1-v} \geq 1 \\ 0, & \sqrt{1-u}+\sqrt{1-v} < 1. \end{cases}$$

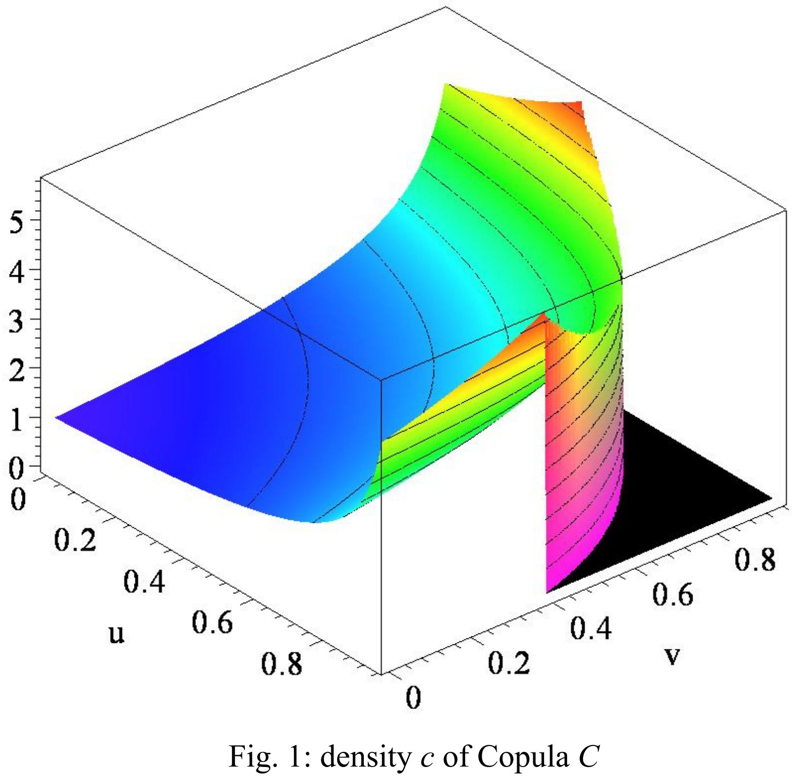

Fig. 1: density c of Copula C

Let now (U,V) be distributed according to the copula C. We can think of (U,V) as a non-linear transformation of (X,Y) via $U = X(2-X),\ V = Y(2-Y)$. Since U and V are both uniformly distributed, their correlation is given by

$$\operatorname{corr}(U,V) = \frac{\operatorname{Cov}(U,V)}{\sqrt{\operatorname{Var}(U)\operatorname{Var}(V)}} = 12\cdot E(UV) - 3 = 12\cdot E\big(XY(2-X)(2-Y)\big) - 3.$$

Now

$$E\big(XY(2-X)(2-Y)\big) = 2 \iint\limits_{\substack{x+y\le 1\\ 0\le x,y\le 1}} xy(2-x)(2-y)\,dx\,dy = 2\int_0^1 x(2-x)\int_0^{1-x} y(2-y)\,dy\,dx$$

$$= \frac{2}{3}\int_0^1 x(2-x)(1-x)^2(2+x)\,dx = \frac{19}{90},$$

hence $\operatorname{corr}(U,V) = 12\cdot E\big(XY(2-X)(2-Y)\big) - 3 = -\frac{7}{15}$. The following construction serves to find another pair of random variables $\big(U,W\big)$ (with copula *D*, say) which has this same correlation $\operatorname{corr}(U,W) = -\frac{7}{15}$, but has a totally different joint distribution than $\big(U,V\big)$. For this purpose, we consider the approach $W = U + b \bmod 1$ with some appropriate constant $b \in \big(0,1\big)$, i.e. we consider

$$W = \begin{cases} U+b, & U+b\le 1 \\ U+b-1, & U+b>1 \end{cases} = U + b - \mathbf{1}_{U+b>1}$$

with the indicator r.v. $\mathbf{1}_A$ for some set *A*. Note that by shift-invariance of Lebesgue measure, *W* is also uniformly distributed but in general dependent of *U*. [Such constructions are well-known in the literature, see e.g. Nelsen [17], chapter 3.2.] For the correlation $\operatorname{corr}(U,W)$, this means

$$\operatorname{corr}(U,W) = 12\cdot E(UW) - 3 = 12\cdot E\Big(U\big(U+b-\mathbf{1}_{X+b>1}\big)\Big) - 3 = 1 + 6b - 12\int_{1-b}^{1} x\,dx$$

$$= -5 + 6b + 6(1-b)^2 = 6b^2 - 6b + 1 = 6b(b-1) + 1.$$

Solving this equation for $\operatorname{corr}(U,W) = r$, we obtain $b = \frac{1}{2} + \frac{1}{6}\sqrt{3+6r}$ in the range $r \in \left[-\frac{1}{2}, 1\right]$. For the case above, this gives the value $b = \frac{1}{2} + \frac{\sqrt{5}}{30} = 0{,}57543\ldots$

The following graph shows 5000 simulations of pairs of the type $\big(U,V\big)$ (solid points) and $\big(U,W\big)$ (dotted points) each.

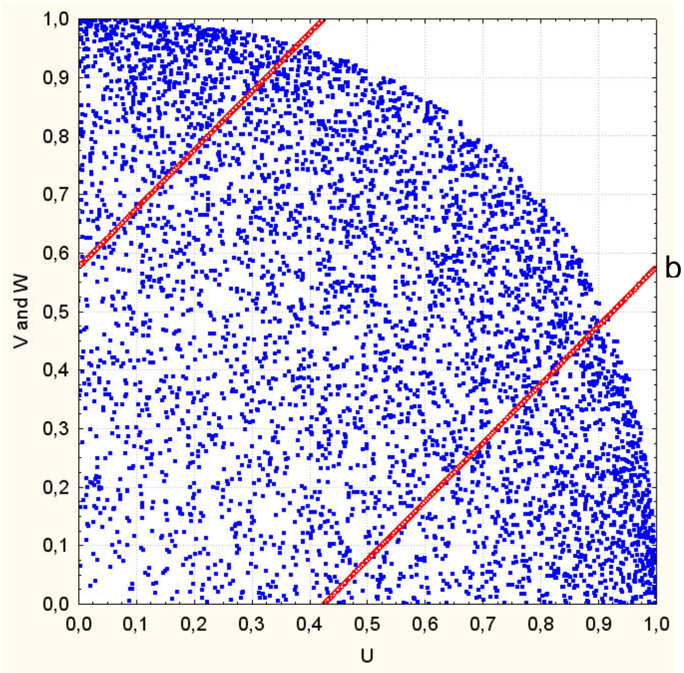

Fig. 2: simulation study of copulas C and D

As is clearly to be seen, correlation does not at all characterize the joint distributions of this example. Note even that the support of the distribution of the pair (U,W) is singular w.r.t. Lebesgue measure (i.e. concentrated on a set of measure zero), so that no joint density exists here. Some easy calculation shows that the copula D belonging to (U,W) is given by

$$D(u,v) = P(U \le u, W \le v) = \begin{cases} 0, & u \le 1-b,\ v \le b \\ \min(u, v-b), & u \le 1-b,\ v > b \\ \min(u+b-1, v), & u > 1-b,\ v \le b \\ u+v-1, & u > 1-b,\ v > b. \end{cases}$$

Note that both of the copulas C and D coincide partly with the Fréchet-Hoeffding lower bound $\mathcal{W}$, namely in the range $\sqrt{1-u}+\sqrt{1-v}<1$ for C and in the ranges $u \le 1-b,\ v \le b$ and $u > 1-b,\ v > b$ for D (due to the fact that no probability mass is given to the corresponding regions in the "north-east corner" each). On the contrary, for the limiting cases $b \in \{0,1\}$, D coincides everywhere with the upper Fréchet-Hoeffding bound $\mathcal{M}$, i.e. in the case where $W = U$ almost surely.

3. Copulas and dependence measures

In the bivariate case, both the Fréchet-Hoeffding bounds as well as the independence copula have the following stochastic representation, as was already noticed by Hoeffding [12]:

Theorem 3.1. Let U and V be random variables uniformly distributed over the unit interval [0,1]. Then their joint c.d.f. restricted to the unit square $[0,1]^2$ is equal to

- $\mathcal{W}$, if and only if U is almost surely a decreasing function of V,
- Π, if and only if U and V are independent,
- $\mathcal{M}$, if and only if U is almost surely an increasing function of V.

I.e., every bivariate dependence structure lies somewhere between two extremes, the perfect negative and the perfect positive dependence. In light of this fact copulas can be partially ordered in the following way:

Definition 3.2. A copula C_1 is *smaller* than a copula C_2, in symbols: $C_1 \prec C_2$, if for any $\mathbf{u}$ in the unit square $[0,1]^2$, $C_1(\mathbf{u}) \leq C_2(\mathbf{u})$.

As mentioned above, copulas reflect the dependence structure between the margins, and are therefore of great use in various dependence and association concepts. For instance, the well known bivariate concordance measures Kendall's τ and Spearman's ρ (rank correlation) as well as the likelihood ratio dependence and tail dependence concepts (see section 5 later) can be expressed in terms of the underlying copula alone. However, the role played by copulas in the study of multivariate dependence is much more complex and far less well understood (for further details see Mari and Kotz [16], Nelsen [17] or Joe [13] and the references given therein).

Motivated by the introductory example and its historic importance, we will first focus on the *Pearson correlation* as a special bivariate well-known dependence measure, given by

$$r = \operatorname{corr}(X,Y) = \frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}(X)}\sqrt{\operatorname{Var}(Y)}}.$$

It is easily seen that r does not depend on the underlying copula alone and will therefore be influenced by the marginal distributions as well. However, the following result from Hoeffding [12] suggests that the role played by copulas in this setting will nevertheless be important.

Theorem 3.3. Let (X,Y) be a bivariate random vector with a copula C and marginal distribution functions F and G. Then the covariance between X and Y can be expressed as

$$\begin{aligned}\operatorname{Cov}(X,Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [C(F(x),G(y)) - F(x)G(y)]\,dx\,dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [C(F(x),G(y)) - \Pi(F(x),G(y))]\,dx\,dy.\end{aligned}$$

This result together with the abovementioned Fréchet-Hoeffding inequality has the following consequence for the correlation coefficient.

Theorem 3.4. Let (U,V) be a bivariate random vector with uniform marginal distributions. The corresponding correlation coefficient will be denoted by r_C if the underlying copula is C. Then

1. r_C is always bounded by the correlation coefficients corresponding to the Fréchet-Hoeffding bounds, $r_{\mathcal{W}} \le r_C \le r_{\mathcal{M}}$;
2. if C_1 and C_2 are copulas, then the relation $C_1 \prec C_2$ yields $r_{C_1} \le r_{C_2}$;
3. each number in the interval $[r_{\mathcal{W}}, r_{\mathcal{M}}]$ is equal to r_C for some copula C.

The proof of this theorem relies on the observation that as in the example above,

$$r = \text{corr}(U,V) = \frac{\text{Cov}(U,V)}{\sqrt{\text{Var}(U)}\sqrt{\text{Var}(V)}} = 12 \cdot \int_0^1 \int_0^1 \big(C(u,v) - uv\big)\, du\, dv.$$

Further, if we choose some α from [0,1] and define a copula C_α by

$$C_\alpha := \alpha \cdot \mathcal{W} + (1-\alpha) \cdot \mathcal{M},$$

then the corresponding correlation coefficient r_{C_α} is $\alpha \cdot r_{\mathcal{W}} + (1-\alpha) \cdot r_{\mathcal{M}}$. The so constructed one-parameter (mixture) family of copulas includes the Fréchet-Hoeffding lower and upper bound and allows for both negative and positive correlation. However, it is worth noticing that $\alpha = \dfrac{\rho_{\mathcal{M}}}{\rho_{\mathcal{M}} - \rho_{\mathcal{W}}}$ yields a zero correlation and a copula which does not correspond to the independence of *U* and *V*. Moreover, the independence copula can never be constructed using the above method.

One of the consequences of Theorem 3.4 is that it is in general not possible to construct pairs of random variables (X,Y) with given marginals *F* and *G* and *arbitrary* correlation. In particular, if the variances of *X* and *Y* are finite, then a more general statement analogous to Theorem 3.2 is possible, i.e. minimal and maximal correlation are obtained for the general Fréchet-Hoeffding bounds. (This theorem is a central part in Hoeffding's 1940 paper [12]; see also Embrechts et al. [8], Theorem 4.) Nice examples with applications in the finance world are e.g. given in Embrechts et al. [8], p. 206 ff., where also the role of correlation w.r.t. coherent risk measures is discussed.

Since *Spearman's rank correlation* ρ is defined by

$$\rho = \rho(X,Y) = \text{corr}\big(F(X), G(Y)\big)$$

and $F(X)$ and $G(Y)$ are each uniformly distributed in case of continuity, we can easily extend Therem 3.4 to arbitrary random variables.

Corollary 3.5. Let (X,Y) be a bivariate random vector with arbitrary fixed continuous marginal distribution functions. The corresponding rank correlation coefficient will be denoted by ρ_C if the underlying copula is C. Then

1. ρ_C is always bounded by the rank correlation coefficients corresponding to the Fréchet-Hoeffding bounds, $\rho_{\mathcal{W}} \le \rho_C \le \rho_{\mathcal{M}}$;
2. if C_1 and C_2 are copulas, then the relation $C_1 \prec C_2$ yields $\rho_{C_1} \le \rho_{C_2}$;
3. each number in the interval $[\rho_{\mathcal{W}}, \rho_{\mathcal{M}}]$ is equal to ρ_C for some copula C.

In fact, we can use the mixture copula $C_\alpha := \alpha \cdot \mathcal{W} + (1-\alpha)\cdot \mathcal{M}$ from above to construct the pair (X,Y) via the (pseudo-)inverted c.d.f.'s F^{-1} and G^{-1} to achieve the rank correlation $\alpha \cdot \rho_{\mathcal{W}} + (1-\alpha)\cdot \rho_{\mathcal{M}}$ for $\alpha \in [0,1]$.

Likewise, *Kendall's* τ can in general be expressed solely through the underlying copula (cf. e.g. Embrechts et al. [8], p. 195f):

$$\tau(X,Y) = 4\int_0^1 \int_0^1 C(u,v)\, dC(u,v) - 1.$$

Similar measures of dependence have been proposed in the literature by Schweizer and Wolf [25]; they are all of the form $\delta(C,\Pi)$ for some metric δ on the space of (continuous) functions over the unit square $[0,1]^2$ (see e.g. Embrechts et al. [8], p. 197 for details).

4. Parametric families of copulas

Unfortunately it is seldom possible to calculate copulas explicitly from a given model of random variables, such as in the introductory example. Two major problems occur frequently in this context:

- The construction of the random variables involved is known and the joint distribution can be written down in closed form, but the copula cannot be described explicitly.
- The copula and the marginal distributions are known, but there is no explicit (simple) probabilistic representation for the corresponding random variables.

A typical example for the first situation is the Gauss-copula (also called Φ-copula), where the random vector $\mathbf{X} = (X_1, \cdots, X_n)$ has a multivariate normal distribution $N(\boldsymbol{\mu}, \Sigma)$ with mean vector $\boldsymbol{\mu}$ and variance-covariance matrix Σ. If Σ is non-singular, then it is possible to represent $\mathbf{X}$ via a linear transformation using a random vector $\mathbf{Y} = (Y_1, \cdots, Y_n)$ with distribution $N(\mathbf{0}, \mathbf{I})$, i.e. all components Y_i are independent and standard-normally distributed (here $\mathbf{I}$ = unit matrix). Since under the assumptions made, Σ is positive-definite, there is a matrix A with the property $\Sigma = A \cdot A^{\mathrm{tr}}$ (to be constructed from the diagonal matrix Δ of the (non-negative) eigenvalues and the orthonormal matrix T of eigenvectors of Σ via $A = T\Delta^{1/2}$), so that $\mathbf{X} = \mathbf{A} \cdot \mathbf{Y} + \boldsymbol{\mu}$. Furthermore, there is a one-to-one correspondence between the pairwise correlations of the components X_i and Σ, given by $\operatorname{corr}\left(X_i, X_j\right) = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}}$ for the $n \times n$-matrix $\Sigma = \left[\sigma_{ij}\right]$. This emphasizes the fact that in the world of normal distributions, pairwise correlations determine the joint (normal) distribution uniquely, and that these correlations can be considered as natural parameters of the corresponding Φ-copula. For a somewhat more general discussion of such topics in the world of elliptical distributions, see e.g. Embrechts et al. [8], section 3.3. If we denote as usual by Φ the c.d.f. for the univariate standard normal distribution, then the corresponding (Φ-)copula C_Φ has the representation

$$C_\Phi(u_1, \cdots, u_n) = \int_{-\infty}^{\Phi^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi^{-1}(u_n)} \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \exp\left(-\frac{1}{2}\mathbf{v}^{\mathrm{tr}}\Sigma^{-1}\mathbf{v}\right) dv_1 \cdots dv_n$$

which cannot be simplified in a reasonable manner. On the other hand, note that for instance with the well-known Box-Muller-transform (cf. e.g. Robert and Casella [22], example 2.2.2), it is easy to generate the Y_i above directly from uniformly distributed and independent r.v.'s U_i and V_i via $Y_i = \sqrt{-2\ln\left(U_i\right)}\cos\left(2\pi V_i\right)$.

Hence if $\mathbf{Z} = \left(Z_1, \cdots, Z_n\right)$ is a random vector with components Z_i having an arbitrary c.d.f. F_i each and a Φ-copula governing their mutual dependence, then $\mathbf{Z}$ can be represented as

$$Z_i = \Phi\left(\sum_{j=1}^{n} a_{ij}\sqrt{-2\ln\left(U_j\right)}\cos\left(2\pi V_j\right)\right), \; i = 1, \cdots, n \text{ with } A = \left[a_{ij}\right].$$

The generation of random variables with such a dependence structure is thus quite easy, although the corresponding Φ-copula cannot be explicitly calculated.

The same is true for the so-called Student- or *t*-copula, which for various reasons is of particular importance especially in the finance world (see e.g. Embrechts et al. [8] or Blum et al. [3]). It arises from the multivariate *t*-distribution with ν degrees of freedom $t_\nu\left(\boldsymbol{\mu}, \Sigma\right)$ and is given by

$$C_{t,\nu}(u_1, \cdots, u_n) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \cdots \int_{-\infty}^{t_\nu^{-1}(u_n)} \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\pi\nu)^n \det(\Sigma)}} \left(1 + \frac{1}{\nu}\mathbf{v}^{\mathrm{tr}}\Sigma^{-1}\mathbf{v}\right)^{\left(-\frac{\nu+n}{2}\right)} dv_1 \cdots dv_n$$

where t_ν denotes the c.d.f. of the univariate standard *t*-distribution with ν degrees of freedom and Σ is a positive-definite and symmetric matrix. Note that here the corresponding variance-covariance matrix is given by $\frac{\nu}{\nu-2}\Sigma$ for $\nu \geq 3$, and that the "limiting" copula for $\nu \to \infty$,

$C_{t,\infty}$ is identical to C_Φ. The stochastic representation of $t_\nu(\boldsymbol{\mu},\Sigma)$-distributed random vectors **X** is equally simple as in the case of the Φ-copula and can be achieved by multiplication of the zero-mean part in the normal distribution with an independent real-valued random factor W_ν which must be chosen such that $\frac{\nu}{W_\nu^2}$ is χ_ν^2-distributed. To be more precise, let **Y** be as above. Then $\mathbf{X} = W \cdot A \cdot \mathbf{Y} + \boldsymbol{\mu}$ is the desired stochastic representation for the $t_\nu(\boldsymbol{\mu},\Sigma)$-distribution. Likewise, random vectors with arbitrary marginals and a *t*-copula governing their mutual dependence can be constructed in a manner completely similar as above.

On the other hand, looking at copulas directly might also be a reasonable approach, since the structure of dependence could be more clearly visible or modeled as in the cases above. Here we shall concentrate on the so-called *Archimedian copulas* (for the history of this naming, see e.g. Nelsen [17], p. 98). These copulas are characterized by their so-called *generator* φ through the following equation (cf. Nelsen [17], chapter 4.6):

$$C(u_1,\cdots,u_n) = \varphi^{-1}\left(\sum_{i=1}^{n}\varphi(u_i)\right) \quad \text{for} \quad u_1,\cdots,u_n \in [0,1].$$

The following result makes a more precise statement about the existence of such copulas (cf. Nelsen [17], Theorem 4.6.2 and Corollary 4.6.3).

Theorem 4.1. Let φ be a continuous strictly decreasing function on the interval $(0,1]$ such that $\varphi(1) = 0$ and $\lim_{z\downarrow 0}\varphi(z) = \infty$, and let φ^{-1} denote its inverse on the interval $[0,\infty)$. If $C = C_n$ is a function on the unit *n*-cube $[0,1]^n$ with $n \geq 3$, fulfilling the above additive relationship, then C_n is a copula for all $n \geq 2$ if and only if φ^{-1} is completely monotonic, i.e.

$$(-1)^k \frac{d^k}{ds^k}\varphi^{-1}(s) \geq 0 \quad \text{for all } k \in \mathbb{N} \text{ and } s > 0.$$

In this case, we have necessarily $C \succ \Pi$ (in the sense of Definition 3.2).

In fact, for dimension $n = 2$, the situation is a little simpler here since just convexity of φ is sufficient for *C* being a copula. Also, $C \prec \Pi$ is possible here. For details, see Theorem 4.1.4 in Nelsen [17].

It is worth mentioning that by an old Theorem of Bernstein [2] and under the above side-conditions, such functions can be represented as the Laplace transforms of non-negative random variables *Z* via

$$\varphi^{-1}(s) = E\left(e^{-sZ}\right), \; s \geq 0,$$

a fact that has also been paid attention to in the literature [see e.g. Nelsen [17], p. 65f, p. 106 and p. 124, or the appendix in Joe [13]]. Indeed, such a representation gives, by the monotone convergence theorem,

$$\varphi^{-1}(0)=1,\ \lim_{s\to\infty}\varphi^{-1}(s)=0,\ \text{and } (-1)^k \frac{d^k}{ds^k}\varphi^{-1}(s)=E\left(Z^k e^{-sZ}\right)\geq 0 \text{ for all } k\in\mathbb{N} \text{ and } s>0.$$

The distribution of Z can also be interpreted as a *mixing distribution* for the parameter z in the family of exponential functions $e^{-sz}, s\geq 0,\ z>0.$ In particular, if we choose $Z\equiv 1,$ then $\varphi^{-1}(s)=e^{-s},\ s\geq 0,$ and the resulting copula is the independence copula Π. If we choose Z to be $\Gamma(\alpha,\alpha)$-distributed with $\alpha>0,$ i.e. Z has a density f_Z of the form

$$f_Z(z)=\frac{z^{\alpha-1}}{\Gamma(\alpha)}\alpha^\alpha e^{-\alpha z},\ z>0,$$

then $\varphi^{-1}(s)=E\left(e^{-sZ}\right)=\left(\frac{\alpha}{\alpha+s}\right)^\alpha,\ s\geq 0,$ hence $\varphi(t)=\alpha\left(t^{-1/\alpha}-1\right).$ Replacing α by $1/\theta$ we obtain the so-called one-parameter family of *Clayton* copulas:

$$C(u_1,\cdots,u_n)=\left[\sum_{i=1}^n u_i^{-\theta}-n+1\right]^{-1/\theta},\ \mathbf{u}\in(0,1]^n,\ \theta>0.$$

Note that for $\theta\to 0,$ i.e. $\alpha\to\infty,$ the mixing Gamma-distribution converges to the Dirac measure concentrated in the point 1, hence the independence copula is obtained as a limit copula here. For $\theta\to\infty,$ i.e. $\alpha\to 0,$ the mixing Gamma-distribution converges to the Dirac measure concentrated in the point zero, which means that the corresponding limit copula is the upper Fréchet –Hoeffding bound $\mathcal{M}$.

As can easily be seen, neither the Φ-copula nor the *t*-copula above are Archimedian except for the independence copula Π included in these models as special cases. In practical applications, the following parametric Archimedian families C_θ of copulas have gained particular interest (cf. Nelsen [17], p. 94ff):

name	copula C_θ	generator φ_θ	mixing distribution[1)]
Clayton	$\left[\sum_{i=1}^n u_i^{-\theta}-n+1\right]^{-1/\theta},\ \theta>0$	$\frac{1}{\theta}\left(t^{-\theta}-1\right)$	Gamma
Gumbel	$\exp\left(-\left\{\sum_{i=1}^n\left(-\ln(u_i)\right)^\theta\right\}^{1/\theta}\right),\ \theta\geq 1.$	$(-\ln t)^\theta$	positive stable
Frank	$-\frac{1}{\theta}\ln\left(1+\left(e^{-\theta}-1\right)\prod_{i=1}^n\left\{\frac{e^{-\theta u_i}-1}{e^{-\theta}-1}\right\}\right),\ \theta>0$	$-\ln\frac{e^{-\theta t}-1}{e^{-\theta}-1}$	log series

1) see Joe [13], p. 374f

The Gumbel copula has for long played a central role in the area of statistics of extremes where it and others can also be motivated by appropriate limit theorems for joint extremes (see e.g. Reiss and Thomas [21] or Kotz and Nadarajah [15], chapter 3). While the Gumbel copula has been used in particular in the analysis of natural catastrophes (see e.g. Pfeifer [18], [19]), the Clayton and Frank copulas have more recently attracted attention in the area of finance (cf. Blum et al. [3] and Junker and May [14]), but also in general insurance applications (cf. Frees and Valdez [10], Belguise and Levi [1] or Charpentier [4]).

It should be mentioned that for $n = 2$, the theory of Archimedian copulas is richer in the sense that less restrictions have to be imposed on the generator (see e.g. Nelsen [17], chapter 4), so that "more" Archimedian copulas exist in two dimensions than in greater generality, especially some with $C \prec \Pi$. However, according to the modern challenges in particular in IRM and DFA, copula models in higher dimensions become more and more important (see e.g. Pfeifer [19] for an insurance-linked example with eight dimensions).

Concerning the remarks at the beginning of this chapter, we would like to add that the generation of random vectors from two-dimensional Archimedian copulas is quite easy and can be performed in the following way, as was shown by Genest and Mackay [11] (cf. also Nelsen [17], p. 108 or Mari and Kotz [16], section 4.10.2): Let U and V be independent uniformly distributed random variables; set

$$S := (\varphi')^{-1}\left(\frac{\varphi'(U)}{V}\right),\ T := \varphi^{-1}\left(\varphi(S) - \varphi(U)\right).$$

Then (S, T) is the desired pair. This procedure works well for the Clayton and Frank family. For the Gumbel copula, see e.g. Reiss and Thomas [21], p. 241.
Unfortunately, the generation of random vectors from (not only Archimedian) copulas in higher dimensions is not as simple and requires different techniques, cf. Embrechts et al. [8], chapter 6, or Robert and Casella [22] for MCMC-methods.

Finally, we would like to mention that there are many possibilities to introduce *sets* of parameters into given families of copulas. Archimedian copulas can immediately be generalized to two-parameter families due to the fact that for appropriate Laplace transforms ψ and ς, $\psi(-\ln(\varsigma))$ also is a Laplace transform (see Joe [13], p. 374ff). This fact can be used to construct such extensions, cf. also Nelsen [17], p. 123f. A nice survey for multivariate extreme value distributions can be found in Kotz and Nadarajah [15]. Perhaps the simplest way to create multi-parameter copula families is by *mixing*, i.e. if $C_1, \cdots, C_m$ are arbitrary (n-dimensional) copulas and $\alpha_1, \cdots, \alpha_m$ are non-negative weights with $\sum_{k=1}^{m} \alpha_k = 1$, then $C = \sum_{k=1}^{m} \alpha_k C_k$ also is an (n-dimensional) copula. Another simple possibility to construct an n-parametric copula $C_{\theta_1, \cdots, \theta_n}$ in $2n$-dimensions is via products of one-parametric two-dimensional copulas $C_{1,\theta_1}, \cdots, C_{n,\theta_n}$ by

$$C_{\theta_1, \cdots, \theta_n}(u_1, \cdots, u_{2n}) = C_{1,\theta_1}(u_1, u_2) \cdot C_{2,\theta_2}(u_3, u_4) \cdot \ldots \cdot C_{n,\theta_n}(u_{2n-1}, u_{2n}).$$

For the more general problem of creating higher dimensional copulas from lower dimensional ones, see e.g. Nelsen [17], chapter 3.4. However, the question arises here whether such approaches make reasonable sense for modeling real-world problems, where usually at least a minimum amount of information about the structure of mutual relationships is available.

5. Further topics

Survival, dual and co-copula. Let U and V be uniformly distributed random variables with a copula C. Then also $1-U$ and $1-V$ are uniformly distributed. We refer to

$$\begin{aligned}\hat{C}(u,v) &:= P(1-U \le u,\, 1-V \le v) = P(U \ge 1-u,\, V \ge 1-v) = P\left(\{U \le 1-u\}^c \cap \{V \le 1-v\}^c\right) \\ &= 1 - P\left(\{U \le 1-u\} \cup \{V \le 1-v\}\right) = u + v - 1 + C(1-u, 1-v),\ \ u, v \in [0,1]\end{aligned}$$

as the *survival copula* induced by C. (Corresponding generalizations to n dimensions are obvious, involving Sylverster's sieve theorem for the evaluation of probabilities for unions of intersecting events.) Note that this notion of a survival copula is different from the joint survival function $\overline{C}$ of the original random variables, but there is of course the relationship

$$\begin{aligned}\overline{C}(u,v) &:= P(U > u,\, V > v) = P(1-U \le 1-u,\, 1-V \le 1-v) = \hat{C}(1-u, 1-v) \\ &= 1 - u - v + C(u,v),\ \ u, v \in [0,1].\end{aligned}$$

From the above it is clear that the survival copula of a survival copula is identical with the original copula, i.e. $\hat{\hat{C}} = C.$ The *dual* $\tilde{C}$ and the *co-copula* C^* of a bivariate copula C are defined as

$$\begin{aligned}\tilde{C}(u,v) &:= u + v - C(u,v) = P\left(\{U \le u\} \cup \{V \le v\}\right) \\ C^*(u,v) &:= 1 - C(1-u, 1-v) = P\left(\{U > u\} \cup \{V > v\}\right),\ \ u, v \in [0,1]\end{aligned}$$

(cf. Nelsen [17], chapter 2.6). Note that these are not copulas in the strict sense of Definition 2.1, but they can be helpful in calculating probabilities for events other than standard intervals. Seemingly, extensions to more than two dimensions are likewise possible.

It is interesting to notice that for Archimedian copulas, survival copulas can also be Archimedian, as is the case for the Gumbel copula. The generator for the Gumbel survival copula is given by $\hat{\varphi}_\theta(t) = \ln(1 - \theta \ln t)$ (cf. Nelsen [17], p. 94).

Copulas and tail-dependence. This topic is of importance for the analysis of joint extremes and is hence of relevance for insurance applications, in particular in connection with reinsurance.

Definition 5.1. Let X and Y be random variables with c.d.f.'s F and G. The coefficient of (upper) tail dependence is in case of its existence given by

$$\lambda := \lim_{\alpha \downarrow 0} P\left(Y > G^{-1}(1-\alpha) \,\middle|\, X > F^{-1}(1-\alpha)\right).$$

For $\lambda > 0,$ we say that Y is *asymptotically dependent* of X (in the upper tail); for $\lambda = 0,$ we say that X and Y are *asymptotically independent*.

In case of continuity of F and G, λ can be expressed through the survival copula $\hat{C}$ induced by X and Y as

$$\lambda = \lim_{\alpha \downarrow 0} \frac{P\big(G(Y) > 1-\alpha\,,\, F(X) > 1-\alpha\big)}{P\big(F(X) > 1-\alpha\big)} = \lim_{\alpha \downarrow 0} \frac{\overline{C}(1-\alpha, 1-\alpha)}{\alpha} = \lim_{\alpha \downarrow 0} \frac{\hat{C}(\alpha,\alpha)}{\alpha}$$

(see e.g. Embrechts et al. [8] and the discussion on further topics related to this). Some recent applications of this concept to insurance problems, in particular w.r.t. the Clayton survival copula (HRT-copula, "heavy right tail copula"), can be found in Belguise and Levi [1] and Charpentier [4]. Denoting by C_θ the original Clayton copula, we obtain for the HRT-copula $\hat{C}_\theta$, after some simple analysis,

$$\lambda_\theta = \lim_{\alpha \downarrow 0} \frac{\hat{\hat{C}}_\theta(\alpha,\alpha)}{\alpha} = \lim_{\alpha \downarrow 0} \frac{C_\theta(\alpha,\alpha)}{\alpha} = 2^{-1/\theta} \;\text{ for }\; \theta > 0.$$

Note that taking the limit $\theta \downarrow 0$ (which corresponds to the independence copula Π), we obtain $\lambda_0 = 0$ here; likewise, $\lambda_\infty = 1$. Thus for the HRT-copula, θ can be regarded as a "natural" dependence parameter.

C o p u l a s   a n d   s t o c h a s t i c   p r o c e s s e s. It seems to be natural to apply the theory of copulas not only to pairs or *n*-tuples of random variables as above, but rather to its infinitely-dimensional analogue, *stochastic processes*. Some progress has been made in particular in the framework of the "simplest" of such dependence structures, Markov processes (see e.g. Nelsen [17], chapter 6.3). A nice exposition with applications to extreme value theory can be found in the recent Ph.-D. Thesis of Schmitz [23]. In actuarial science, the classical risk processes of the type

$$X(t) = \sum_{n=1}^{N(t)} Y_n,\; t \ge 0$$

are of particular interest, where $\{N(t) \mid t \ge 0\}$ is a counting process (for the claims frequency in the time period $[0,t]$) and $\{Y_n \mid n \in \mathbb{N}\}$ is a family of non-negative random variables (individual claim sizes). Note that no independence assumptions are necessary to describe such a model, although this is usually the case in order to be able calculate certain event probabilities (in particular in connection with ruin). For practical purposes, it might be desirable to "couple" claim sizes and claims frequency together, which can of course be done via copulas. A more general approach to this kind of problems is through point process theory. In a recent paper of Pfeifer and Nešlehová [20], some ideas on how to couple risk processes of such type together (with "local" or "global" dependence) are outlined, with special emphasis on negative correlation for the frequency part (which is non-trivial). Such problems occur e.g. in modern DFA where time-dependence is an essential ingredient.

6. Conclusion and outlook

Copulas have turned out to be a powerful tool in modeling stochastic dependencies in various fields related to insurance and finance. They are in particular useful for simulation studies within the framework of IRM and DFA, to mention some of the more recent applications. However, going into higher dimensions is still a challenge, since a lot of problems occuring there have not yet been sufficiently tackled. The most urgent task is perhaps to develop mathematical tools for a proper identification of suitable copulas in higher dimensions, and statistical tools for testing hypotheses on copulas or estimating their parameters. Some progress has been made for certain classes of copulas such as the Frank family (cf. Junker and May [14]), or the multivariate extreme value copulas (cf. Kotz and Naradajah [15]). The recent paper by Deheuvels and Martynov [6] is very promising in this respect because the approach chosen there allows for the development of distribution-free goodness-of-fit tests for copulas in general.

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S u m m a r y

This paper contains a survey over the mathematical foundations, properties and potential applications of copulas in insurance and finance. Special emphasis is put on relationships between copulas and correlation as well as dependence measures, parametric families of copulas, Archimedian copulas (in particular in higher dimensions), tail dependence and general stochastic processes.

Z u s a m m e n f a s s u n g

Diese Arbeit gibt eine Übersicht über die mathematischen Grundlagen, Eigenschaften und möglichen Anwendungsfelder von Copulas in der Finanz- und Versicherungswelt. Besonderes Gewicht liegt dabei auf dem Zusammenhang zwischen Copulas und der Korrelation bzw. allgemeineren stochastischen Abhängigkeitsmassen, parametrischen Familien von Copulas, Archimedischen Copulas (insbesondere in höheren Dimensionen), Tail-Abhängigkeit und dem Bezug zu stochastischen Prozessen.

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