ON THE DISTANCE BETWEEN THE DISTRIBUTIONS OF RANDOM SUMS

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Abstract

In this paper, we consider the total variation distance between the distributions of two random sums S_M and S_N with different random summation indices M and N. We derive upper bounds, some of which are sharp. Further, bounds with so-called magic factors are possible. Better results are possible when M and N are stochastically or stop-loss ordered. It turns out that the solution of this approximation problem strongly depends on how many of the first moments of M and N coincide. As approximations, we therefore choose suitable finite signed measures, which coincide with the distribution of the approximating random sum S_N, if M and N have the same first moments.

Keywords: Binomial distribution; discrete Taylor formula; finite signed measures; Poisson approximation; random sums; sharpness results; stochastic order; stop-loss order; total variation distance.

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1. Introduction

1.1. Motivation

Random sums have many applications in different disciplines, such as probability theory, statistics, risk theory, reliability, queueing, biology, and others [see, for example, the books by Gut (1988), Gnedenko and Korolev (1996), Rahimov (1995), and

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Kalashnikov (1997)].

If the random summation index assumes large values with positive probability, the distribution of the corresponding random sum is often difficult to calculate and an approximation is necessary. One can find many contributions on normal approximations (for example, see the books by Gut (1988), Kruglov and Korolev (1990), and Gnedenko and Korolev (1996) and the references therein). But relatively little is known about the approximation by the distribution of another random sum, even if only the random summation index is changed. In this paper, we show that this assumption already allows non-trivial results.

Throughout this paper, we use the following notation: Let $X, X_1, X_2, \ldots$ be independent and identically distributed random variables in $\mathbf{R}$. Set $p = 1 - q = P(X \neq 0) \in [0,\, 1]$ and $S_n = \sum_{j=1}^n X_j$ for $n \in \mathbf{Z}_+ = \{0, 1, 2, \ldots\}$. Let M and N be two random variables in $\mathbf{Z}_+$ independent of the X_j. We give answers to the following question: how large is the distance between the distributions P^{S_M} and P^{S_N} of S_M and S_N? Indeed, the answers strongly depend on how many of the first moments of M and N are the same. As an approximation of P^{S_M}, we therefore choose suitable finite signed measures, which coincide with P^{S_N}, if M and N have the same first moments.

As a measure of accuracy, we use the total variation distance, which is defined by

$$d_{\mathrm{TV}}(Q_1,\, Q_2) = \sup_B |Q_1(B) - Q_2(B)|,$$

where Q_1 and Q_2 are two finite signed measures on the Borel sets of $\mathbf{R}$ and the sup is over all Borel measurable sets $B \subseteq \mathbf{R}$. In the case that Q_i is the distribution of a random variable Z_i, we also write $d_{\mathrm{TV}}(Z_1,\, Z_2) = d_{\mathrm{TV}}(Q_1,\, Q_2)$. In particular, we are interested in bounds for

$$d_\tau := d_{\mathrm{TV}}(S_M,\, S_N).$$

By using characteristic functions, it is easily shown that $P^{S_M} = P^{S_N}$, if and only if M and N have the same distributions or $p = 0$. The inequalities we search for should reflect this fact. In particular, our main interest lies in good bounds for a small p.

We encountered the present problem in the excess of loss (XL) reinsurance (see, for example, Heilmann (1988, 6.2.2, p. 212)): Here, we start from an arbitrary collective model with claims $Y_1, Y_2, \ldots$ and claim number M. The aggregate claim is then given by $\sum_{j=1}^M Y_j$. Clearly, we assume that, in this context, the Y_j are non-negative, independent and identically distributed random variables, which are also independent of M. In the excess of loss reinsurance, each claim Y_j ($j \in \{1, \ldots, M\}$) is divided between the ceding company and the reinsurer, i.e. the reinsurer has to pay the excess $X_j = \max\{Y_j - t, 0\}$ over an agreed retention (priority) $t > 0$, whereas the ceding company has to pay the remaining amount $Y_j - X_j$. The reinsurer's aggregate claim is now given by $S_M = \sum_{j=1}^{M} X_j$, whereby we may assume that $p = P(X_1 > 0) = P(Y_1 > t)$ is small.

The paper is structured as follows. The next two subsections are devoted to the comparison of our results with those given in the literature; here we consider the general case and the case that $\mathrm{E}(M) = \mathrm{E}(N) < \infty$. In Subsection 1.4, we derive suitable finite signed measures, which will be used as approximations of P^{S_M}. In Section 2, we present the main error bounds, some of which turn out to be sharp. Furthermore, in the cases when M and N are stochastically or stop-loss ordered we give even better bounds. (It should be mentioned here that, in the theory of stochastic ordering, the stop-loss order is also called increasing convex order, cf. Müller and Stoyan (2002).) In Sections 3 and 4, we present the proofs.

Note that Theorems 1–3 presented in the following Subsections 1.2 and 1.3 are special cases of the more general theorems in Section 2 and show the most important improvements given in this paper; further, Theorems A and B are due to other authors and need not be proved.

1.2. Known results: the general case

With the help of coupling arguments originally due to Doeblin (1938), a simple inequality can be derived (cf. Lindvall (1992, Theorem 5.2, p. 19)): a maximal coupling (M', N') of (M, N) exists such that M' and N' are independent of the X_j and $d_{\mathrm{TV}}(M, N) = P(M' \neq N')$. Using the above notation $d_\tau = d_{\mathrm{TV}}(S_M, S_N)$, this leads to

$$d_\tau \leq P(S_{M'} \neq S_{N'}) \leq P(M' \neq N') = d_{\mathrm{TV}}(M, N), \tag{1}$$

which also holds in the context of dependent and not necessarily identically distributed X_j. See Finkelstein et al. (1990, Lemma 4) and Vellaisamy and Chaudhuri (1996, Lemma 3.1), for elementary proofs of (1) in the case that the X_j have values in $\{0, 1\}$ and $\mathbf{Z}_+$, respectively. Further developments can be found in the paper of Denuit and

Van Bellegem (2001). Since the bound in (1) is independent of p, it is useless in the important case of small p. Better results are possible. Our bounds (16) and (17) below (see also Remark 1) yield the following:

Theorem 1. *If the X_j are real-valued, then, for all $m \in \mathbf{Z}_+$,*

$$d_\tau \leq p \sum_{n=0}^{\infty} |m-n|\,|P(M=n)-P(N=n)|, \tag{2}$$

$$d_\tau \leq \sqrt{\frac{p}{q}} \sum_{n=0}^{\infty} |\sqrt{m}-\sqrt{n}|\,|P(M=n)-P(N=n)|. \tag{3}$$

To avoid trivialities, we may suppose that M and N have different distributions. In this case, it is possible to specify the bounds above by using moments of functions of a $\mathbf{Z}_+$-valued random variable Z with distribution

$$P(Z=n) = \frac{|P(M=n)-P(N=n)|}{\sum_{k=0}^{\infty} |P(M=k)-P(N=k)|}, \qquad (n \in \mathbf{Z}_+). \tag{4}$$

In order to minimize the bound in (2), it suffices to replace m with a median of Z. In contrast to the bound in (1), the upper bounds in Theorem 1 tend to zero as p tends to zero, if we assume the finiteness of the respective moments of Z. Note that, if one is interested in the convergence rate concerning p, (2) is more favourable than (3) only if Z has finite expectation. Otherwise, if $\mathrm{E}(\sqrt{Z}) < \infty$, (3) can be used. Similar considerations hold for the other inequalities given below. Note that all our bounds presented in this paper depend only on p and the distributions of M and N. In Theorem 5, one can find further inequalities, which in contrast to those in Theorem 1 and further results below contain differences of distribution functions instead of differences of point probabilities of M and N, respectively.

For further general results concerning the Kolmogorov metric, see Krajka and Rychlik (1987) and the references therein.

1.3. Known results: the case $\mathbf{E}(M) = \mathbf{E}(N) < \infty$

In this subsection, we always consider the case that

$$\mathrm{E}(M) = \mathrm{E}(N) < \infty. \tag{5}$$

Logunov (1990, p. 588) proved the following theorem.

Theorem A *If* (5) *holds and the* X_j *are Bernoulli random variables with* $P(X_j = 1) = 1 - P(X_j = 0) = p$, *then*

$$d_\tau \le p^2 \sum_{n=0}^{\infty} n(n-1)\,|P(M=n) - P(N=n)|.$$

Vellaisamy and Chaudhuri (1996) gave an extension of this result. In particular, by using Logunov's ideas, they showed the following statement in the proof of their Lemma 2.1.

Theorem B *If* (5) *holds and the* X_j *have values in* $\mathbf{Z}_+$, *then*

$$d_\tau \le \sum_{n=0}^{\infty} h(n)\,|P(M=n) - P(N=n)| \tag{6}$$

with $h(n) := \mathrm{E}(S_n(S_n - 1))$, $(n \in \mathbf{Z}_+)$.

Note that Theorem A is indeed contained in Theorem B, since under the conditions of Theorem A, we have $h(n) = p^2 n(n-1)$. It should be mentioned that, in the proof of (6), Vellaisamy and Chaudhuri do not need the independence of the X_j.

Let us compare Theorem B with our results. The following inequality follows from (16) below.

Theorem 2. *If* (5) *holds and the* X_j *are real-valued, then, for all* $m \in \mathbf{Z}_+$,

$$d_\tau \le p^2 \sum_{n=0}^{\infty} (n-m)\,(n-m-1)\,|P(M=n) - P(N=n)|. \tag{7}$$

To minimize the bound in (7), we consider the random variable Z as given in (4), set

$$\mu = \mathrm{E}(Z), \tag{8}$$

and replace m in (7) with

$$m_1 = \lfloor \mu \rfloor.$$

As usual, here $\lfloor x \rfloor$ denotes the largest integer $\le x$, $(x \in \mathbf{R})$. Note that, if the X_j are $\mathbf{Z}_+$-valued, we have

$$h(n) = \mathrm{E}(S_n(S_n - 1)) = (\mathrm{E}(X))^2 n(n-1) + n\mathrm{E}(X(X-1)) \ge p^2 n(n-1)$$

and $h(n)$ can be large or infinite, whereas the lower bound of $h(n)$ is bounded by $n(n-1)$. Therefore, if we set $m = 0$ in (7), we obtain a bound, which is better than (6). As a result, Theorem 2 is a considerable improvement of Theorem B.

From (18) below, an inequality follows, which is much more interesting than (7):

Theorem 3. *If* (5) *holds and the* X_j *are real-valued, then, for all* $m \in \mathbf{Z}_+$*,*

$$d_\tau \le \frac{p}{\sqrt{2}\,q} \sum_{n=0}^{\infty} \Big(m - n + n \ln\Big(\frac{n+1/2}{m+1/2}\Big)\Big) |P(M=n) - P(N=n)|. \tag{9}$$

To minimize the bound in (9), replace m in (9) with

$$m_2 = \Big\lfloor \frac{1}{e^{1/\mu} - 1} + \frac{1}{2} \Big\rfloor,$$

where μ is defined in (8). Note that $m_2 \sim \mu$ as $\mu \to \infty$ and that $m_2 = \lfloor \mu \rfloor = 0$ for μ small enough. In order to derive a bound easier than (9), it makes therefore sense to use (9) with $m = \lfloor \mu \rfloor$, which yields

$$d_\tau \le \frac{\sqrt{2}\,p}{q}\, d_{\mathrm{TV}}(M,\, N)\, \mathrm{E}\Big[Z \ln\Big(\frac{Z+1/2}{\lfloor \mu \rfloor + 1/2}\Big)\Big] \tag{10}$$

A further bound without a log-term can be derived by the help of the easy inequality

$$\mathrm{E}\Big(\lfloor \mu \rfloor - Z + Z \ln\Big(\frac{Z+1/2}{\lfloor \mu \rfloor + 1/2}\Big)\Big) \le \frac{\mathrm{E}(Z - \lfloor \mu \rfloor)^2}{\lfloor \mu \rfloor + 1/2}.$$

In combination with (9), this leads to

$$d_\tau \le \frac{p}{\sqrt{2}\,q(\lfloor \mu \rfloor + 1/2)} \sum_{n=0}^{\infty} (n - \lfloor \mu \rfloor)^2 |P(M=n) - P(N=n)|. \tag{11}$$

Let us compare the order of the bounds in (7) and (11). Letting $m = m_1 = \lfloor \mu \rfloor$ in (7), we see that (11) contains an additional factor $(pq(\lfloor \mu \rfloor + 1/2))^{-1}$, which leads to a better upper bound in the case of $pq\mu$ being large. Using the terminology of Barbour et al. (1992, p. 5) in the Poisson approximation, this extra factor can be called a magic factor.

Observe that it may be difficult to evaluate μ. There are however two cases, in which we easily obtain a formula for μ: If $M = r \in \mathbf{Z}_+$ is a constant, we have $\mu = 2^{-1}(r + \mathrm{E}(N \,|\, N \ne r))$. Further, if P^N and P^M are mutually singular, i.e. $P(N = n) > 0$ implies $P(M = n) = 0$ for each $n \in \mathbf{Z}_+$, then we have $\mu = 2^{-1}\mathrm{E}(N+M)$.

Example 1. For the rest of this subsection, let us assume that the X_j are Bernoulli random variables with $P(X_j = 1) = p$ and that N has a Poisson distribution $\mathrm{Po}(\lambda)$ with mean $\lambda \in (0, \infty)$. This yields $S_N \sim \mathrm{Po}(p\lambda)$ and from (5) we obtain $\mathrm{E}(M) = \lambda$. Under these conditions, one can find further inequalities in the literature. Barbour et al. (1992, equation (3.8), p. 39) derived the inequality

$$d_{\mathrm{TV}}(S_M,\, \mathrm{Po}(p\lambda)) \le p\Big(1 + \frac{\mathrm{Var}(M)}{\lambda}\Big). \tag{12}$$

This bound can uniformly be sharpened by modifying the proof of Barbour et al. with the help of Theorem 1 in Roos (2001), which yields

$$d_{\mathrm{TV}}(S_M, \mathrm{Po}(p\lambda)) \leq p\Big(1 + \frac{3}{2e}\mathrm{E}\Big(M \ln \frac{M}{\lambda}\Big)\Big). \tag{13}$$

Here, we set $0 \ln 0 = 0$. The bound in (13) is indeed an improvement of (12), since we have $\mathrm{E}(M \ln(M/\lambda)) \leq \mathrm{Var}(M)/\lambda$ (a more general estimate can be found in Roos (2001, Lemma 1)). However, in contrast to (10), the bound in (13) is not small when P^M and $P^N = \mathrm{Po}(\lambda)$ are close. Therefore, our bound (10) (or the better (9)) should be preferred over (13). Note that, from (9), it is also possible to derive the bound

$$d_{\mathrm{TV}}(S_M, \mathrm{Po}(p\lambda)) \leq \frac{p}{\sqrt{2}\,q}\mathrm{E}\Big(M \ln \frac{M + 1/2}{\lfloor\lambda\rfloor + 1/2} + N \ln \frac{N + 1/2}{\lfloor\lambda\rfloor + 1/2}\Big),$$

which has the same order as (13), if p is bounded away from unity. Indeed, this follows from the easy observation that, here,

$$\mathrm{E}\Big(N \ln\Big(\frac{N + 1/2}{\lfloor\lambda\rfloor + 1/2}\Big)\Big) \leq 4.$$

See Yannaros (1991, p. 163), for another upper bound in this context.

1.4. Suitable finite signed measures

We write

$$P^X = q\epsilon_0 + pQ, \qquad Q = P(X \in \cdot \,|\, X \neq 0),$$

where $p = 1 - q = P(X \neq 0)$ as defined above and ϵ_x denotes the Dirac measure at point $x \in \mathbf{R}$. The method we use can be understood as a variant of Lindeberg's (1922) device. It is based on the use of the finite signed measures

$$R_{k,m} := P^{S_N} + \sum_{j=0}^{k} \beta_{j,m}\, p^j (Q - \epsilon_0)^{*j} * P^{S_m}, \qquad (k, m \in \mathbf{Z}_+ \text{ fixed})$$

as approximations of P^{S_M}, where we assume that

$$\sum_{n=0}^{\infty} |P(M = n) - P(N = n)|\, n^k < \infty, \tag{14}$$

such that

$$\beta_{j,m} = \sum_{n=0}^{\infty} (P(M = n) - P(N = n)) \binom{n - m}{j} \tag{15}$$

is absolutely convergent for all $j \in \{0,\dots,k\}$. Here, for a finite signed measure G on $\mathbf{R}$, we let G^{*n}, $(n \in \mathbf{N} = \{1,2,\dots\})$ denote the n–fold convolution of G and set $G^{*0} = \epsilon_0$. Further, $\binom{x}{j} = \prod_{i=1}^{j}[(x-i+1)/i]$ for $j \in \mathbf{Z}_+$ and $x \in \mathbf{R}$. For all $m \in \mathbf{Z}_+$, we have $R_{0,m} = P^{S_N}$, since $\beta_{0,m} = 0$. More generally, $R_{k,m} = P^{S_N}$ is valid for $k, m \in \mathbf{Z}_+$, if $\beta_{0,m} = \cdots = \beta_{k,m} = 0$. In the important case $\mathrm{E}(M) = \mathrm{E}(N) < \infty$ we have $\beta_{0,m} = \beta_{1,m} = 0$ and hence $R_{1,m} = P^{S_N}$ for all $m \in \mathbf{Z}_+$.

Let us now explain how the $R_{k,m}$ can be found: For fixed $k, m \in \mathbf{Z}_+$, we have

$$
\begin{aligned}
P^{S_M} &= \sum_{n=0}^{\infty} P(M=n)(\epsilon_0 + p(Q-\epsilon_0))^{*n} \\
&= \sum_{n=0}^{\infty} P(M=n)(U_{n,k,m} + V_{n,k,m}) * (\epsilon_0 + p(Q-\epsilon_0))^{*m},
\end{aligned}
$$

where we assume for a short while that $p \in [0, 1/2)$ and set

$$
U_{n,k,m} = \sum_{j=0}^{k} \binom{n-m}{j} p^j (Q-\epsilon_0)^{*j}, \quad V_{n,k,m} = \sum_{j=k+1}^{\infty} \binom{n-m}{j} p^j (Q-\epsilon_0)^{*j}.
$$

Clearly, $U_{n,k,m}$ is a finite signed measure and, since $p \in [0, 1/2)$, this also holds for $V_{n,k,m}$. The idea is now to replace $P(M=n)V_{n,k,m}$ with $P(N=n)V_{n,k,m}$, i.e. P^{S_M} will be approximated by

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \Big(P(M=n)U_{n,k,m} + P(N=n)V_{n,k,m}\Big) * (\epsilon_0 + p(Q-\epsilon_0))^{*m} \\
&= P^{S_N} + \sum_{n=0}^{\infty} \Big(P(M=n) - P(N=n)\Big) U_{n,k,m} * P^{S_m} \\
&= P^{S_N} + \sum_{j=0}^{k} \bigg(\sum_{n=0}^{\infty} \Big(P(M=n) - P(N=n)\Big) \binom{n-m}{j} \bigg) p^j (Q-\epsilon_0)^{*j} * P^{S_m} \\
&= R_{k,m},
\end{aligned}
$$

where these equalities hold, whenever (14) is valid. Since the total variation norm of $V_{n,k,m}$ tends to zero as $k \to \infty$, we expect that the accuracy of the approximation is increasing in k. The arbitrary parameter m can be chosen to minimize some of our bounds. Note that, in the presence of condition (14), $R_{k,m}$ is a finite signed measure for all $p \in [0, 1]$; since we do not need the $V_{n,k,m}$ any longer, we drop the above assumption on p and consider from now on $p \in [0, 1]$.

2. Main results

Here and henceforth, we use the following notation. For $m, n \in \mathbf{Z}_+$, we set

$$a_n = P(M = n) - P(N = n), \qquad A_n = P(M \le n) - P(N \le n),$$

$m \wedge n = \min\{m, n\}$, and $m \vee n = \max\{m, n\}$. Further, for $x \in \mathbf{R}$ and $n \in \mathbf{Z}_+$, let $(x)_n = \prod_{j=0}^{n-1}(x+j)$ denote the Pochhammer symbol. As indicated above, whenever we use the finite signed measure $R_{k,m}$ for $k, m \in \mathbf{Z}_+$ given, we assume that $\sum_{n=0}^{\infty} |a_n| n^k < \infty$.

Theorem 4. *Let $k, m \in \mathbf{Z}_+$. Then*

$$d_{\mathrm{TV}}(S_M,\, R_{k,m}) \le 2^k\, p^{k+1} \sum_{n=0}^{\infty} |a_n| \left| \binom{n-m}{k+1} \right|. \tag{16}$$

If additionally $k \ge 2$, $c_0 = 1$, and

$$c_j = \frac{1}{j - \sqrt{j(j-1)}}, \qquad (j \in \mathbf{N}),$$

then

$$\begin{aligned} d_{\mathrm{TV}}(S_M,\, S_N) &\le \frac{1}{2}\sqrt{\frac{p}{q}} \sum_{n=0}^{\infty} c_{m \vee n}\, |a_n|\, |\sqrt{m} - \sqrt{n}|, && (17)\\ d_{\mathrm{TV}}(S_M,\, R_{1,m}) &\le \frac{p}{\sqrt{2}\, q} \sum_{n=0}^{\infty} |a_n| \left(m - n + n \ln\left(\frac{n+1/2}{m+1/2} \right) \right), && (18)\\ d_{\mathrm{TV}}(S_M,\, R_{k,m}) &\le \frac{2\sqrt{(k+1)!}}{\sqrt{(m+1)_{k-1}}} \left(\frac{p}{q}\right)^{(k+1)/2} \sum_{n=0}^{\infty} \frac{|a_n|\, |m-n|^{k+1}}{(\sqrt{m+1} + \sqrt{n+1})^2}. && (19) \end{aligned}$$

In the following theorem, we present bounds, which are comparable with those given in Theorem 4. The main difference is that we replace point probabilities with distribution functions, i.e. we use $A_n = P(M \le n) - P(N \le n)$ instead of $a_n = P(M = n) - P(N = n)$.

Theorem 5. *Let $k, m \in \mathbf{Z}_+$. Then*

$$d_{\mathrm{TV}}(S_M,\, R_{k,m}) \le 2^k\, p^{k+1} \sum_{n=0}^{\infty} |A_n| \left| \binom{n-m}{k} \right|. \tag{20}$$

If additionally $k \ge 2$, $C_0 = 0$, and

$$C_j = \frac{1}{\sqrt{j(j+1)}\ln((j+1/2)/(j-1/2))}, \qquad (j \in \mathbf{N}),$$

then

$$d_{\mathrm{TV}}(S_M,\, S_N) \;\le\; \frac{1}{2}\sqrt{\frac{p}{q}}\sum_{n=0}^{\infty}\frac{|A_n|}{\sqrt{n+1}}, \tag{21}$$

$$d_{\mathrm{TV}}(S_M,\, R_{1,m}) \;\le\; \frac{p}{\sqrt{2}\,q}\sum_{n=0}^{\infty} C_{m\vee n}\,|A_n|\left|\ln\left(\frac{n+1/2}{m+1/2}\right)\right|, \tag{22}$$

$$d_{\mathrm{TV}}(S_M,\, R_{k,m}) \;\le\; \frac{2\sqrt{(k+1)!}}{\sqrt{(m+1)_{k-1}}}\left(\frac{p}{q}\right)^{(k+1)/2}\sum_{n=0}^{\infty}\frac{|A_n|\,|m-n|^k}{(\sqrt{m+1}+\sqrt{n+1})^2}. \tag{23}$$

Remark 1. It is easy to prove that c_j in Theorem 4 is increasing in $j \in \mathbf{Z}_+$ and that $1 \le c_j \le 2$ for all j. Furthermore, as shown in the proof of (29) (see Section 4 below), C_j in Theorem 5 is increasing in $j \in \mathbf{Z}_+$ and satisfies the inequalities $0 \le C_j \le 1$. The upper bounds for c_j and C_j can be used to obtain inequalities simpler than (17) and (22). The terms $c_{m\vee n}$ and $C_{m\vee n}$ in (17) and (22) enable us to show the sharpness of the respective bounds (see Proposition 1 below). The inequalities of Theorem 1 follow from (16) and (17). If $\mathrm{E}(M) = \mathrm{E}(N) < \infty$, then, as mentioned above, $R_{1,m} = P^{S_N}$ for all $m \in \mathbf{Z}_+$. Therefore, in this case (16) and (18) yield the inequalities in Theorems 2 and 3.

Proposition 1. *If $M = 0$ and $N = 1$ almost surely, $m = 1$, and $Q = \epsilon_x$ with $x \neq 0$, then in (16) and (20) equalities hold. If additionally $p = 1/2$, then also in (17), (21), and (22) equalities hold.*

In the case that M and N have finite and nearly the same expectations, the single bounds in (16) with $k = 0$ and (17) can be sharpened. This can be achieved by using the triangle inequality

$$d_\tau = d_{\mathrm{TV}}(S_M,\, S_N) \le d_{\mathrm{TV}}(S_M,\, R_{k,m}) + d_{\mathrm{TV}}(R_{k,m},\, S_N)$$

with $k = 1$ and arbitrary $m \in \mathbf{Z}_+$. In particular the following result is valid.

Theorem 6. *If M and N have finite but not necessarily equal means, then, for all $m \in \mathbf{Z}_+$,*

$$d_\tau \;\le\; p|\mathrm{E}(M-N)| + p^2\sum_{n=0}^{\infty}|a_n|(n-m)(n-m-1),$$

$$d_\tau \;\le\; \frac{1}{2}\sqrt{\frac{p}{(m+1)q}}|\mathrm{E}(M-N)| + \frac{p}{\sqrt{2}\,q}\sum_{n=0}^{\infty}|a_n|\left(m-n+n\ln\left(\frac{n+1/2}{m+1/2}\right)\right).$$

The argumentation above can easily be generalized to moments of higher order. For example, it is possible to derive inequalities when M and N have the same expectation and nearly the same variance. Furthermore, similar refinements of (20) with $k = 0$ and (21) can be shown.

We finally present some results in the case that M and N are stochastically or stop-loss ordered. It turns out that here better results are possible. As usual, we say that M is stochastically smaller than N, written $M \leq_{\mathrm{ST}} N$, if $P(M > t) \leq P(N > t)$ for all $t \in \mathbf{R}$. Further, M is smaller than N in the stop-loss order, written $M \leq_{\mathrm{SL}} N$, if $\mathrm{E}(M - t)_+ \leq \mathrm{E}(N - t)_+$ for all $t \in [0, \infty)$. Here $x_+ = \max\{x, 0\}$ for $x \in \mathbf{R}$. For an extensive treatment of stochastic orderings, we refer the interested reader, for example, to Shaked and Shanthikumar (1994) or Müller and Stoyan (2002).

It is well-known that the distributions of M and N coincide, if one of the following two assumptions hold:

$$M \leq_{\mathrm{ST}} N \quad \text{and} \quad \mathrm{E}(M^\alpha) = \mathrm{E}(N^\alpha) < \infty \quad \text{for some} \quad \alpha > 0 \quad \text{or}$$
$$M \leq_{\mathrm{SL}} N, \quad \mathrm{E}(M) = \mathrm{E}(N) < \infty \quad \text{and} \quad \mathrm{Var}(M) = \mathrm{Var}(N) < \infty.$$

If one of these assumptions is valid, then obviously $d_\tau = 0$. The following theorem reflects this fact. Observe that, due to the form of the bounds below, no arbitrary parameter m occurs.

Theorem 7. (a) *Let us assume that* $M \leq_{\mathrm{ST}} N$. *If* M *and* N *have finite mean, then*

$$d_\tau \leq p\,\mathrm{E}(N - M).$$

If $\mathrm{E}(\sqrt{M})$ *and* $\mathrm{E}(\sqrt{N})$ *are finite, then*

$$d_\tau \leq \sqrt{\frac{p}{q}}\,\mathrm{E}(\sqrt{N} - \sqrt{M}).$$

(b) *Let us assume that* $M \leq_{\mathrm{SL}} N$. *If* $\mathrm{E}(M^2)$ *and* $\mathrm{E}(N^2)$ *are finite, then*

$$d_\tau \leq p\,\mathrm{E}(N - M) + p^2\mathrm{E}\Big(N(N-1) - M(M-1)\Big).$$

If $\mathrm{E}(M \ln M)$ *and* $\mathrm{E}(N \ln N)$ *are finite, then*

$$d_\tau \leq p\,\mathrm{E}(N - M) + \frac{p}{\sqrt{2}\,q}\,\mathrm{E}\Big(N\ln(N) - M\ln(M)\Big).$$

Corollary 1. *Let us assume that* $M \leq_{\mathrm{SL}} N$ *and* $\mathrm{E}(M) = \mathrm{E}(N) < \infty$*. Then, if the right-hand sides exist,*

$$\begin{aligned} d_\tau &\leq p^2\Big(\mathrm{Var}(N) - \mathrm{Var}(M)\Big), \\ d_\tau &\leq \frac{p}{\sqrt{2}\,q}\,\mathrm{E}\Big(N\ln(N) - M\ln(M)\Big). \end{aligned}$$

Remark 2. Seemingly, the results of the present paper can be extended using the discrete s-(increasing) convex orderings, which extend stochastic dominance and stop-loss order. Indeed, quantities like the $\beta_{j,m}$ (see 15) play an important role in that framework [e.g. Denuit and Lefèvre, 1997; Denuit, Lefèvre, and Mesfioui, 1999; Denuit, Lefèvre, and Utev, 1999]. This will be investigated in a subsequent paper.

3. Proofs of the main results

We need some notation. For $m, n \in \mathbf{Z}_+$, let

$$(-1)_{n<m} = \begin{cases} -1, & \text{if } n < m, \\ 1, & \text{if } n \geq m. \end{cases}$$

Further, let

$$b(m,n,p) = \Delta^0 b(m,n,p) = \begin{cases} \binom{n}{m}\,p^m\,q^{n-m}, & \text{if } n, m \in \mathbf{Z}_+,\ m \leq n, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\Delta^j b(m,n,p) = \Delta^{j-1} b(m-1,n,p) - \Delta^{j-1} b(m,n,p) \qquad \text{for } j \in \mathbf{N}.$$

The discrete Taylor formula for the binomial counting density presented in the following lemma plays the key rôle in the method of this paper.

Lemma 1. *For* $k, m, n, r \in \mathbf{Z}_+$ *and* $p \in [0,1]$*, we have*

$$\begin{aligned} & b(r,n,p) - \sum_{j=0}^{k} \binom{n-m}{j} p^j \Delta^j b(r,m,p) \\ &\quad = p^{k+1}\,(-1)_{n<m} \sum_{j=m\wedge n}^{(m\vee n)-1} \binom{n-j-1}{k}\,\Delta^{k+1} b(r,j,p). \end{aligned} \tag{24}$$

Proof. The assertion can easily be verified, by using that, for $j, n \in \mathbf{Z}_+$, $p \in [0,1]$, and complex $z \in \mathbf{C}$,

$$\sum_{r=0}^{\infty} \Delta^j b(r,n,p)\, z^r = (z-1)^j (1+p(z-1))^n$$

(cf. Roos (2000, formula (26))). Indeed, letting $m, n \in \mathbf{Z}_+$, $p \in [0,1]$, and $y, z \in \mathbf{C}$ with $|y| < 1/(1+|z|)$, we have

$$\begin{aligned} &\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\Big[b(r,n,p) - \sum_{j=0}^{k}\binom{n-m}{j} p^j \Delta^j b(r,m,p)\Big] y^k\, z^r \\ &= \frac{(1+yp(z-1))^n}{1-y}\bigg[\Big(\frac{1+p(z-1)}{1+yp(z-1)}\Big)^n - \Big(\frac{1+p(z-1)}{1+yp(z-1)}\Big)^m\bigg] \\ &= \sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\Big[p^{k+1}\,(-1)_{n<m} \sum_{j=m\wedge n}^{(m\vee n)-1}\binom{n-j-1}{k}\,\Delta^{k+1} b(r,j,p)\Big] y^k\, z^r. \end{aligned}$$

Comparing the power series, we see that (24) is valid. ■

Lemma 2. *Let $k, m \in \mathbf{Z}_+$. If $\sum_{n=0}^{\infty} |a_n|\, n^k < \infty$, we have*

$$P^{S_M} - R_{k,m} = p^{k+1} \sum_{j=0}^{\infty} g_{j,k,m}\, H_{j,k},$$

where $H_{j,k} = (Q-\epsilon_0)^{*(k+1)} * P^{S_j}$ and

$$g_{j,k,m} = \begin{cases} -\displaystyle\sum_{n=0}^{j} a_n \binom{n-j-1}{k}, & \text{if } 0 \le j < m, \\ \displaystyle\sum_{n=j+1}^{\infty} a_n \binom{n-j-1}{k}, & \text{if } m \le j. \end{cases}$$

Proof. Using Lemma 1, we obtain

$$\begin{aligned} P^{S_M} - R_{k,m} &= \sum_{n=0}^{\infty} a_n \Big[P^{S_n} - \sum_{j=0}^{k}\binom{n-m}{j} p^j (Q-\epsilon_0)^{*j} * P^{S_m}\Big] \\ &= \sum_{n=0}^{\infty} a_n \sum_{r=0}^{\infty}\Big[p^{k+1}(-1)_{n<m} \sum_{j=m\wedge n}^{(m\vee n)-1}\binom{n-j-1}{k}\Delta^{k+1} b(r,j,p)\Big] Q^{*r} \\ &= p^{k+1} \sum_{n=0}^{\infty}\sum_{j=m\wedge n}^{(m\vee n)-1} a_n\, (-1)_{n<m}\binom{n-j-1}{k} H_{j,k} \\ &= p^{k+1}\sum_{j=0}^{\infty} g_{j,k,m}\, H_{j,k}, \end{aligned}$$

where the latter equality follows from the easy observation that

$$\begin{aligned}&n \in \mathbf{Z}_+ \setminus \{m\} \text{ and } j \in \{m \wedge n, \dots, (m \vee n) - 1\}\\ \Leftrightarrow &\begin{cases} j \in \{0, \dots, m-1\} \text{ and } n \in \{0, \dots, j\} \text{ or}\\ j \in \{m, m+1, \dots\} \text{ and } n \in \{j+1, j+2, \dots\}.\end{cases}\end{aligned}$$

The assertion is shown. ∎

Lemma 3. *Let $k, m \in \mathbf{Z}_+$. We assume that $\sum_{n=0}^{\infty} |a_n| n^k < \infty$. Let $g_{j,k,m}$ be defined as in Lemma 2 and set, for $j \in \mathbf{Z}_+$ and $p \in (0,1)$,*

$$f_{j,k,p} = \min\left\{\frac{\sqrt{(k+1)!}}{2\sqrt{(j+1)_{k+1}}\,(pq)^{(k+1)/2}},\ 2^k\right\}.$$

Then

$$d_{\mathrm{TV}}(S_M,\, R_{k,m}) \leq p^{k+1} \sum_{j=0}^{\infty} |g_{j,k,m}|\, f_{j,k,p}. \tag{25}$$

Proof. Let $k, m \in \mathbf{Z}_+$. From Lemma 2, we derive

$$d_{\mathrm{TV}}(S_M,\, R_{k,m}) \leq p^{k+1} \sum_{j=0}^{\infty} |g_{j,k,m}| \sup_B |H_{j,k}(B)|.$$

In Roos (2000, Lemma 4 and formula (41)), it was shown by the help of the Krawtchouk polynomials, that, for $j, k \in \mathbf{Z}_+$ and $p \in (0,1)$,

$$\sum_{r=0}^{\infty} |\Delta^{k+1} b(r, j, p)| \leq 2\, f_{j,k,p}. \tag{26}$$

Note that in (26) equality holds for $j = 0$, $p = 1/2$, and arbitrary $k \in \mathbf{Z}_+$. Later on we will make use of this observation to show Proposition 1. Now we see that, for $j \in \mathbf{Z}_+$,

$$\sup_B |H_{j,k}(B)| = \sup_B \left| \sum_{r=0}^{k+1+j} \Delta^{k+1} b(r, j, p) \Big(Q^{*r}(B) - \frac{1}{2}\Big) \right| \leq f_{j,k,p},$$

which leads to (25). ∎

The following two lemmas are needed in the proofs of Theorems 4 and 5.

Lemma 4. *For all $k, m, n \in \mathbf{Z}_+$, we have*

$$\sum_{j=m\wedge n}^{(m\vee n)-1} \left| \binom{n-j-1}{k} \right| = \left| \binom{n-m}{k+1} \right|. \tag{27}$$

The proof of the preceeding lemma is easily done by using generating functions.

Lemma 5. *Let $k,m,n \in \mathbf{Z}_+$, $k \geq 2$, and c_j, C_j, $(j \in \mathbf{Z}_+)$ be as in Theorem 4 and Theorem 5, respectively. Then*

$$\sum_{j=m\wedge n}^{(m\vee n)-1} \frac{1}{\sqrt{j+1}} \leq c_{m\vee n}\,|\sqrt{m}-\sqrt{n}|, \tag{28}$$

$$\sum_{j=m\wedge n}^{(m\vee n)-1} \frac{1}{\sqrt{(j+1)(j+2)}} \leq C_{m\vee n}\left|\ln\left(\frac{n+1/2}{m+1/2}\right)\right|, \tag{29}$$

$$\sum_{j=m\wedge n}^{(m\vee n)-1} \frac{|n-j-1|}{\sqrt{(j+1)(j+2)}} \leq m-n+n\ln\left(\frac{n+1/2}{m+1/2}\right), \tag{30}$$

$$\sum_{j=m\wedge n}^{(m\vee n)-1} \frac{|(n-j-k+1)_{k-1}|}{\sqrt{(j+1)_{k+1}}} \leq \frac{4\,(k-1)!\,|m-n|^k}{\sqrt{(m+1)_{k-1}}(\sqrt{m+1}+\sqrt{n+1})^2}, \tag{31}$$

$$\sum_{j=m\wedge n}^{(m\vee n)-1} \frac{|(n-j-k)_k|}{\sqrt{(j+1)_{k+1}}} \leq \frac{4\,k!\,|m-n|^{k+1}}{\sqrt{(m+1)_{k-1}}(\sqrt{m+1}+\sqrt{n+1})^2}. \tag{32}$$

The proof of Lemma 5 is somewhat lengthy and is therefore deferred to Section 4 below.

Proof of Theorem 4. Let us first show that, for $k,m \in \mathbf{Z}_+$,

$$d_{\mathrm{TV}}(S_M,\,R_{k,m}) \leq p^{k+1}\sum_{n=0}^{\infty}|a_n|\sum_{j=m\wedge n}^{(m\vee n)-1}\left|\binom{n-j-1}{k}\right| f_{j,k,p}. \tag{33}$$

Indeed, using (25), we derive

$$\begin{aligned}
&d_{\mathrm{TV}}(S_M,\,R_{k,m})\\
&\leq\ p^{k+1}\left[\sum_{j=0}^{m-1}\left|\sum_{n=0}^{j}a_n\binom{n-j-1}{k}\right|f_{j,k,p}+\sum_{j=m}^{\infty}\left|\sum_{n=j+1}^{\infty}a_n\binom{n-j-1}{k}\right|f_{j,k,p}\right]\\
&\leq\ p^{k+1}\left[\sum_{n=0}^{m-1}|a_n|\sum_{j=n}^{m-1}\left|\binom{n-j-1}{k}\right|f_{j,k,p}+\sum_{n=m}^{\infty}|a_n|\sum_{j=m}^{n-1}\left|\binom{n-j-1}{k}\right|f_{j,k,p}\right]\\
&=\ p^{k+1}\sum_{n=0}^{\infty}|a_n|\sum_{j=m\wedge n}^{(m\vee n)-1}\left|\binom{n-j-1}{k}\right|f_{j,k,p}.
\end{aligned}$$

If we look at the definition of $f_{j,k,p}$ (see Lemma 3), we see that, in order to complete the proof we must treat the terms

$$\sum_{j=m\wedge n}^{(m\vee n)-1}\left|\binom{n-j-1}{k}\right| \quad\text{and}\quad \sum_{j=m\wedge n}^{(m\vee n)-1}\frac{|(n-j-k)_k|}{\sqrt{(j+1)_{k+1}}}$$

for $k, m, n \in \mathbf{Z}_+$. In Lemma 4, we gave a formula for the first term. The second term was estimated in (28), (30), and (32). The proof of the theorem can easily be completed with the help of these inequalities and (33). ∎

Proof of Theorem 5. For $k = 0$, Lemma 3 leads to

$$d_{\mathrm{TV}}(S_M, S_N) \le p \sum_{n=0}^{\infty} |P(M \le n) - P(N \le n)| \min\left\{\frac{1}{2\sqrt{(n+1)pq}}, 1\right\},$$

which shows (20) in the case $k = 0$ and (21). Let us now assume that $k \ge 1$. In this case we have

$$g_{j,k,m} = \begin{cases} \sum_{n=0}^{j} A_n \binom{n-j-1}{k-1}, & \text{if } 0 \le j < m, \\ \sum_{n=j+1}^{\infty} A_n \binom{n-j-1}{k-1}, & \text{if } 0 \le m \le j, \end{cases}$$

since

$$\begin{aligned} -\sum_{n=0}^{j} a_n \binom{n-j-1}{k} &= -\sum_{n=0}^{j} A_n \left(\binom{n-j-1}{k} - \binom{n-j}{k}\right) \\ &= \sum_{n=0}^{j} A_n \binom{n-j-1}{k-1}, \\ \sum_{n=j+1}^{\infty} a_n \binom{n-j-1}{k} &= -\sum_{n=j+1}^{\infty} \Big(\sum_{i=n+1}^{\infty} a_i\Big)\left(\binom{n-j-1}{k} - \binom{n-j}{k}\right) \\ &= \sum_{n=j+1}^{\infty} A_n \binom{n-j-1}{k-1}. \end{aligned}$$

Using the above representation of $g_{j,k,m}$ with Lemma 3, we obtain similarly as in the proof of (33)

$$d_{\mathrm{TV}}(S_M, R_{k,m}) \le p^{k+1} \sum_{n=0}^{\infty} |A_n| \sum_{j=m\wedge n}^{(m\vee n)-1} \left|\binom{n-j-1}{k-1}\right| f_{j,k,p}.$$

Using (27), (29) and (31), we arrive at (20) in the case $k \ge 1$ and at (22) and (23). ∎

Proof of Proposition 1. From Lemma 2 we obtain, under the present conditions, that

$$P^{S_M} - R_{k,m} = (-p)^{k+1} \sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^{k+1-j} Q^{*j}.$$

Since $Q = \epsilon_x$ with $x \neq 0$, we therefore get, for $k \in \mathbf{Z}_+$,

$$d_{\mathrm{TV}}(S_M,\, R_{k,m}) = \frac{p^{k+1}}{2} \sum_{j=0}^{k+1} \binom{k+1}{j} = 2^k\, p^{k+1},$$

which coincides with the right-hand sides of (16) and (20). Now it is easy to verify that, in the case $p = 1/2$, in (17), (21), and (22) equalities hold. ■

We omit the proof of Theorem 6, since it is easily done by using some of the assertions above.

Proof of Theorem 7. Assertion (a) follows from (25). In particular, assuming that $M \leq_{\mathrm{ST}} N$, the second part of (a) is shown as follows:

$$\begin{aligned} d_{\mathrm{TV}}(S_M,\, S_N) &\leq \sum_{j=1}^{\infty} (P(N \geq j) - P(M \geq j)) \frac{1}{2} \sqrt{\frac{p}{jq}} \\ &\leq \sqrt{\frac{p}{q}} \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} (P(N = i) - P(M = i))(\sqrt{j} - \sqrt{j-1}) \\ &= \sqrt{\frac{p}{q}} \sum_{i=1}^{\infty} \sqrt{i}(P(N = i) - P(M = i)) = \sqrt{\frac{p}{q}}\, \mathrm{E}(\sqrt{N} - \sqrt{M}). \end{aligned}$$

Here, we used that

$$\frac{1}{\sqrt{j}} \leq 2(\sqrt{j} - \sqrt{j-1}), \qquad (j \in \mathbf{N}), \tag{34}$$

which can be derived from (28). To prove (b), we assume that $M \leq_{\mathrm{SL}} N$. In particular, we have here $\mathrm{E}(N - M) \geq 0$. Since $R_{1,0} = P^{S_N} + p\,\mathrm{E}(M - N)(Q - \epsilon_0)$, it follows from the triangle inequality that

$$d_{\mathrm{TV}}(S_M,\, S_N) \leq d_{\mathrm{TV}}(S_M,\, R_{1,0}) + p\,\mathrm{E}(N - M).$$

It therefore suffices to derive some bounds for $d_{\mathrm{TV}}(S_M,\, R_{1,0})$. Since, for all $j \in \mathbf{Z}_+$,

$$g_{j,1,0} = \sum_{n=j+1}^{\infty} a_n (n - j - 1) = \mathrm{E}(M - j - 1)_+ - \mathrm{E}(N - j - 1)_+ \leq 0,$$

we obtain from Lemma 3 the inequality

$$\begin{aligned} d_{\mathrm{TV}}(S_M,\, R_{1,0}) &\leq -2\,p^2 \sum_{j=0}^{\infty} g_{j,1,0} = -2\,p^2 \sum_{j=0}^{\infty} \sum_{n=j+1}^{\infty} a_n (n - j - 1) \\ &= -2\,p^2 \sum_{n=1}^{\infty} a_n \sum_{j=0}^{n-1} (n - j - 1) \\ &= p^2\, \mathrm{E}\Big(N(N-1) - M(M-1)\Big). \end{aligned}$$

A further bound can be shown by means of the inequality

$$\frac{1}{x} \le \ln\left(\frac{(x-1)^{x-1}(x+1)^{x+1}}{x^{2x}}\right) =: h_1(x), \qquad (x \in [1,\infty)),$$

with the convention $0^0 = 1$ (see Mitrinović (1970, 3.9.48)). Due to the fact that $g_{j,1,0} \le 0$ for $j \in \mathbf{Z}_+$, we now have

$$\begin{aligned} d_{\mathrm{TV}}(S_M,\, R_{1,0}) &\le -\frac{p}{\sqrt{2}\,q}\sum_{j=0}^{\infty}\frac{g_{j,1,0}}{j+1} \le -\frac{p}{\sqrt{2}\,q}\sum_{j=0}^{\infty} g_{j,1,0}\, h_1(j+1) \\ &= -\frac{p}{\sqrt{2}\,q}\sum_{n=1}^{\infty} a_n \sum_{j=0}^{n-1}(n-j-1)\, h_1(j+1). \end{aligned}$$

Since

$$\begin{aligned} \sum_{j=0}^{n-1}(n-j-1)\, h_1(j+1) &= \sum_{j=0}^{n-1}\sum_{i=j+1}^{n-1} h_1(j+1) = \sum_{i=1}^{n-1}\sum_{j=0}^{i-1} h_1(j+1) \\ &= \ln\left[\prod_{i=1}^{n-1}\prod_{j=0}^{i-1}\frac{j^j(j+2)^{j+2}}{(j+1)^{2(j+1)}}\right] = n\ln n, \end{aligned}$$

we arrive at

$$d_{\mathrm{TV}}(S_M,\, R_{0,1}) \le \frac{p}{\sqrt{2}\,q}\mathrm{E}\Big(N\ln(N) - M\ln(M)\Big).$$

Combining the inequalities above, we see that (b) is valid. ■

4. Proof of Lemma 5

Proof of (28). Since c_j is increasing in $j \in \mathbf{Z}_+$, we obtain (28) in the following way:

$$\begin{aligned} \sum_{j=m\wedge n}^{(m\vee n)-1}\frac{1}{\sqrt{j+1}} &= \sum_{j=m\wedge n}^{(m\vee n)-1} c_{j+1}\,(\sqrt{j+1}-\sqrt{j}) \\ &\le c_{m\vee n}\,(\sqrt{m\vee n}-\sqrt{m\wedge n}) = c_{m\vee n}\,|\sqrt{m}-\sqrt{n}|. \end{aligned}$$

In the proofs of (29) and (30), the following Lemma is needed, the proof of which can be found in Mitrinović (1970, 3.6.19 and 2.27).

Lemma 6. *The function*

$$h_2(x) = (x+1)\ln\left(\frac{x+3/2}{x+1/2}\right), \qquad (x \in [0,\infty))$$

is decreasing in x and satisfies the inequalities

$$1 \le h_2(x) \le 1 + \frac{1}{12(x+1)^2 - 3}.$$

Proof of (29). By Lemma 6, we see that C_j is increasing in $j \in \mathbf{Z}_+$, since $C_0 = 0$ and

$$\frac{1}{C_j} = \sqrt{j(j+1)}\ln\left(\frac{j+1/2}{j-1/2}\right) = \sqrt{1+\frac{1}{j}}\, j\, \ln\left(\frac{j+1/2}{j-1/2}\right)$$

is decreasing in $j \in \mathbf{N}$. Hence, for $m, n \in \mathbf{Z}_+$,

$$\sum_{j=m\wedge n}^{(m\vee n)-1} \frac{1}{\sqrt{(j+1)(j+2)}} = \sum_{j=m\wedge n}^{(m\vee n)-1} C_{j+1}\ln\left(\frac{j+3/2}{j+1/2}\right) \le C_{m\vee n}\left|\ln\left(\frac{n+1/2}{m+1/2}\right)\right|,$$

which shows the validity of (29).

Proof of (30). For (30) it suffices to show that for all $j, n \in \mathbf{Z}_+$

$$\frac{|n-j-1|}{\sqrt{(j+1)(j+2)}} \le (-1)_{n-1<j}\left(n\ln\left(\frac{j+3/2}{j+1/2}\right) - 1\right), \tag{35}$$

since this leads to

$$\begin{aligned}\sum_{j=m\wedge n}^{(m\vee n)-1} \frac{|n-j-1|}{\sqrt{(j+1)(j+2)}} &\le \sum_{j=m\wedge n}^{(m\vee n)-1} (-1)_{n<m}\left(n\ln\left(\frac{j+3/2}{j+1/2}\right) - 1\right)\\ &= m-n+n\ln\left(\frac{n+1/2}{m+1/2}\right).\end{aligned}$$

Inequality (35) can be verified as follows: For $n = 0$ and arbitrary $j \in \mathbf{Z}_+$ the assertion is trivial. Let us assume that $n \ge 1$. If $j \le n-1$, we have

$$\frac{|n-j-1|}{\sqrt{(j+1)(j+2)}} \le \frac{n}{j+1} - 1 \le n\ln\left(\frac{j+3/2}{j+1/2}\right) - 1,$$

where we made use of Lemma 6. For $j \ge n$, we obtain

$$\begin{aligned}\frac{|n-j-1|}{\sqrt{(j+1)(j+2)}} &= 1 - \frac{n}{\sqrt{(j+1)(j+2)}}\left(1 + \frac{\sqrt{j+1}}{n}(\sqrt{j+2}-\sqrt{j+1})\right)\\ &\le 1 - \frac{n}{j+1}\left(1 + \frac{1}{j\sqrt{j+2}}(\sqrt{j+2}-\sqrt{j+1})\right),\end{aligned}$$

since $1/n \ge 1/j$. Application of (34) yields

$$\begin{aligned}\frac{|n-j-1|}{\sqrt{(j+1)(j+2)}} &\le 1 - \frac{n}{j+1}\left(1 + \frac{1}{2j(j+2)}\right)\\ &\le 1 - \frac{n}{j+1}\left(1 + \frac{1}{12(j+1)^2-3}\right)\\ &\le 1 - n\ln\left(\frac{j+3/2}{j+1/2}\right),\end{aligned}$$

where we used Lemma 6 again. This completes the proof of (35) and hence that of (30).

For the proof of (31) and (32), we need the following lemma, the proof of which is easy and therefore omitted.

Lemma 7. *For $j, m, n \in \mathbf{Z}_+$ and $1 \le r_1 \le r_2$, we have*

$$\max_{m\wedge n\le j\le (m\vee n)-1} \frac{|n-j-r_1|}{\sqrt{j+r_2}} \le \frac{r_1\,|m-n|}{\sqrt{m+r_2-1}}, \tag{36}$$

$$\max_{m\wedge n\le j\le (m\vee n)-1} |n-j-r_1| \le r_1|m-n|. \tag{37}$$

Proof of (31). Let us first observe that it suffices to prove (31) only for $k=2$. This can easily be verified by using (36): If the assertion is valid for $k=2$, then, for $k \ge 2$,

$$\begin{aligned}\sum_{j=m\wedge n}^{(m\vee n)-1} \frac{|(n-j-k+1)_{k-1}|}{\sqrt{(j+1)_{k+1}}} &= \sum_{j=m\wedge n}^{(m\vee n)-1} \frac{|n-j-1|}{\sqrt{(j+1)(j+2)(j+3)}} \prod_{i=2}^{k-1} \frac{|n-j-i|}{\sqrt{j+i+2}} \\ &\le \frac{4|m-n|^2}{\sqrt{m+1}(\sqrt{m+1}+\sqrt{n+1})^2} \prod_{i=2}^{k-1} \frac{i|m-n|}{\sqrt{m+i+1}} \\ &\le \frac{4\,(k-1)!\,|m-n|^k}{\sqrt{(m+1)_{k-1}}(\sqrt{m+1}+\sqrt{n+1})^2}.\end{aligned}$$

To prove (31) for $k=2$, it suffices to show that, for $j, n \in \mathbf{Z}_+$,

$$\begin{aligned}\eta(j,n) &:= \frac{|n-j-1|}{\sqrt{(j+1)(j+2)(j+3)}} \\ &\quad - 4(-1)_{n-1<j}\Big[(n+1)\Big(\frac{1}{\sqrt{j+1}} - \frac{1}{\sqrt{j+2}}\Big) + \sqrt{j+1} - \sqrt{j+2}\Big] \\ &\le 0,\end{aligned} \tag{38}$$

since this leads to

$$\begin{aligned}&\sum_{j=m\wedge n}^{(m\vee n)-1} \frac{|n-j-1|}{\sqrt{(j+1)(j+2)(j+3)}} \\ &\le \sum_{j=m\wedge n}^{(m\vee n)-1} 4(-1)_{n<m}\Big[(n+1)\Big(\frac{1}{\sqrt{j+1}} - \frac{1}{\sqrt{j+2}}\Big) + \sqrt{j+1} - \sqrt{j+2}\Big] \\ &= \frac{4(m-n)^2}{\sqrt{m+1}(\sqrt{m+1}+\sqrt{n+1})^2}.\end{aligned}$$

We now prove (38). For $j, n \in \mathbf{Z}_+$, we have

$$\begin{aligned}\eta(j,n) &= (-1)_{n-1<j}\Big[\frac{n(1-4\sqrt{j+3}(\sqrt{j+2}-\sqrt{j+1}))-j-1}{\sqrt{(j+1)(j+2)(j+3)}} \\ &\quad - 4\Big(\frac{1}{\sqrt{j+1}} - \frac{1}{\sqrt{j+2}} + \sqrt{j+1} - \sqrt{j+2}\Big)\Big].\end{aligned}$$

Since

$$1 - 4\sqrt{j+3}(\sqrt{j+2} - \sqrt{j+1}) < 0$$

for all $j \in \mathbf{Z}_+$, we see that

$$\eta(j,0) < \eta(j,1) < \cdots < \eta(j,j), \qquad \eta(j,j+1) > \eta(j,j+2) > \ldots .$$

Therefore it suffices to show that $\eta(j,n) \le 0$ holds for $n \in \{j, j+1\}$. If $n = j$, we obtain

$$\begin{aligned}\eta(j,n) &= \frac{1}{\sqrt{j+2}}\left[\frac{1}{\sqrt{(j+1)(j+3)}} - \frac{4}{(\sqrt{j+1}+\sqrt{j+2})^2}\right] \\ &\le \frac{1}{\sqrt{j+2}}\left[\frac{1}{j+3/2} - \frac{4}{2(2j+3)}\right] = 0.\end{aligned}$$

If $n = j+1$, we have

$$\eta(j,n) = \frac{-4(\sqrt{j+2} - \sqrt{j+1})^2}{\sqrt{j+1}} \le 0.$$

The proof of (31) is completed.

Proof of (32). For $k, m, n \in \mathbf{Z}_+$ with $k \ge 2$, we obtain by using (37) and (31)

$$\begin{aligned}\sum_{j=m\wedge n}^{(m\vee n)-1} \frac{|(n-j-k)_k|}{\sqrt{(j+1)_{k+1}}} &\le k|m-n| \sum_{j=m\wedge n}^{(m\vee n)-1} \frac{|(n-j-k+1)_{k-1}|}{\sqrt{(j+1)_{k+1}}} \\ &\le \frac{4\,k!\,|m-n|^{k+1}}{\sqrt{(m+1)_{k-1}}(\sqrt{m+1}+\sqrt{n+1})^2}.\end{aligned}$$

Therefore (32) is valid. The proof of Lemma 5 is completed. ∎

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