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Maximum Likelihood Estimators in a Statistical Model of Natural Catastrophe Claims with Trend

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Abstract. A statistical model to analyse stochastically increasing claims arising out of natural catastrophes is presented. Based on record values, the exponential trends over time can be identified. A more specific three-parameter model involving such a trend is also proposed. Observed claims are modeled as a stochastically increasing sequence of Fréchet distributed random variables. Consistency and asymptotic normality of the joint maximum likelihood estimator are shown. Possible applications in forecasting of claims are indicated. In particular claims data from U.S. hurricanes and Japanese taifuns are discussed.

Key words. catastrophe claims, Fréchet distribution, maximum likelihood estimator, Nevzorov's record model, trend

AMS 2000 subject classifications. Primary—62P05, 62F12, Secondary—62H12

1. Introduction

It is evident that insurance claims due to the occurrence of natural catastrophes have raised enormously over the past decades all over the world, in particular w.r.t. wind storm losses. We present a particular approach to the investigation of catastrophe claims in the presence of a trend, which is based on a combinations of parametric and semi-parametric methods. In the first step, the type of trend is analyzed using the number of record values in the times series of claims data, and in the second step, a maximum-likelihood estimator (MLE) is constructed from the data taking into account what type of trend has been detected before. In order to check the validity of the model assumptions, the estimates for the trend parameter obtained from both steps can be compared.

The proposed combination of semi-parametric and parametric models is due to Pfeifer (1997). In the present paper the asymptotic properties of the estimators are presented. They are important in forecasting of claims.

In some articles issued during last time there were some attempts to investigate this area. However no author succeeded in proving consistency in non-i.i.d case. Thus in Smith and Goodman (2000) insurance data claims obtained from a large company are analyzed to determine the distribution of tail values. The effect of possible trends in the observed data is considered. In McNeil and Saladin (2000) the peaks-over-threshold method is used to derive a natural model for the point process of large losses exceeding a high threshold. This model is used to obtain a joint description of the frequency and the severity with which large losses occur. In Coles (2001) the model diagnostics for a non-homogeneous in time model is considered. In Rootzén and Tajvidi (1997) the statistical extreme value theory is reviewed and some examples are given which show how to use it in large claims insurance.

A weaker version of the asymptotic results was announced in Kukush (1999). We mention that a goodness-of-fit test for both semi-parametric and three-parametric models was constructed in Kukush and Chernikov (2001), and in Kukush and Chernikov (2002) it is shown that in both models the MLE are asymptotically efficient in the sense of Hajék bound.

The present paper is organized as follows. In Section 2 Nevzorov's record model is introduced. In Section 3 the consistency and asymptotic normality results of the semi-parametric MLE are formulated. Section 4 contains the three-parametric model and the corresponding consistency and asymptotic normality results. In Section 5 the semi-parametric and the three-parametric approaches in data analysis are compared, Section 6 contains simulations. Implications for insurance applications are considered in Section 7, Section 8 concludes, and the proofs are given is Section 9.

2. Nevzorov's record model

A record model has been studied by Nevzorov (1988) and Borovkov and Pfeifer (1995). Assume that the yearly catastrophe claims considered here are realizations of an independent sequence $\{X_n, n \geq 1\}$ of random variables (r.v.) with support $\mathbf{R}^+ := [0, \infty)$ and continuous cumulative d.f. $\{F_n, n \geq 1\}$, s.t.

$$F_n = F^{\gamma_n}, \text{ with } \gamma_n := \gamma^{n-1}, \ \gamma \geq 1. \tag{2.1}$$

Here F is a fixed cumulative d.f. with F(0) = 0. Define record indicators by

$$I_1 := 1, \ I_n := \begin{cases} 1, \text{ if } X_n > \max\{X_1, \ldots, X_{n-1}\} \\ 0, \text{ otherwise} \end{cases} \quad \text{for } n \geq 2,$$

i.e., $I_n = 1$ iff observation X_n is a record value in the sequence. Under the above assumptions, the record indicators are independent r.v. with

$$P_n(\gamma) := P_\gamma(I_n = 1) = \frac{\gamma_n}{\gamma_1 + \ldots + \gamma_n} = \frac{1}{1 + \gamma^{-1} + \ldots + \gamma^{-n+1}}.$$

Consider also the number S_n of record values in a finite number of observations,

$$S_n := \sum_{i=1}^{n} I_i, \; n \geq 1.$$

The record times $T_1, \ldots T_{S_n}$ denote the observation times at which record values occur:

$$T_1 := 1, \; T_{k+1} := \min\{i \leq n | X_i > X_{T_k}\}, \; 1 \leq k < S_n.$$

The unknown parameter γ, see (2.1), is called trend parameter. If $\gamma = 1$, then we have the i.i.d. situation (**no trend**), while for $\gamma > 1$, the r.v. $\{X_n\}$ are stochastically increasing (**positive trend**). Given the observations $I_1, \ldots, I_n$, $n \geq 2$, of record indicators in a sequence of data, the log-likelihood function $L(\gamma)$ for $\gamma \geq 1$ is given by

$$\begin{aligned} L(\gamma) &= \ln\left(\prod_{i=2}^{n} p_i(\gamma)^{I_i}(1 - p_i(\gamma))^{1-I_i}\right) \\ &= \sum_{i=2}^{n} I_i \ln(p_i(\gamma)) + \sum_{i=2}^{n} (1 - I_i) \ln(1 - p_i(\gamma)). \end{aligned} \tag{2.2}$$

For $\gamma > 1$ it is possible to rewrite it in a way, which is more comfortable for numerical optimization:

$$L(\gamma) = S_n \ln(\gamma - 1) - \ln(\gamma^n - 1) - \sum_{k=2}^{S_n} \ln\left(1 - \gamma^{1-T_k}\right). \tag{2.3}$$

The semi-parametric MLE $\hat{\gamma} = \hat{\gamma}_n$ is defined as a measurable function of $I_1, \ldots, I_n$, for which

$$\hat{\gamma} \in \arg\max_{\gamma \geq 1} L(\gamma). \tag{2.4}$$

If $I_1, \ldots, I_n \neq (1, 1, \ldots, 1)$, then maximum in (2.4) is attained. Otherwise the maximum in (2.4) is not attained, and in that case we set $\hat{\gamma} := +\infty$. It happens with probability tending to zero as $n \to +\infty$.

3. Asymptotic properties of semi-parametric MLE

Theorem 1: *The MLE $\hat{\gamma}$ is strongly consistent, namely $\hat{\gamma}_n \to \gamma$, as $\to \infty$, a.s.*

Theorem 2: *Let $\gamma > 1$. Then the MLE $\hat{\gamma}$ is asymptotically normal, namely the normalized estimator $\sqrt{n}(\hat{\gamma}_n - \gamma)$ converges in distribution to a normal law with mean 0 and variance $\sigma_\infty^2 = \gamma^2(\gamma - 1)$.*

In Borovkov and Pfeifer (1995) the following result was obtained for the efficiency in the semi-parametric model. We shall understand efficiency here in the sense of Hajék bound, see Ibragimov and Has'minskii (1981).

Introduce the class $W_{e,2}$ bell-shaped loss functions. These functions $w(u)$, $u \in \mathbf{R}$, satisfy the following conditions:

a) $w(u) \geq 0$, $u \in \mathbf{R}$; $w(0) = 0$, w is continuous at $u = 0$ and is not identically 0.
b) w is even function.
c) w is non-decreasing for $u \geq 0$.
d) The growth of w as $u \to +\infty$ is slower than any one of the functions $\exp(\varepsilon u^2)$, $\varepsilon > 0$.

Denote by δ a standard Gaussian r.v.

Theorem 3: *Let $\gamma_0 > 1$, and the function w: $\mathbf{R} \to \mathbf{R}$ be bounded, Borel measurable and continuous a.e. with respect to Lebesgue measure. Then*

1. $$\lim_{\delta\to 0}\liminf_{n\to\infty}\sup_{\gamma:|\gamma-\gamma_0|<\delta} E_\gamma w\left(\sqrt{\frac{n}{\gamma_0 - 1}} \times \frac{\hat{\gamma}_n - \gamma}{\gamma_0}\right) = Ew(\xi). \tag{3.1}$$

2. *For any family γ_n^* of estimators of γ, based on the observations $I_1, \ldots, I_n$, and for any loss function $w \in W_{e,2}$, the inequality holds:*

$$\lim_{\delta\to 0}\liminf_{n\to\infty}\sup_{\gamma:|\gamma-\gamma_0|<\delta} E_\gamma w\left(\sqrt{\frac{n}{\gamma_0 - 1}} \times \frac{\hat{\gamma}_n - \gamma}{\gamma_0}\right) \geq Ew(\xi). \tag{3.2}$$

The inequality gives a lower bound for the loss of arbitrary normalized estimator. Theorem 3 shows that the MLE has asymptotically the smallest possible averaged loss. The proof of the theorem is given in Kukush and Chernikov (2002).

4. The three-parametric model

Since by economic arguments it is reasonable to assume that a possible trend in the data is of exponential type, we shall base the parametric model on a combination of Nevzorov's record model and the parametric class of Fréchet distributions (one of the extreme-value distribution classes). Thus we assume now that the cumulative d.f. F_n for the yearly claims are of the form

$$F_n(x) = \exp\left(-\gamma^{n-1}(Ax)^{-\alpha}\right),\ n = 1, 2, \ldots,\ x > 0.$$

Here $A > 0$, $\alpha > 0$ and $\gamma \geq 1$ are parameters of interest. In order to avoid economically meaningless parameter constellation we restrict our considerations only to a scale family with a scale parameter A rather than to a combined scale and location family.

For the above parametric family, the log-likelihood function $L(A, \alpha, \gamma)$ for the observed data set $X_1, \ldots, X_n$ is given by

$$L(A,\alpha,\gamma) = \tfrac{n(n-1)}{2}\ln\gamma - (\alpha+1)\sum_{i=1}^{n}\ln X_i - \sum_{i=1}^{n}\gamma^{i-1}(AX_i)^{-\alpha} + n\ln(\alpha A^{-\alpha}). \tag{4.1}$$

Choose a parameter set

$$\Theta = (0,+\infty)\times(0,+\infty)\times[1,+\infty)$$

and define the joint MLE of the parameters of interest as a measurable vector function $(\hat{A},\hat{\alpha},\hat{\gamma})$ of $X_1,\ldots,X_n$, for which

$$(\hat{A},\hat{\alpha},\hat{\gamma}) \in \arg\max_{(A,\alpha,\gamma)\in\Theta} L(A,\alpha,\gamma).$$

Further it will be shown that the maximum here is attained with probability tending to 1 as $n\to\infty$. Denote $\beta := (A, \alpha, \gamma)$.

Theorem 4: *The joint MLE is strongly consistent, moreover*

$$\hat{A}\to A,\ \hat{\alpha}\to\alpha,\ n(\hat{\gamma}-\gamma)\to 0,\ \text{as } n\to\infty, a.s.$$

Thus if the model of observations is valid, the MLE approximates the true values of parameters as the sample size grows. The trend parameter γ is better estimable than the other parameters.

Theorem 5: *If $\gamma > 1$, then the joint MLE is asymptotically normal, namely the normalized estimator*

$$\sqrt{n}\left(\mathbf{R}_n\mathbf{T}\mathbf{R}_n'\right)^{1/2}\begin{pmatrix}\hat{A}-A\\ \hat{\alpha}-\alpha\\ n(\ln\hat{\gamma}-\ln\gamma)\end{pmatrix}$$

converges in distribution to a normal law with mean $\mathbf{0}$ *and a unit covariance matrix, where*

$$\mathbf{R}_n := \begin{pmatrix}\frac{\alpha}{A} & 0 & 0\\ 0 & -\frac{1}{\alpha} & n\cdot\frac{\ln\gamma}{\alpha}\\ 0 & 0 & -1\end{pmatrix},$$

$$\mathbf{T} = \begin{pmatrix}1 & 1-\gamma_e & \frac{1}{2}\\ 1-\gamma_e & \frac{1}{6}\pi^2+\gamma_e^2-2\gamma_e+1 & \frac{1}{2}(1-\gamma_e)\\ \frac{1}{2} & \frac{1}{2}(1-\gamma_e) & \frac{1}{3}\end{pmatrix}, \tag{4.2}$$

$\mathbf{R}_n'$ *is* $\mathbf{R}_n$ *transposed, and* γ_e *is Euler's constant,* $\gamma_e \doteq 0.5772$.

This result can be applied to forecast claims. Introduce the transformed observations

$$Z_i := (AX_i)^{\alpha}\left(\gamma^{-\frac{1}{\alpha}}\right)^{i-1},\ i = 1, 2, \ldots$$

It is an i.i.d. sequence with standard Fréchet distribution $F(x) = \exp(-x^{-1})$, $x > 0$. The observations are represented as

$$X_i = A^{-1}\left(\gamma^{\frac{1}{\alpha}}\right)^{i-1} Z_i^{\frac{1}{\alpha}}.$$

We interpret the trend as a trend in the median of X_i. The forecast of claims for the year $k > n$ will be

$$\hat{X}_k = \hat{A}^{-1}\left(\hat{\gamma}^{\frac{1}{\alpha}}\right)^{k-1} \cdot \operatorname{med}\left(Z_1^{\frac{1}{t}}\right)\Big|_{t=\hat{\alpha}} \tag{4.3}$$

Theorem 5 makes it possible to construct a confidence interval for the forecast via the confidence region for the true value of $\beta = (A, \alpha, \gamma)$.

In Kukush and Chernikov (2002) the theorem analogous to the theorem 3 for the three-parameter model is proved.

5. Comparison of semi-parametric and parametric approaches in data analysis

Two sets of data were analyzed in Pfeifer (1997) by above mentioned methods:

a) yearly claims in Million U.S. \$ from U.S. hurricane events from 1949 to 1992 (source: Catastrophe Reinsurance Newsletter (1993)),
b) yearly claims in 1000 JYen from Japanese taifun events from 1977 to 1991 (source: personal communication, the data set is presented in Pfeifer (1997)).

Since in general the functions (2.3) and (4.1) cannot be maximized by elementary calculations, a particular stochastic search procedure was performed for the explicit data analysis. The graphical data displayed in logarithmic scale show that the assumption of an exponential trend in the data was reasonable, see Figures 1 and 2. The Tables 1 and 2, which we take from Pfeifer (1997), present the estimated trend parameters $\hat{\gamma}$ from the three approaches:

a) semi-parametric (s.-par.)—via record values,
b) joint maximum likelihood (jML),
c) least squares (l.-sq.)—from the graphical analysis; here $\hat{\gamma} := \exp(\hat{\alpha}\hat{m})$ where $\hat{m}$ is the estimated slope for the regression line in logarithmic scale and $\hat{\alpha}$ is the estimator of α from the joint MLE.

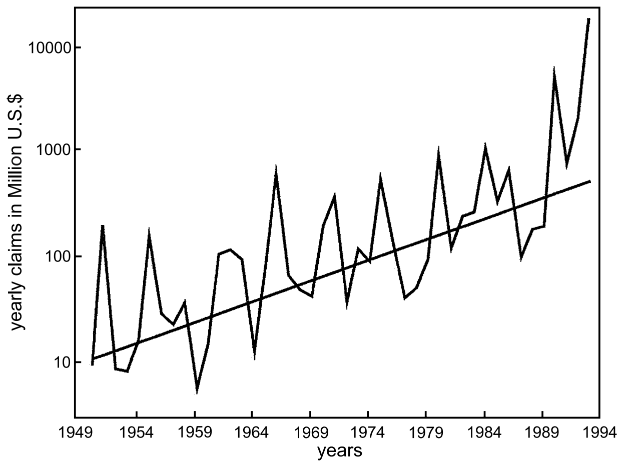

Figure 1. Prediction line for the U.S. data in logarithmic scale.

While for the U.S. data, all the approaches give nearly the same estimator for γ, the situation is not so clear for the Japan data. For the U.S., we have $\hat{\alpha} \approx 1.06$. For Japan, we have $\hat{\alpha} \approx 0.91$. The prediction line (4.3) in logarithmic scale is shown in the Figures 1 and 2. For Japan, the prediction looks unreasonable because of poor fitting of the three-parametric model.

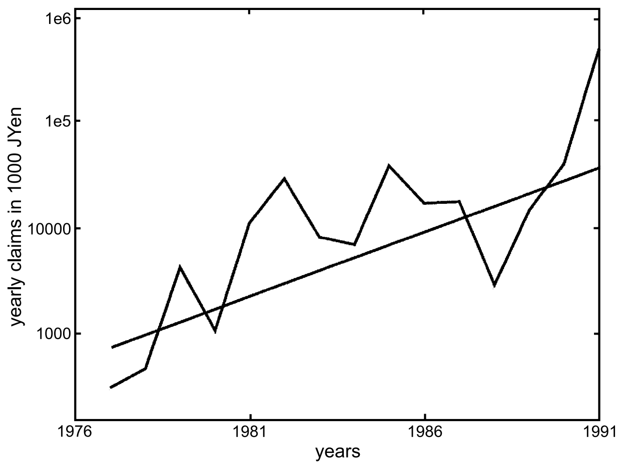

Figure 2. Prediction line for the Japan data in logarithmic scale.

Table 1. U.S data.

Method	s.-par	jML	l.-sq.
$\hat{\gamma}$	1.15	1.10	1.11

6. Asymptotic confidence regions and simulations

In this section we check the efficacy of the proposed model.

We estimate parameters of the distribution and construct the asymptotic confidence region basing on data of yearly claims from U.S. hurricane events (1949–1992) and Japan taifun events (1977–1991) mentioned above. The confidence interval and region are constructed in a standard way based on Theorems 2 and 4, respectively. One can see that for U.S. data the confidence region is quite small, i.e., "real" value of the parameter varies in quite small interval (ellipsoid). For Japan data, the confidence region is larger.

6.1. U.S. hurricane events

Semi-parametric case: The estimated value is $\hat{\gamma} = 1.1499$. The confidence interval is $\gamma_0 \in (1.0184,\ 1.2814)$.

The three-parametric case: The estimated vector is $\left(\hat{A}, \hat{\alpha}, \hat{\gamma}\right) = (0.1204,\ 1.0675,\ 1.1023)$. The projections of the confidence ellipsoid are:

$$A_0 \in (0.0729,\ 0.1679);$$

$$\alpha_0 \in (0.8858,\ 1.2492);$$

$$\gamma_0 \in (1.0857,\ 1.1188).$$

6.2. Japan taifun events

Semi-parametric case: The estimated value is $\hat{\gamma} = 1.8099$. The confidence interval is $\gamma_0 \in (0.9856,\ 2.6341)$.

Table 2. Japan data.

Method	s.-par	jML	l.-sq.
$\hat{\gamma}$	1.81	1.30	1.34

Table 3. Percentage of data having hit into the 95 percent confidence region. $(A, \alpha, \gamma) = (0.1204, 1.0675, 1.1023)$.

n	Semi-parametric model	The 3-parametric model
20	92.9	84.3
44	88.5	89.8
100	88.9	92.2
500	93.9	93.4
1000	94.5	95.2

The three-parametric case: The estimated vector is $\left(\hat{A}, \hat{\alpha}, \hat{\gamma}\right) = (0.0016,\ 0.9095,\ 1.2981)$. The projections of the confidence ellipsoid are:

$$A_0 \in (0.0003,\ 0.0029);$$

$$\alpha_0 \in (0.6236,\ 1.1953);$$

$$\gamma_0 \in (1.2147,\ 1.3814).$$

The following simulation shows the efficacy from another point of view. We simulate 1000 series of n random Fréchet distributed values with parameters (A, α, γ) similar to one estimated basing on real data. We use four sets of parameters: (0.1204, 1.0675, 1.1023), (0.14, 1.02, 1.12), (0.0016, 0.9095, 1.2981), (0.0116, 0.9295, 1.3). For these parameters we compute the statistics corresponding the Theorems 2 and 5, and check how many simulated data fall into the 95% confidence region. The Tables 3–6 show the percentage of the data having hit into the confidence region for both models.

The simulations above show that the proposed methods can be applied even for small data samples, that insurance companies deal with, though the empirical coverage probability seems often to be a bit less than 0.95 for 95% regions.

7. Implications for insurance applications

An important approach to the mathematical analysis of losses caused by climatic events is the modelling of the corresponding physical forces and their impact on the insurance

Table 4. Percentage of data having hit into the 95 percent confidence region. $(A, \alpha, \gamma) = (0.14, 1.02, 1.12)$.

n	Semi-parametric model	The 3-parametric model
20	95	85.4
44	97.7	90
100	97	94.4
500	94.4	93.8
1000	95.3	94.8

Table 5. Percentage of data having hit into the 95 percent confidence region. $(A, \alpha, \gamma) = (0.0016, 0.9095, 1.2981)$.

n	Semi-parametric model	The 3-parametric model
20	86	82.6
44	91.7	85.2
100	92.6	91.4
500	95.8	93.7
1000	95.3	95.1

industry. One of the first companies to develop such models was Applied Insurance Research (AIR) who have in particular concentrated on claims caused by hurricanes in the south-east of the U.S. For a survey, see e.g., Clark (1997). Besides a detailed study of relevant physical parameters such as air pressure, wind speed and direction, geographical locations of storm centers etc. the model also relies on a large data base with informations on the location, type and content of insured buildings. With the aid of high-speed computers the model simulates storm events on the basis of weather records dating back until the early 1900's; a typical study comprises about 1000 simulations which are considered to be representative for future occurrences of such events. By means of suitable mathematical functions the simulated meteorological and physical parameters are then linked to the possible damages at or in the buildings under consideration. This results in the generation of loss potentials which are considered to be representative for today's and future claim scenarios, and allow for an empirical estimate for some PML (Probable Maximum Loss), which mathematically corresponds to a (in general high) quantile of the overall loss distribution. For practical purposes this quantile is usually expressed in terms of the so-called return period T, which denotes the time interval within which on average one exceedance of the PML is expected; i.e. we have $T = 1/(1 - q)$ where q denotes the exceedance probability of the PML. Clark (1997) provides the table (see Table 7) for the overall loss potential due to hurricanes (insured claims, basis 1993) per one yearly hurricane event.

From a statistical point of view, however, the empirical PML's particularly for large return periods (above 200 years) are critical, since they rely only on 5 simulated

Table 6. Percentage of data having hit into the 95 percent confidence region. $(A, \alpha, \gamma) = (0.03, 0.9, 1.35)$.

n	Semi-parametric model	The 3-parametric model
20	85.3	80.6
44	90.5	89.2
100	93.6	93
500	94.2	91.5
1000	94.7	94.2

Table 7. Overall loss potential due to hurricanes per one yearly hurricane event.

Return period T (years)	q	PML (in Mio. U.S. $)
10	0.90	7800
20	0.95	13200
50	0.98	23600
100	0.99	30700
200	0.995	34500
500	0.998	50900
1000	0.999	51500

(observed) values. Also, the knowledge of only a few such PML estimations does not provide sufficient information about the underlying loss distribution as a whole, which however would be possible if all of the simulated values were taken into account. In contrast to the meteorological and geophysical models the statistical approach to the problem of forecasting potential future losses and PML's is to analyze past or historic data. There is some criticism by the physical modellers and in part also by the insurance industry in particular w.r.t. PML estimates for return periods of 200 years and above since no or only sparse loss observations are available here. In principle, however, this objection also applies to the physical models since they base on comparable historic storm events which are likewise extrapolated into the future. Interestingly, due to the historical hurricane loss data set used above, ending with the 15 billion U.S.$ record loss caused by hurricane Andrew, it is in some sense possible to compare both approaches. Since the data are strongly affected by an exponential trend with a rate of about 10% yearly average increase (as seen by the analysis above) the data have to be detrended and adjusted to the year 1993 before they can be compared to the AIR study. In this context it is worth while to think a moment about possible candidates for fitting distributional models. Since most commercial statistical software packages offer a great variety of alternatives here, one should take some theoretical results into account which have been derived for large claims, e.g., in the framework of statistics of extremes (see e.g., Reiss and Thomas (2001)). Here, the Fréchet distribution has turned out to be extremely efficient, in particular when fitting losses from windstorm events; cf. Pfeifer (2001) and Rootzén and Tajvidi (1997). However, it makes sense to include also other distributional classes into the analysis, in particular those which exhibit a tail behaviour similar to that of the Fréchet or other extreme value distributions. Such classes include for instance the

Table 8. The estimated parameters for the detrended hurricane data set.

Model	Fréchet distribution	Pearson type V	Loglogistic	Lognormal
Scale parameter	506,8325	566,37823	802,31944	6,77273
Shape parameter	1,05681	1,09325	1,50267	1,17497

Table 9. Approximation results.

Return period T	q	Distribution class		PML (Mio. U.S. $)		
		Fréchet	Pearson type V	Loglogistic	Lognormal	AIR
10	0.90	4262	4201	3462	3938	7800
20	0.95	8422	8167	5692	6035	13200
50	0.98	20340	19244	10694	9757	23600
100	0.99	39381	36520	17076	13441	30700
200	0.995	76063	69088	27176	18020	34500
500	0.998	181276	160091	50105	25706	50900
1000	0.999	349459	302041	79525	32980	51500

Pearson type V (inverse Gamma) or the loglogistic distribution, which are available in some professional fitting packages (see Law and Kelton (1991)). The Table 8 contains the estimated parameters (scale and shape parameter) for some of these distribution classes, for the detrended hurricane data set. The ordering of the models is according to the goodness-of-fit, i.e., the Fréchet distribution provides the best result here.

Once an appropriate model fitting has been done it is possible to obtain corresponding PML estimates from that. In the case of a Fréchet distribution model, it is even possible to express the PML in terms of the return period T explicitly:

$$\mathrm{PML}(T) = \sigma\{-ln(1 - 1/T)\}^{-1/\alpha} \approx \sigma T^{1/\alpha},$$

where σ denotes the scale parameter, and α denotes the shape parameter. The above approximation is sufficiently precise for return periods above 20 years already. The obtained results for the four distribution classes are put into Table 9.

Seemingly there is a qualitatively good coincidence between the PML estimates of AIR and those from the Fréchet or Pearson type V model in the range of up to 100 years for the return period T. For larger values of T, however, there are substantial differences in the estimates; one possible aspect is here that the PML estimates of AIR are for a single storm event per year only while the statistical analysis considers the aggregate claims over the whole year. This might explain for a PML estimate which is roughly twice as high in the Fréchet and Pearson type V model compared with the AIR value for a return period of 200 years since the average frequency of hurricanes is definitely more than one per year.

8. Conclusions

The Nevzorov's record model was considered, which includes the existence of trend. Two approaches were used. In the semi-parametric case the cumulant distribution function was not specified, while in the three-parametric case the cumulant distribution function was chosen to be the Frechét distribution, and the trend, shape and scale parameters were estimated simultaneously. Theorems about the consistency of the MLE's are the central results of the paper. They do not follow directly from well-known general properties of MLE, because the considered model is non-regular and contains non-identically distributed observations. The asymptotic normality results are proven as well, and the results about the asymptotic efficiency of the estimators are stated.

It would be interesting to expand the results in parametric setting to other cumulant distribution functions. The model based on Fréchet distribution fits the losses from the U.S. hurricanes, but it is not the case for Japanese taifuns. It would be interesting to adapt the Nevzorov's model for Japanese events.

9. Proofs

The following simple lemma gives the way to prove the asymptotic normality. The proof of the lemma is standard and uses Taylor expansion, compare Cramér (1999). One can find it, for example, in Kukush and Chernikov (2001).

Lemma 1: *Let* $\Theta \subset \mathbf{R}^{\mathbf{d}}$, $\boldsymbol{\theta}_0$ *be an interior point of* Θ, $\{Q_n(\boldsymbol{\theta}), \boldsymbol{\theta} \in \Theta, n \geq 1\}$ *be a sequence of random fields, which are twice differentiable in the neighborhood of* $\boldsymbol{\theta}_0$. *Let* $\boldsymbol{\theta}_n$ *be a random vector defined by*

$$\boldsymbol{\theta}_n = \arg\max_{\boldsymbol{\theta}\in\Theta} Q_n(\boldsymbol{\theta}),$$

and suppose that $\boldsymbol{\theta}_n \to \boldsymbol{\theta}_0$ *in probability. Assume also that:*

a) $\sqrt{n}Q_n'(\boldsymbol{\theta}_0)$ *converges in law to a random vector* γ,
b) $Q_n''(\boldsymbol{\theta}_0) \to \mathbf{S}$ *in probability, where* $\mathbf{S}$ *is nonsingular matrix,*
c) *For each* $\delta > 0$

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty} \mathrm{P}\left(\sup_{\|\boldsymbol{\theta}-\boldsymbol{\theta}_0\|\leq\epsilon} \|Q_n''(\boldsymbol{\theta}) - Q_n''(\boldsymbol{\theta}_0)\| > \delta\right) = 0.$$

Then $\delta_n := \sqrt{n}(\boldsymbol{\theta}_n - \boldsymbol{\theta}_0) \to -\mathbf{S}^{-1}\gamma$ *in law.*

9.1. Proof of Theorem 1

(i) *Limit functional.* Introduce the normalized log-likelihood function Q,

$$Q = Q_n(\gamma) := \frac{1}{n}L(\gamma) = Q_{(n,1)} + Q_{(n,2)}, \tag{9.1.1}$$

with

$$Q_{(n,1)} := \frac{1}{n}\sum_{i=2}^{n}\left(I_i - p_i^0\right)\ln\frac{p_i}{1-p_i} \tag{9.1.2}$$

and

$$Q_{(n,2)} := \frac{1}{n}\sum_{i=2}^{n}\left[p_i^0 \ln\frac{p_i}{1-p_i} + \ln(1-p_i)\right]. \tag{9.1.3}$$

Then functional $Q_{(n,\ 1)}(\gamma) \to 0$, $n \to \infty$ a.s., for each $\gamma \geq 1$. Indeed, by the Rosenthal inequality for independent random variables (see Rosenthal (1970)) we have

$$\begin{aligned} E_{\gamma_0} Q^4_{(n,1)} &\leq \frac{\text{const}}{n^4}\left(\sum_{i=2}^{n} \ln^2\frac{p_i}{1-p_i}\right)^2 \max_{2\leq i\leq n} E_{\gamma_0}\left(I_i - p_i^0\right)^4 \\ &\leq \frac{\text{const}}{n^4}\left(\sum_{i=2}^{n} \ln^2\frac{p_i}{1-p_i}\right)^2. \end{aligned} \tag{9.1.4}$$

If $\gamma > 1$ then the sequence $\ln^2\frac{p_i}{1-p_i}$, $i \geq 2$ is bounded, therefore

$$E_{\gamma_0} Q^4_{(n,1)} \leq \frac{\text{const}}{n^2}. \tag{9.1.5}$$

If $\gamma = 1$ then $\ln^2\frac{p_i}{1-p_i} = \ln^2(i-1)$, and

$$E_{\gamma_0} Q^4_{(n,1)} \leq \frac{\text{const}\ \ln^4 n}{n^2}. \tag{9.1.6}$$

By the Chebyshev inequality we have $P_{\gamma_0}\left\{\left|Q_{(n,1)}\right| > \delta\right\} \leq \frac{E_{\gamma_0} Q^4_{(n,1)}}{\delta^4}$; hence in both cases (9.1.5) or (9.1.6)

$$\sum_{n=2}^{\infty} E_{\gamma_0} Q^4_{(n,1)} < \infty,$$

and by the Borel–Cantelli lemma $Q_{(n,\ 1)} \to 0$, $n \to \infty$ a.s.

The deterministic part $Q_{(n,\ 2)}$ converges to the limit

$$Q_\infty(\gamma,\gamma_0) = \begin{cases} p_i^0 \ln\frac{p_i}{1-p_i} + \ln(1-p_i), & \text{if } \gamma > 1,\ \gamma_0 \geq 1 \\ 0, & \text{if } \gamma = \gamma_0 = 1 \text{ or} \\ \quad -\infty, & \text{if } \gamma = 1,\ \gamma_0 > 1. \end{cases}$$

$$Q_\infty(\gamma,\gamma_0) = \begin{cases} \left(1-\gamma_0^{-1}\right)\ln(\gamma-1) - \ln\gamma, & \text{if } \gamma > 1,\ \gamma_0 \geq 1 \\ 0, & \text{if } \gamma = \gamma_0 = 1 \\ -\infty, & \text{if } \gamma = 1,\ \gamma_0 > 1 \end{cases} \tag{9.1.7}$$

Therefore, for each $\gamma \geq 1$

$$Q_n(\gamma) \to Q_\infty(\gamma,\gamma_0),\ n \to \infty \text{ a.s.} \tag{9.1.8}$$

(ii) *Uniform convergence of Q_n.* Fix $\epsilon > 0$ and $C > 1 + \epsilon$. For fixed ω, the functional sequence

$$Q_{(n,1)}(\gamma) := \frac{1}{n}\sum_{i=2}^{n}\left(I_i - p_i^0\right)\ln\frac{p_i}{1-p_i},\quad \gamma\in[1+\epsilon, C];\ n\geq 1,$$

is equicontinuous, therefore

$$P_{\gamma_0}\left(\sup_{1+\epsilon\leq\gamma\leq C}\left|Q_{(n,1)}(\gamma)\right|\to 0,\ n\to\infty\right) = 1.$$

Then, $Q_{(n,2)}(\gamma)\to Q_\infty(\gamma,\gamma_0)$, $n\to\infty$ uniformly for $\gamma\in[1+\epsilon, C]$. Therefore

$$P_{\gamma_0}\left(\sup_{1+\epsilon\leq\gamma\leq C}\left|Q_n(\gamma) - Q_\infty(\gamma,\gamma_0)\right|\to 0,\ n\to\infty\right) = 1, \tag{9.1.9}$$

i.e., we obtain uniform convergence a.s. for γ belonging to a bounded interval, separated from 1.

(iii) *Maximum point of Q_∞.* Formula (9.1.7) implies directly the contrast inequality:

$$Q_\infty(\gamma,\gamma_0) < Q_\infty(\gamma_0,\gamma_0),\ \text{for}\ \gamma\neq\gamma_0. \tag{9.1.10}$$

(iv) *Behavior of Q_n for large and small γ.* Let $\gamma\geq C > 1$. From (2.2) we obtain

$$Q_n(\gamma)\leq\frac{1}{n}\sum_{i=2}^{n}(1-I_i)\ln(1-p_i(C))\to -\gamma_0^{-1}\ln C,\ n\to\infty\ \text{a.s.}$$

Therefore a.s.

$$\lim_{C\to+\infty}\limsup_{n\to\infty}\sup_{\gamma\geq C}Q_n(\gamma) = -\infty. \tag{9.1.11}$$

Now, let $\gamma_0 > 1$ and $\gamma\leq 1+\epsilon$, with fixed $\epsilon > 0$. Again use (2.2):

$$Q_n(\gamma)\leq\frac{1}{n}\sum_{i=2}^{n}I_i\ln p_i(1+\epsilon)\to\left(1-\gamma_0^{-1}\right)\ln\left(1-(1+\epsilon)^{-1}\right),\ n\to\infty\ \text{a.s.}$$

Therefore, if $\gamma_0 > 1$ then

$$\lim_{\epsilon\to 0}\limsup_{n\to\infty}\sup_{\gamma\leq 1+\epsilon}Q_n(\gamma) = -\infty. \tag{9.1.12}$$

(v) *Strong consistency for $\gamma_0 > 1$.* Choose $n_0(\omega)$, s.t. $\hat{\gamma} < \infty$ for $n\geq n_0(\omega)$. Then for $n\geq n_0(\omega)$ we have

$$Q_n(\hat{\gamma}_n)\geq Q_n(\gamma_0) = Q_\infty(\gamma_0,\gamma_0) + o(1),\ n\to\infty.$$

From (9.1.11), (9.1.12) we get that for some $\epsilon > 0$, $C > 0$, $n_1(\omega)$

$$\hat{\gamma}_n \in [1+\epsilon, C], \ \text{for } n \geq n_1(\omega). \tag{9.1.13}$$

If $|\hat{\gamma}_n(\omega) - \gamma_0| \geq \delta > 0$ and $n \geq n_1(\omega)$, then

$$Q_n(\hat{\gamma}_n) \leq \sup_{|\gamma-\gamma_0| \geq \delta,\ \gamma \in [1+\epsilon, C]} Q_n(\gamma) = \sup_{|\gamma-\gamma_0| \geq \delta,\ \gamma \in [1+\epsilon, C]} Q_\infty(\gamma, \gamma_0) + o(1), n \to \infty.$$

Hence

$$Q_\infty(\gamma_0, \gamma_0) \leq \sup_{|\gamma-\gamma_0| \geq \delta,\ \gamma \in [1+\epsilon, C]} Q_\infty(\gamma, \gamma_0) + o(1).$$

But because of (9.1.10) this can hold only for a finite set of numbers n. Therefore there exists $n_2 = n_2(\omega)$, s.t. for all $n \geq n_2(\omega)$, $|\hat{\gamma}_n(\omega) - \gamma_0| < \delta$. Hence $\hat{\gamma}_n \to \gamma_0$ a.s.

(vi) *Consistency for* $\gamma_0 = 1$. Similarly in the case $\gamma_0 = 1$ we have

$$\hat{\gamma}_n \in [1, C], \ \text{for } n \geq n_1(\omega).$$

If $|\hat{\gamma}_n(\omega) - 1| \geq \delta$ and $n \geq n_1(\omega)$, then

$$Q_n(\hat{\gamma}_n) \leq \sup_{\gamma \in [1+\delta, C]} Q_n(\gamma) = \sup_{\gamma \in [1+\delta, C]} Q_\infty(\gamma, 1) + o(1), \ n \to \infty,$$

and

$$Q_\infty(1,1) \leq \sup_{\gamma \in [1+\delta, C]} Q_\infty(\gamma, 1) + o(1).$$

From (9.1.10) we obtain again that for $n \geq n_2(\omega)$, $|\hat{\gamma}_n(\omega) - 1| < \delta$.

9.2. *Proof of Theorem 2*

Here we have $\gamma_0 > 1$. We apply Lemma 1.

(i) *Convergence of the first derivative*. From (9.1.2) we have

$$\sqrt{n} Q'_{(n,1)}(\gamma_0) = \frac{1}{\sqrt{n}} \sum_{i=2}^{n} \left(I_i - p_i^0\right) \frac{p'_i(\gamma_0)}{p_i^0\left(1 - p_i^0\right)}.$$

But $\lim_{i \to \infty} p_i'(\gamma_0) = \gamma_0^{-2}$, therefore

$$\begin{aligned} &\operatorname{Var}\left(\sqrt{n} Q'_{(n,1)}(\gamma_0)\right) \\ &= \frac{1}{n} \sum_{i=2}^{n} \frac{\left(p_i'(\gamma_0)\right)^2}{p_i^0\left(1 - p_i^0\right)} \to \frac{\gamma_0^{-4}}{\gamma_0^{-1}\left(1 - \gamma_0^{-1}\right)} = \frac{1}{\gamma_0^2(\gamma_0 - 1)}, \ n \to \infty. \end{aligned} \tag{9.2.1}$$

By the CLT in Lyapunov form we get

$$\sqrt{n}Q'_{(n,1)}(\gamma_0) \to N\left(0, \frac{1}{\gamma_0^2(\gamma_0 - 1)}\right) \text{ in distribution.} \tag{9.2.2}$$

Now, the function

$$\phi(p_i) := p_i^0 \ln \frac{p_i}{(1-p_i)} + \ln(1-p_i)$$

has a minimum point $p_i = p_i^0$, therefore

$$\frac{d}{d\gamma}\phi(p_i)\Big|_{\gamma=\gamma_0} = \frac{d\phi(p_i^0)}{dp_i}\frac{dp_i(\gamma_0)}{d\gamma} = 0$$

and

$$Q'_{(n,2)}(\gamma_0) = 0. \tag{9.2.3}$$

Now, (9.2.2) and (9.2.3) imply

$$\sqrt{n}Q'_n(\gamma_0) \xrightarrow{d} N\left(0, \frac{1}{{\gamma_0}^2(\gamma_0 - 1)}\right). \tag{9.2.4}$$

(ii) *Convergence of the second derivative*. We have

$$Q''_{(n,1)}(\gamma_0) = \frac{1}{n}\sum_{i=2}^{n}\left(I_i - p_i^0\right)\left(\frac{d}{d\gamma}\frac{p'_i}{p_i(1-p_i)}\Big|_{\gamma=\gamma_0}\right) \tag{9.2.5}$$

The derivatives in (9.2.5) form a bounded sequence, and using the second moment one can easily show that

$$Q''_{(n,1)}(\gamma_0) \to 0 \text{ in probability } P_{\gamma_0}. \tag{9.2.6}$$

Now,

$$\frac{d^2\phi(p_i)}{d\gamma^2}\Big|_{\gamma=\gamma_0} = \phi''\left(p_i^0\right)\left(p'_i(\gamma_0)\right)^2 + \phi'\left(p_i^0\right)p''_i(\gamma_0) = \phi''\left(p_i^0\right)\left(p'_i(\gamma_0)\right)^2,$$

and $\phi''\left(p_i^0\right) = -\dfrac{1}{p_i^0\left(1-p_i^0\right)}$. Then

$$\lim_{i\to\infty}\frac{d^2\phi(p_i)}{d\gamma^2}\Big|_{\gamma=\gamma_0} = -\lim_{i\to\infty}\frac{\left(p'_i(\gamma_0)\right)^2}{p_i^0\left(1-p_i^0\right)} = -\frac{1}{{\gamma_0}^2(\gamma_0 - 1)}.$$

Therefore

$$\lim_{n\to\infty} Q''_{(n,2)}(\gamma_0) = -\frac{1}{{\gamma_0}^2(\gamma_0 - 1)}. \tag{9.2.7}$$

Finally, (9.2.6) and (9.2.7) imply the convergence

$$Q''_{(n)}(\gamma_0) \to -\frac{1}{{\gamma_0}^2(\gamma_0 - 1)} \text{ in probability } P_{\gamma_0}. \tag{9.2.8}$$

(iii) *Oscillations of* Q''_n. Fix $\epsilon > 0$, $C > 1 + \epsilon$. From (2.2) we get for $\gamma \in [1 + \epsilon, C]$

$$Q'''_n(\gamma) = \frac{1}{n}\sum_{i=2}^{n} I_i(\ln p_i)''' + \frac{1}{n}\sum_{i=2}^{n} (\ln(1 - p_i))'''(1 - I_i),$$

and there exists a constant M, s.t. for all $n \geq 1$, all $\gamma \in [1 + \epsilon, C]$

$$|Q'''_n(\gamma)| \leq M. \tag{9.2.9}$$

Now we are able to apply the above mentioned Lemma 1.

$$\begin{aligned}\sqrt{n}(\hat{\gamma} - \gamma_0) &\to \gamma_0^2(\gamma_0 - 1)N\left(0,\ \frac{1}{\gamma_0^2(\gamma_0 - 1)}\right) \\ &= N\big(0,\ \gamma_0^2(\gamma_0 - 1)\big) \text{ in distribution.}\end{aligned} \tag{9.2.10}$$

9.3. *Proof of Theorem 4*

(i) *Reparametrization*. We prove the statement of the theorem using the notation $\beta = \beta_0 = (A_0, \alpha_0, \gamma_0)$. Let $B_0 = A_0^{-\alpha_0}$, $B = A^{-\alpha}$, $\mu_0 = \ln\gamma_0 \geq 0$, $\mu = \ln\gamma$. Rewrite the function (4.1) using the i.i.d. sequence $z_i = (A_0 X_i)^{\alpha_0}\left(\gamma_0^{-1}\right)^{i-1}$, $i = 1, 2, \ldots$, and cancelling summands which do not depend upon B, α, μ. We get

$$\begin{aligned}L_1(B, \alpha, \mu) = {} & \frac{n(n-1)}{2}\left(\mu - \frac{\alpha\mu_0}{\alpha_0}\right) + n\left(\ln B - \frac{\alpha \ln B_0}{\alpha_0}\right) + n\ln\alpha \\ & - \frac{\alpha}{\alpha_0}\sum_{i=1}^{n} \ln z_i - BB_0^{-\frac{\alpha}{\alpha_0}}\sum_{i=1}^{n} e^{(i-1)\left(\mu - \mu_0\frac{\alpha}{\alpha_0}\right)} z_i^{-\frac{\alpha}{\alpha_0}}.\end{aligned}$$

Now, let $c = BB_0^{-\frac{\alpha}{\alpha_0}} = \left(\frac{A_0}{A}\right)^{\alpha}$, $\tau = \frac{\alpha}{\alpha_0}$, $\nu = \mu - \frac{\alpha}{\alpha_0}\mu_0$. Rewrite L_1 and cancel the summands which do not depend upon the new arguments:

$$L_2(c, \tau, \nu) = \frac{n(n-1)}{2}\nu + n\ln c + n\ln\tau - \tau\sum_{i=1}^{n} \ln z_i - c\sum_{i=1}^{n} e^{(i-1)\nu} z_i^{-\tau}.$$

Now, $E \ln z_1 = -\Gamma'(1) = \gamma_e$, see Kukush and Chernikov (2001), $Ez_1^{-\tau} = \Gamma(1+\tau)$. Then

$$\frac{1}{n}L_2(c,\tau,\nu) = \frac{n-1}{2}\nu + (\ln c + \ln\tau - \tau\,\gamma_e) - \frac{c}{n}\Gamma(1+\tau)\sum_{i=1}^{n} e^{(i-1)\nu} + R_1 + R_2,$$

where

$$R_1 = -\frac{\tau}{n}\sum_{i=1}^{n}(\ln z_i - E\ln z_i), \tag{9.3.1}$$

$$R_2 = -\frac{c}{n}\sum_{i=1}^{n} e^{(i-1)\nu}\left(z_i^{-\tau} - Ez_i^{-\tau}\right). \tag{9.3.2}$$

Finally, let $\phi = n\nu$. Rewrite L_2:

$$L_3(c,\tau,\phi) = \frac{1}{n}L_2\left(c,\tau,\frac{\phi}{n}\right) = \frac{n-1}{2n}\phi + (\ln c + \ln\tau - \tau\,\gamma_e)$$
$$- \frac{c}{n}\Gamma(1+\tau)\frac{e^{\phi}-1}{e^{\frac{\phi}{n}}-1} + R_1 + R_2. \tag{9.3.3}$$

For $\phi = 0$ we assume here and further formally that $\frac{1}{n}\frac{e^{\phi}-1}{e^{\frac{\phi}{n}}-1} = 1$. A new parameter set is

$$\Theta = \{(c,\tau,\phi) : 0 < c < \infty, 0 < \tau < \infty, \phi \in \mathbf{R}\}.$$

Denote $\left(\hat{c},\hat{\tau},\hat{\phi}\right) = \arg\max_{(c,\tau,\phi)\in\Theta} L_3(c,\tau,\phi)$. Obviously, if $\hat{\phi} \geq -n\frac{\hat{\alpha}}{\alpha_0}\ln\gamma_0$ then

$$\hat{c} = \left(\frac{A_0}{\hat{A}}\right)^{\hat{\alpha}},\ \hat{\tau} = \frac{\hat{\alpha}}{\alpha_0},\ \hat{\phi} = n\left(\ln\hat{\gamma} - \frac{\hat{\alpha}}{\alpha_0}\ln\gamma_0\right). \tag{9.3.4}$$

The true value β_0 corresponds to the values $c_0 = \tau_0 = 1$, $\phi_0 = 0$. We must prove, that a.s. in P_{β_0}

$$\left(\hat{c},\hat{\tau},\hat{\phi}\right) \to (c_0,\tau_0,\phi_0),\ \text{as } n \to \infty.$$

(ii) *Limit function*. Consider $(c, \tau, \phi) \in \Theta^R$, where Θ^R is a compact subset of Θ. Uniformly in Θ^R we have

$$L_3(c,\tau,\phi) = L_\infty(c,\tau,\phi) + R_1 + R_2 + o(1),$$

with the limit function

$$L_\infty(c,\tau,\phi) = \frac{\phi}{2} + \ln c + \ln\tau - \tau\gamma_e - c\Gamma(1+\tau)\frac{e^{\phi}-1}{\phi}.$$

Here for $\phi = 0$ we assume that $\frac{e^{\phi}-1}{\phi} = 1$. The function $L_3(c, \tau, \phi)$ converges to $L_\infty(c, \tau, \phi)$ uniformly a.s., when $(c, \tau, \phi) \in \Theta^R$. Looking at (9.3.3), it is enough to prove that with probability 1 R_1 and R_2 converge to 0 uniformly, $n \to \infty$, when $(c, \tau, \phi) \in \Theta^R$.

R_1 converges to 0 uniformly a.s., because $\frac{1}{n}\sum_{i=1}^{n}(\ln z_i - E \ln z_i) \to 0$ a.s. by SLLN.

To prove that R_2 converges to 0 uniformly a.s., it is enough to prove that

$$\frac{S_n(\phi,\tau)}{n} = \frac{1}{n}\sum_{i=1}^{n} e^{\frac{(i-1)\phi}{n}}\left(z_i^{-\tau} - Ez_i^{-\tau}\right)$$

converges to 0 uniformly a.s. One can use the 4-th moment and the Rosenthal inequality (Rosenthal, 1970), the Chebyshev inequality and the Borel–Cantelli lemma to obtain that $\frac{S_n(\phi,\tau)}{n} \to 0$, $n \to \infty$ a.s., for all ϕ, τ. Uniform convergence a.s. follows from the relations:

$$\frac{1}{n}\frac{\partial S_n(\phi,\tau)}{\partial\phi} = \frac{1}{n}\sum_{i=1}^{n}\frac{i-1}{n}\, e^{\frac{(i-1)\phi}{n}}\left(z_i^{-\tau} - Ez_i^{-\tau}\right), \text{ and}$$

$$\sup_{n\geq 1}\ \sup_{\phi,\tau\in\text{compact}}\left|\frac{1}{n}\frac{\partial S_n(\phi,\tau)}{\partial\phi}\right| \leq C(\omega);$$

the same for

$$\frac{1}{n}\frac{\partial S_n(\phi,\tau)}{\partial\tau} = \frac{1}{n}\sum_{i=1}^{n} e^{\frac{(i-1)\phi}{n}}\left(-\ln(z_i)z_i^{-\tau} + E\ln(z_i)z_i^{-\tau}\right).$$

(iii) L_∞ *attains its maximum at the unique point* $(c_0, \tau_0, \phi_0) = (1, 1, 0)$. Find the maximum point of L_∞. We have

$$\frac{\partial L_\infty}{\partial c} = \frac{1}{c} - \Gamma(1+\tau)\frac{e^{\phi}-1}{\phi}.$$

If the maximum of L_∞ exists then it is attained on the curve $\frac{\partial L_\infty}{\partial c} = 0$, or

$$c = \left(\Gamma(1+\tau)\frac{e^{\phi}-1}{\phi}\right)^{-1}.$$

Consider L_∞ on this curve,

$$\tilde{L}_\infty(\tau,\phi) = L_\infty(c(\tau,\phi),\ \tau,\phi) = -\ln\Gamma(1+\tau) - \ln\frac{e^{\phi}-1}{\phi} + \ln\tau - \tau\,\gamma_e + \frac{\phi}{2} - 1.$$

Prove that $g(\phi) := \frac{\phi}{2} - \ln\frac{e^{\phi}-1}{\phi}$ attains the maximum at the unique point $\phi = 0$: $g(0) = 0$. Indeed,

$$g(\phi) := \frac{\phi}{2} - \ln\frac{e^{\phi}-1}{\phi} \leq 0 \Leftrightarrow \frac{e^{\phi}-1}{\phi} \geq e^{\frac{\phi}{2}} \Leftrightarrow$$

$$\Leftrightarrow \left[\begin{array}{l} h = e^{\phi} - \phi e^{\frac{\phi}{2}} - 1 \geq 0,\ \phi \geq 0\ , \\ h = e^{\phi} - \phi e^{\frac{\phi}{2}} - 1 \leq 0,\ \phi < 0 \end{array}\right. \tag{9.3.5}$$

$g(0) = 0$. We have

$$h' = e^{\phi} - e^{\frac{\phi}{2}} - \frac{\phi}{2}e^{\frac{\phi}{2}} \geq 0, \text{ as } e^{\frac{\phi}{2}} \geq 1 + \frac{\phi}{2}, \text{ for all } \phi \in \mathbf{R}.$$

Equality holds here only for $\phi = 0$. Therefore, the only maximum point of g is zero.

We have to prove that the function $f(\tau) := \ln\tau - \ln\Gamma(1 + \tau) - \tau\gamma_e$, $\tau > 0$ has the unique maximum point $\tau = 1$. Consider the inequality

$$\ln\tau - \ln\Gamma(1+\tau) - \tau\gamma_e \leq -\gamma_e \Leftrightarrow \ln\Gamma(\tau) \geq (1-\tau)\gamma_e.$$

But the function $\ln\Gamma(\tau)$ is strictly convex, and $y = (\tau - 1)\Gamma'(1)$ is a tangent line to the graph $y = \ln\Gamma(\tau)$ at the point $\tau = 1$. Therefore, for $\tau \neq 1$ $\ln\Gamma(\tau) > (1 - \tau)\gamma_e$, and the only maximum point of f is $\tau = 1$. So, we proved that the only maximum point of L_∞ is (1, 1, 0).

(iv) $\hat{\boldsymbol{\theta}}$ *is stochastically bounded*. Consider the transformed function given in (9.3.3)

$$L_3(c, \tau, \phi) = \frac{\phi}{2}\left(1 - \frac{1}{n}\right) + (\ln c + \ln\tau - \tau\ \gamma_e) + R_1 - \frac{c}{n}\sum_1^n e^{(i-1)\frac{\phi}{n}} z_i^{-\tau},$$

$c > 0$, $\tau > 0$, $\phi \in \mathbf{R}$, and find the curve on which it attains its maximum upon c:

$$\frac{\partial L_3}{\partial c} = \frac{1}{c} - \frac{1}{n}\sum_1^n e^{(i-1)\frac{\phi}{n}} z_i^{-\tau} = 0,$$

and $c = \frac{n}{\sum_1^n e^{(i-1)\frac{\phi}{n}} z_i^{-\tau}}$. Consider first the case $\phi \geq 0$. On this curve the function equals

$$\tilde{L}_3(\tau, \phi) = L_3(c(\tau, \phi), \tau, \phi) = \frac{\phi}{2}\left(1 - \frac{1}{n}\right) - \ln\left(\frac{1}{n}\sum_1^n e^{(i-1)\frac{\phi}{n}} z_i^{-\tau}\right) + \ln\tau - \tau\ \gamma_e$$

$$+ R_1 - 1 < \frac{\phi}{2}\left(1 - \frac{1}{n}\right) - \ln\left(\frac{1}{n}e^{\frac{2\phi}{3}}\sum_{\left[\frac{2n}{3}\right]+2}^{n} z_i^{-\tau}\right) + \ln\tau - \tau\ \gamma_e + R_1,$$

$$\tilde{L}_3(\tau, \phi) < -\frac{\phi}{2n} - \frac{\phi}{6} - \ln\left(\frac{1}{n}\sum_{z_i < e^{\gamma_e - 2},\ \left[\frac{2n}{3}\right]+2 \leq i \leq n} z_i^{-\tau}\right) + \ln\tau - \tau\gamma_e + R_1$$

$$< \ln\tau - \tau\ \gamma_e + R_1 - \frac{\phi}{6} - \ln\left(\frac{1}{n}\sum_{z_i < e^{\gamma_e - 2},\ \left[\frac{2n}{3}\right]+2 \leq i \leq n} z_i^{-\tau}\right). \tag{9.3.6}$$

Remind that $\{z_i\}$ are i.i.d. realizations of a Fréchet distribution with

$$F(x) = \exp(-x^{-1}),\ x > 0,\ \mathrm{P}\left\{z_i < e^{\gamma_e - 2}\right\} = \exp(-\exp(-\gamma_e + 2)) := p. \tag{9.3.7}$$

Denote by $\nu_{n/3}$ the number of z_i such that $z_i < e^{\gamma_e - 2}$, $\left[\frac{2n}{3}\right] + 2 \le i \le n$.

Then by SLLN $\frac{\nu_{n/3}}{n/3} \to p$, as $n \to \infty$, a.s.

So, we obtain that for any $0 < \epsilon_1 < p$ there exists $n_0 = n_0(\epsilon_1, \omega)$ such that for any $n > n_0 |\frac{\nu_{n/3}}{n/3} - p| < \epsilon_1$, a.s., therefore $\nu_{n/3} > \frac{n}{3}(-\epsilon_1 + p)$ a.s. Also we have that a.s. for $n > n_0$

$$\begin{aligned}\tilde{L}_3(\tau,\phi) &< \ln\tau - \tau\,\gamma_e - \frac{\phi}{6} + R_1 - 2\tau + \tau\,\gamma_e - \ln\frac{1}{3}(-\epsilon_1 + p)\\ &= \ln\tau - 2\tau - \frac{\phi}{6} - C_1 + R_1.\\ R_1 &= -\frac{\tau}{n}\sum_{i=1}^{n}(\ln z_i - E\ln z_i)\\ &\le \tau\frac{1}{n}\left|\sum_{i-1}^{n}(\ln z_i - E\ln z_i)\right|.\end{aligned}$$

There exists a number $n_2 = n_2(\omega)$ such that $\frac{1}{n}\left|\sum_{i-1}^{n}(\ln z_i - E\ln z_i)\right| < 1$ a.s., as soon as $n > n_2(\omega)$.

Therefore for any $n > n_0 = \max(n_1, n_2)$

$$\tilde{L}_3(\tau,\phi) < \ln\tau - \tau - \frac{\phi}{6} - C_1$$

Also we have

$$\begin{aligned}&\sup_{n>n_0}\sup_{\tau>C'}\sup_{\phi\ge 0}\tilde{L}_3(\tau,\phi) \to -\infty \text{ a.s. when } C' \to +\infty,\\ &\sup_{n>n_0}\sup_{\tau<\delta}\sup_{\phi\ge 0}\tilde{L}_3(\tau,\phi) \to -\infty \text{ a.s. when } \delta \to 0,\\ &\sup_{n>n_0}\sup_{\tau>0}\sup_{\phi>C'}\tilde{L}_3(\tau,\phi) \to -\infty \text{ a.s. when } C' \to +\infty.\end{aligned} \tag{9.3.8}$$

Similarly consider the function $\tilde{L}_3(\tau,\phi)$ when $\phi < 0$:

$$\begin{aligned}\tilde{L}_3(\tau,\phi) = L_3(c(\tau,\phi),\tau,\phi) &= \frac{\phi}{2}\left(1 - \frac{1}{n}\right) - \ln\left(\frac{1}{n}\sum_{1}^{n} e^{(i-1)\frac{\phi}{n}} z_i^{-\tau}\right) + \ln\tau - \tau\,\gamma_e\\ &+ R_1 - 1 < \frac{\phi}{2}\left(1 - \frac{1}{n}\right) - \ln\left(\frac{1}{n}e^{\frac{\phi}{3}}\sum_{1}^{\left[\frac{n}{3}\right]+1} z_i^{-\tau}\right) + \ln\tau - \tau\,\gamma_e + R_1,\end{aligned}$$

$$\begin{aligned}\tilde{L}_3(\tau,\phi) &< -\frac{\phi}{2n} - \frac{\phi}{6} - \ln\left(\frac{1}{n}\sum_{z_i < e^{\gamma_e-2},\ \left[1\le i\le\frac{n}{3}\right]+1} z_i^{-\tau}\right) + \ln\tau - \tau\,\gamma_e + R_1\\ &< \ln\tau - \tau\,\gamma_e + R_1 - \frac{\phi}{6} - \frac{\phi}{2n} - \ln\left(\frac{1}{n}\sum_{z_i < e^{\gamma_e-2},\ 1\le i\le\left[\frac{n}{3}\right]+1} z_i^{-\tau}\right).\end{aligned} \tag{9.3.9}$$

Denote by $\nu'_{n/3}$ the number of z_i such that $z_i < e^{\gamma_e - 2}$, $1 \le i \le \left[\frac{n}{3}\right] + 1$.

Then by SLLN $\frac{\nu'_{n/3}}{n/3} \to p$, as $n \to \infty$, a.s., where p is given in (9.3.7).

For any $0 < \epsilon_1 < p$ there exists $n_0 = n_0(\epsilon_1, \omega)$ such that for any $n > n_0$ $\left|\frac{\nu'_{n/3}}{n/3} - p\right| < \epsilon_1$, a.s., therefore $\nu'_{n/3} > \frac{n}{3}(-\epsilon_1 + p)$ a.s. And we have that a.s. for $n > n_0$

$$\begin{aligned}\tilde{L}_3(\tau, \phi) &< \ln \tau - \tau\, \gamma_e + \frac{\phi}{6} - \frac{\phi}{2n} + R_1 - 2\tau + \tau\, \gamma_e - \ln \frac{2}{3}(-\epsilon_1 + p) \\ &= \ln \tau - 2\tau + \frac{\phi}{6} - \frac{\phi}{2n} - C_1 + R_1. \\ R_1 &= -\frac{\tau}{n} \sum_{i-1}^{n} (\ln z_i - E \ln z_i) \le \tau \frac{1}{n} \left| \sum_{i-1}^{n} (\ln z_i - E \ln z_i) \right|.\end{aligned}$$

For any $\epsilon_2 > 0$ there exists $n_2(\epsilon_2, \omega)$ such that for any $n > n_2$ $\frac{1}{n}\left|\sum_{i-1}^{n} (\ln z_i - E\ln z_i)\right| < \epsilon_2$ a.s. Let $\epsilon_2 = 1$. There exists a number $n_2 = n_2(\omega)$ such that $\frac{1}{n}\left|\sum_{i-1}^{n}(\ln z_i - E\ln z_i)\right| < 1$ a.s., as soon as $n > n_2(\omega)$. Therefore for any $n > n_0 = \max(n_1, n_2, 6)$

$$\tilde{L}_3(\tau, \phi) < \ln \tau - \tau + \frac{\phi}{12} - C_1.$$

Also we have

$$\begin{aligned}&\sup_{n>n_0} \sup_{\tau>C'} \sup_{\phi<0} \tilde{L}_3(\tau, \phi) \to -\infty \text{ a.s. when } C' \to +\infty, \\ &\sup_{n>n_0} \sup_{\tau>\delta} \sup_{\phi<0} \tilde{L}_3(\tau, \phi) \to -\infty \text{ a.s. when } \delta \to 0, \\ &\sup_{n>n_0} \sup_{\tau>0} \sup_{\phi<-C'} \tilde{L}_3(\tau, \phi) \to -\infty \text{ a.s. when } C' \to +\infty.\end{aligned} \tag{9.3.10}$$

Since $L_n(\hat{\boldsymbol{\theta}}_n) \ge L_n(\boldsymbol{\theta}_0) = L_\infty(\boldsymbol{\theta}_0, \boldsymbol{\theta}_0) + o(1)$, $n \to \infty$, from (9.3.8) and (9.3.10) we can conclude that for some random bounds δ, $C > 0$ and some number $n_0(\omega)$ for each $n \ge n_0(\omega)$

$$\hat{\boldsymbol{\theta}}_n \in [\delta, C] \times [\delta, C] \times [-C, C], \text{a.s.} \tag{9.3.11}$$

(v) *Convergence of* $\left(\hat{c}, \hat{\tau}, \hat{\phi}\right)$.

$$L_3(\hat{c}, \hat{\tau}, \hat{\phi}) \ge L_3(c_0, \tau_0, \phi_0) = L_\infty(c_0, \tau_0, \phi_0) + o(1).$$

From (9.3.11) we get that $\|\hat{\boldsymbol{\theta}}\| \le a$, with $a = a(\omega)$ for $n \ge n_0(\omega)$. Then for $n \ge n_0(\omega)$

$$L_3(\hat{\boldsymbol{\theta}}) \le \sup_{\|\boldsymbol{\theta}\| \le a} L_3(\boldsymbol{\theta}) = \sup_{\|\boldsymbol{\theta}\| \le a} L_\infty(\boldsymbol{\theta}) + o(1).$$

If $(\hat{c}-c_0)^2+(\hat{\tau}-\tau_0)^2+\left(\hat{\phi}-\phi_0\right)^2 \geq \delta^2$, then

$$\sup_{(c-c_0)^2+(\tau-\tau_0)^2+(\phi-\phi_0)^2\geq\delta^2, \|\boldsymbol{\theta}\|\leq a} L_\infty(\boldsymbol{\theta}) \geq L_\infty(\boldsymbol{\theta}_0)+o(1),$$

and for $n \geq n_1(\omega)$ this is impossible, because of (iii).

Hence $(\hat{c}-c_0)^2+(\hat{\tau}-\tau_0)^2+\left(\hat{\phi}-\phi_0\right)^2 \leq \delta^2$, $n \geq n_1(\omega)$ and therefore $\left(\hat{c},\hat{\tau},\hat{\phi}\right) \to \left(c_0,\tau_0,\phi_0\right)$ a.s., $n\to\infty$. Now from (9.3.4) the statement of Theorem 3 follows.

9.4. *Proof of Theorem 5*

We apply Lemma 1 to $Q_n = L_3$ given in (9.3.3), (9.3.1) and (9.3.2) with $\boldsymbol{\theta} = (c, \tau, \phi)$, $\Theta = (0, \infty) \times (0, \infty) \times \mathbf{R}$ and $\boldsymbol{\theta}_0 = (c_0, \tau_0, \phi_0) = (1, 1, 0)$. Rewrite

$$L_3(c,\tau,\phi) = L_\infty(c,\tau,\phi) + A_n(c,\tau,\phi) + R_1 + R_2 - \frac{\phi}{2n}, \tag{9.4.1}$$

where $A_n(c, \tau, 0) = 0$ and in the case $\phi \neq 0$

$$A_n(c,\tau,\phi) = c\Gamma(1+\tau)\left(e^\phi-1\right)\left(\frac{1}{\phi}-\frac{1}{n\left(e^{\frac{\phi}{n}}-1\right)}\right).$$

By Theorem 1 the maximum point $(\hat{c},\hat{\tau},\hat{\phi})$ of L_3 converges to $\boldsymbol{\theta}_0 = (c_0, \tau_0, \phi_0)$ a.s. We check the conditions of Lemma 1.

(i) *Convergence of* $\sqrt{n}Q_n'(\theta_0)$. We have $L_\infty'(\boldsymbol{\theta}_0) = 0$. Consider A_n with $t := \frac{\phi}{n}$,

$$A_n(c,\tau,\phi) = c\Gamma(1+\tau)\frac{(e^\phi-1)}{\phi}\left(1-\frac{t}{e^t-1}\right).$$

Let $h(t) = 1-\frac{t}{e^t-1}$ if $t \neq 0$, and $h(0) = 0$. Then

$$\frac{\partial A_n(\boldsymbol{\theta}_0)}{\partial\phi} = c_0\Gamma(1+\tau_0)\tfrac{\partial}{\partial\phi}h(t)|_{t=0} = \frac{1}{n}h'(0) = \frac{1}{2n}.$$

Therefore $\sqrt{n}\frac{\partial A_n(\theta_0)}{\partial\phi} \to 0$. The other partial derivatives of A_n equal zero at $\boldsymbol{\theta}_0$. Thus $\sqrt{n}Q_n'(\boldsymbol{\theta}_0) = o(1)+\sqrt{n}(R_1+R_2)'(\boldsymbol{\theta}_0)$,

$$\sqrt{n}Q_n'(\theta_0) = o(1) - \frac{1}{\sqrt{n}}\sum_{i=1}^n \begin{pmatrix} z_i^{-1}-Ez_i^{-1} \\ \left(1-z_i^{-1}\right)\ln z_i - E\left(1-z_i^{-1}\right)\ln z_i \\ \frac{i-1}{n}\left(z_i^{-1}-Ez_i^{-1}\right) \end{pmatrix}. \tag{9.4.2}$$

Now by the multivariate CLT in Lyapunov form

$$\sqrt{n}Q_n'(\boldsymbol{\theta}_0) \to N(0,\mathbf{T}) \tag{9.4.3}$$

in distribution, where **T** is 3×3 matrix. Check the Lyapunov condition for the third component (9.4.2). For the sum

$$S_n = \sum_{i=1}^{n} \xi_{ni} = \sum_{i=1}^{n} \frac{i-1}{n\sqrt{n}} \left(z_i^{-1} - E z_i^{-1} \right)$$

we have

$$\operatorname{Var} S_n = \frac{1}{n} \sum_{i=1}^{n} \frac{(i-1)^2}{n^2} \to \frac{1}{3}, \; n \to \infty,$$

and

$$\frac{1}{(\operatorname{Var} S_n)^{3/2}} \sum_{i=1}^{n} E|\xi_{ni}|^3 = \frac{\text{Const}}{n^4\sqrt{n}} \sum_{i=1}^{n} (i-1)^3 \to 0, \; n \to \infty.$$

To calculate the variance–covariance matrix **T**, introduce the i.i.d. random vectors

$$\boldsymbol{\zeta}_i := \begin{pmatrix} z_i^{-1} - E z_i^{-1} \\ \left(1 - z_i^{-1}\right) \ln z_i - E\left(1 - z_i^{-1}\right) \ln z_i \\ \dfrac{i-1}{n} \left(z_i^{-1} - E z_i^{-1} \right) \end{pmatrix}$$

and weighting matrices

$$\boldsymbol{\Lambda}_i := \operatorname{diag}\left(1, \; 1, \frac{i-1}{n}\right).$$

Then $\sqrt{n} Q_n'(\boldsymbol{\theta}_0) = o(1) - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \boldsymbol{\Lambda}_i \boldsymbol{\zeta}_i$,

$$\operatorname{Var} \boldsymbol{\zeta}_i = \Gamma = \begin{pmatrix} 1 & 1-\gamma_e & 1 \\ 1-\gamma_e & \dfrac{\pi^2}{6} + (1-\gamma_e)^2 & 1-\gamma_e \\ 1 & 1-\gamma_e & 1 \end{pmatrix},$$

$$\operatorname{Var}\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \boldsymbol{\Lambda}_i \boldsymbol{\zeta}_i \right) = \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{\Lambda}_i \boldsymbol{\Gamma} \boldsymbol{\Lambda}_i^T \to T = \begin{pmatrix} 1 & 1-\gamma_e & \dfrac{1}{2} \\ 1-\gamma_e & \dfrac{\pi^2}{6} + (1-\gamma_e)^2 & \dfrac{1-\gamma_e}{2} \\ \dfrac{1}{2} & \dfrac{1-\gamma_e}{2} & \dfrac{1}{3} \end{pmatrix}.$$

This **T** is the covariance matrix of the normal law in (9.4.3).

(ii) *Convergence of* $Q_n''(\boldsymbol{\theta}_0)$. Direct calculations show that $L_{\infty} = -T$. This matrix is non-singular because $\det \mathbf{T} = \frac{1}{12} + \frac{1}{12}\left(\Gamma''(2) - \left(\Gamma'(2)\right)^2\right) > 0$. Here $\Gamma'(2) = 1 - \gamma_e$, $\Gamma''(2) = \frac{\pi^2}{6} + \gamma_e^2 - 2\gamma_e$, see, e.g., Abramowitz and Stegun (1992). The second derivatives of the other summands in (9.4.1) tend to zero in probability, hence $Q_n''(\boldsymbol{\theta}_0) \to \mathbf{S} = -\mathbf{T}$ in probability.

(iii) *Increments of the second derivative*. For the limit function we have $L_\infty(c, \tau, \phi) \in C^2(\Theta)$, therefore condition c) of Lemma 1 holds if we substitute L_∞ for Q_n. For the other summands of (9.4.1) it is also simple to check this condition. Consider for instance the increments with respect to ϕ of

$$\frac{\partial^2 R_2}{\partial \phi^2} = -\frac{c}{n}\sum_{i=1}^{n} \frac{(i-1)^2}{n^2} e^{\frac{(i-1)\phi}{n}} \left(z_i^{-\tau} - Ez_i^{-\tau}\right).$$

We have to check that for each $\delta > 0$,

$$\lim_{\epsilon \to 0} \limsup_{n\to\infty} P\left(\sup_{\substack{\boldsymbol{\theta}\in\Theta,\\ \|\boldsymbol{\theta}-\boldsymbol{\theta}_0\|\le\epsilon}} \left| \frac{c}{n}\sum_{i=1}^{n} \frac{(i-1)^2}{n^2} \left(e^{\frac{(i-1)\phi}{n}} - 1\right) \left(z_i^{-\tau} - Ez_i^{-\tau}\right) \right| > \delta \right) = 0. \tag{9.4.4}$$

The following inequalities hold:

$$\left| e^{\frac{(i-1)\phi}{n}} - 1 \right| \le \frac{i-1}{n}\epsilon \cdot \text{const}, \text{ when } |\phi - \phi_0| = |\phi| \le \epsilon;$$

$$\left|z_i^{-\tau}\right| \le z_{i*} = \begin{cases} z_i^{-1/2}, \text{ if } \mathrm{z_i} > 1 \\ z_i^{-3/2}, \text{ if } 0 < \mathrm{z_i} \le 1 \end{cases}, \quad \text{when } |\tau - \tau_0| = |\tau - 1| \le \frac{1}{2}.$$

Hence the supremum in (9.4.4) is less or equal to $\epsilon \cdot O_p(1)$, and

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \mathrm{P}\left(\epsilon \cdot O_p(1) > \delta\right) = 0,$$

which induces (9.4.4). Condition c) of Lemma 1 holds.

(iv) *Change of variables*. By Lemma 1

$$\sqrt{n}\begin{pmatrix} \hat{c} - 1 \\ \hat{\tau} - 1 \\ \hat{\phi} \end{pmatrix} \to \mathbf{T}^{-1}\gamma = \zeta_\infty$$

in law, with $\gamma \sim N(0, \mathbf{T})$, and $\zeta_\infty \sim N(0, \mathbf{T}^{-1})$. Return to the variables (A, α, γ). We have

$$\begin{pmatrix} A \\ \alpha \\ n \ln\gamma \end{pmatrix} = \begin{pmatrix} A_0 c_{-\frac{1}{\alpha_0 \tau}} \\ \alpha_0 \tau \\ \phi + \tau \ln \gamma_0 \end{pmatrix} := \mathbf{g}(c, \tau, \phi),$$

and

$$\begin{pmatrix} A_0 \\ \alpha_0 \\ n \ln \gamma_0 \end{pmatrix} = \mathbf{g}(1, 1, 0).$$

The normalized estimators equal

$$\sqrt{n}\begin{pmatrix} \hat{A} - A_0 \\ \hat{\alpha} - \alpha_0 \\ n(\ln \hat{\gamma} - \ln \gamma_0) \end{pmatrix} = \sqrt{n}\Big(\mathbf{g}\Big(\hat{c}, \hat{\tau}, \hat{\phi}\Big) - \mathbf{g}(1, 1, 0)\Big)$$

$$= \sqrt{n}\mathbf{g}'(1, 1, 0)\begin{pmatrix} \hat{c} - 1 \\ \hat{\tau} - 1 \\ \hat{\phi} \end{pmatrix} + o_p(1) \to \mathbf{g}'(1, 1, 0)\zeta_\infty \qquad (9.4.5)$$

in law. But

$$\Big(\mathbf{g}'(1, 1, 0)\Big)^{-1} = \mathrm{h}'(A_0, \alpha_0, n \ln \gamma_0), \qquad (9.4.6)$$

where the inverse transformation **h** equals

$$\mathbf{h}(A, \alpha, \phi) = \begin{pmatrix} (A/A_0)^{-\alpha} \\ \alpha/\alpha_0 \\ \psi - \alpha n \ln \gamma_0/\alpha_0 \end{pmatrix}.$$

Then

$$\mathbf{h}^{'}(A_0, \alpha_0, n \ln \gamma_0) = \begin{pmatrix} -\frac{\alpha_0}{A_0} & 0 & 0 \\ 0 & \frac{1}{\alpha_0} & 0 \\ 0 & -\frac{n \ln \gamma_0}{\alpha_0} & 1 \end{pmatrix} = -\mathbf{R}_n'(A_0, \alpha_0, \gamma_0), \qquad (9.4.7)$$

with $\mathbf{R}_n$ given in Theorem 2, $\mathbf{R}_n'$ is $\mathbf{R}_n$ transposed. Finally, from (9.4.5)–(9.4.7) we obtain that the sequence

$$\sqrt{n}\big(\mathbf{R}_n \mathbf{T} \mathbf{R}_n'\big)^{1/2}\begin{pmatrix} \hat{A} - A_0 \\ \hat{\alpha} - \alpha_0 \\ n(\ln \hat{\gamma} - \ln \gamma_0) \end{pmatrix}$$

converges in law to standard normal distribution.

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