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DEPENDENCE MATTERS!

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ABSTRACT

The actual discussions of appropriate risk measures to be used for the calculation of capital requirements in the Solvency II process have concentrated mainly on *Value-at-Risk* (VaR) and *Expected Shortfall* (ES). However, only recently the possible influence of dependence structures between the various types of risk or lines of business on such risk measures has drawn more attention [see e.g. WÜTHRICH (2003) or EMBRECHTS, HÖING AND PUCCETTI (2005) for a detailed discussion in connection with VaR]. The purpose of this paper is to investigate in more detail how the total risk distribution depends on different underlying dependence structures (copulas) while keeping the marginal distributions fixed, and how at least approximately such distributions can be calculated explicitly. We give several examples of uncorrelated (but dependent) risks with the same marginals, which show a completely different behavior for the aggregated risk distribution, in particular for the corresponding VaR and ES. Further, the influence of co- and counter-monotonicity of the marginal risks is shown to be totally different in the cases where the expectation of the individual risks is finite or infinite. These observations make it clear that the concept of correlation which is widely used e.g. in geophysical modeling and other professional DFA tools is not an appropriate dependence measure when risk aggregation or reinsurance of combined risks is considered.

1. INTRODUCTION

In the paper "Design of a future prudential supervisory system in the EU" (in short: "Solvency II"), being published by the European Commission, Internal Market DG in March 2003[1], one can find some general statements which are – in condensed form – given below:

"The new system should provide supervisors with the appropriate tools to assess the "overall solvency" of an insurance undertaking. This means that the system should not only consist of a number of quantitative ratios and indicators, but also cover qualitative aspects that influence the risk-standing of an undertaking (management, internal risk control, competitive situation

[1] Source: http://europa.eu.int/comm/internal_market/insurance/docs/markt-2509-03/markt-2509-03_en.pdf

etc.). … The solvency system should encourage and give an incentive to insurance undertakings to measure and manage their risks. In this regard, there is a clear need for developing common EU principles on risk management and supervisory review. Furthermore the quantitative solvency requirements should cover the most significant risks to which an insurance undertaking is exposed. This risk-oriented approach would lead to the recognition of internal models (either partial or full) provided these improve the undertaking's risk management and better reflect its true risk profile than a standard formula."

In the light of the last sentence, the possibility of developing "internal models" by the insurance companies themselves is particularly challenging due to the high complexity of mutual dependencies of risks *within* the liability and asset side each, but also *between* risks of these two categories. This fact is, however, still not sufficiently reflected by the present-day commercial and non-commercial DFA software tools. For instance, the meanwhile commonly accepted geophysical simulation software packages (see e.g. PFEIFER (2004) for more technical details) do not allow for a proper consideration of dependencies between different type of risks (such as windstorm and flooding or hailstorm) or between different regions. Likewise, software tools especially designed for Solvency II purposes can frequently not account for more sophisticated dependence structures due to the modular programming technique that is mainly underlying those products. In this paper, we want to show that the proper consideration of risk dependencies beyond correlation is of essential importance in the Solvency II discussion. In particular, we emphasize that the concept of correlation which is wide-spread in Solvency models such as the Swiss Solvency Test (see e.g. KELLER AND LUDER (2004)), but also in geophysical simulation software (see e.g. DONG (2001)) is not appropriate for the description of the distributional properties of aggregated risks. See also BLUM, DIAS AND EMBRECHTS (2002), p. 353f for a case study, or EMBRECHTS, STRAUMANN AND MCNEIL (2000) and EMBRECHTS, MCNEIL AND STRAUMANN (2002) for a more substantial discussion.

2. SUMS OF DEPENDENT RISKS

The problem of determining explicitly the distribution of a sum of two ore more dependent risks is generally non-trivial outside the world of normal distributions. In the latter case, it is clear that if $X_1, \cdots, X_n$ are jointly normally distributed random variables with mean vector $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_n)^{tr} \in \mathbb{R}^n$ and variance-covariance matrix $\Sigma = [\sigma_{ij}] \in \mathbb{R}^{n \times n}$ for some $n \in \mathbb{N}$, then $S_n := \sum_{i=1}^{n} X_i$ is also normally distributed with mean $\sum_{i=1}^{n} \mu_i$ and variance $\mathbf{1}^{tr} \cdot \Sigma \cdot \mathbf{1} = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}$, where $\mathbf{1}$ is the column vector consisting of the entry 1 in every component. This is in general no longer true if the joint distribution is not normal, even if the marginals are still normal. Cases in which explicit expressions for the distribution of the sum of dependent random variables are known are rare, except for the trivial case of identical summands $X_i = X_1$ for $i = 2, \cdots, n$, which corresponds to the case of perfect comonotonicity. In most cases, Monte Carlo simulations are performed from which the cumulative distribution function for the aggregated risk is estimated; see e.g. BLUM, DIAS AND EMBRECHTS (2002) for an example.

A central role in modeling dependencies between risks is played by the concept of *copulas*, which is nowadays widely used in Risk Management and Finance (see e.g. EMBRECHTS, STRAUMANN AND MCNEIL (2000), EMBRECHTS, MCNEIL AND STRAUMANN (2002), or CHERUBINI, LUCIANO AND VECCHIATO (2004)). A copula is essentially a multivariate distribution function restricted to the unit cube that has continuous uniform marginals.

Definition 2.1. A *copula* is a function C of d variables on the unit d-cube $[0,1]^d$ with the following properties:

1. the range of C is the unit interval $[0,1]$;
2. $C(\mathbf{u})$ is zero for all $\mathbf{u}$ in $[0,1]^d$ for which at least one coordinate equals zero;
3. $C(\mathbf{u}) = u_k$ if all coordinates of $\mathbf{u}$ are 1 except the k-th one;
4. C is d-increasing in the sense that for every $\mathbf{a} \le \mathbf{b}$ in $[0,1]^d$ the measure $\Delta C_{\mathbf{a}}^{\mathbf{b}}$ assigned by C to the d-box $[\mathbf{a},\mathbf{b}] = [a_1,b_1]\times\cdots\times[a_d,b_d]$ is nonnegative, i.e.

$$\Delta C_{\mathbf{a}}^{\mathbf{b}} := \sum_{(\varepsilon_1,\cdots,\varepsilon_n)\in\{0,1\}^d} (-1)^{\sum_{i=1}^{d}\varepsilon_i} C\left(\varepsilon_1 a_1 + (1-\varepsilon_1)b_1,\cdots,\varepsilon_d a_d + (1-\varepsilon_d)b_d\right) \ge 0.$$

Copulas have many useful properties, among them uniform continuity and (almost everywhere) existence of all partial derivatives (see e.g. NELSEN (1999), Theorem 2.2.4 and Theorem 2.2.7). Moreover, every copula lies between the so-called *Fréchet-Hoeffding* bounds, i.e.

$$\max(u_1+\cdots+u_d-d+1,0) \le C(u_1,\cdots,u_d) \le \min(u_1,\cdots,u_d)$$

which are commonly denoted by $\mathcal{W}$ and $\mathcal{M}$ in the literature. In two dimensions, both of the *Fréchet-Hoeffding* bounds are copulas themselves, but in higher dimensions, the *Fréchet-Hoeffding* lower bound $\mathcal{W}$ is no longer n-increasing. However, the inequality on the left-hand side cannot be improved, since for any $\mathbf{u}$ from the unit d-cube, there exists a copula $C_{\mathbf{u}}$ such that $\mathcal{W}(\mathbf{u}) = C_{\mathbf{u}}(\mathbf{u})$ (see NELSEN (1999), Theorem 2.10.12).

For the mathematical foundation of copulas we refer, in addition to the references mentioned above, to the monograph of NELSEN (1999) or PFEIFER AND NESLEHOVA (2004).

Copula models considered in the literature so far are typically parametric, such as the Gaussian and t-copulas, and the family of Archimedian copulas comprising the Gumbel, Frank and Clayton copulas, to mention some. In many cases these copulas are symmetric, which does frequently not match the observed data situation, or the number of parameters is small in comparison with the dimensionality of the data. Further, for practically all non-trivial copula models of the above type, it is impossible to derive explicit expressions for the sum of dependent risks for which the dependence structure is given by such a copula. The recent paper by EMBRECHTS, HÖING AND PUCCETTI (2004) is one of the few that deals with explicit (and not just asymptotic) representations of the distribution of aggregate dependent risks for two or three summands. However, very specific copula models are considered here which arise from the problem of finding dependence structures that produce extreme Value-at-Risk (VaR) scenarios.

In this section, we follow a different approach which essentially consists of an approximation of the underlying copula by certain grid-type copulas, for which the distribution of the sum of two or three (or even more) risks can be explicitly calculated in terms of piecewise defined polynomials. This enables also an explicit (approximate) calculation of a VaR and its corresponding Expected Shortfall (ES), at least if the risks involved have compact support. This approach is related to considerations in the paper by EMBRECHTS, HÖING AND JURI (2003), section 4.2.

Definition 2.2. Let $d,n \in \mathbb{N}$ and define intervals $I_{i_1,\cdots,i_d}(n) := \bigtimes_{j=1}^{d} \left(\frac{i_j - 1}{n}, \frac{i_j}{n} \right]$ for all possible choices $i_1,\cdots,i_d \in N_n := \{1,\cdots,n\}$. If $a_{i_1,\cdots,i_d}(n)$ are non-negative real numbers with the property

$$\sum_{(i_1,\cdots,i_d)\in J(i_k)} a_{i_1,\cdots,i_d}(n) = \frac{1}{n} \text{ for all } k \in \{1,\cdots,d\} \text{ and } i_k \in \{1,\cdots,n\},$$
$$\text{with } J(i_k) := \left\{ (j_1,\cdots,j_n) \in N_n{}^d \mid j_k = i_k \right\},$$

then the function $c_n := n^d \sum_{(i_1,\cdots,i_d)\in N_n{}^d} a_{i_1,\cdots,i_d}(n)\, \mathbb{1}_{I_{i_1,\cdots,i_d}(n)}$ is the density of a *d*-dimensional copula, called *grid-type copula* with parameters $\left\{ a_{i_1,\cdots,i_d}(n) \mid (i_1,\cdots,i_d) \in N_n{}^d \right\}$. Here $\mathbb{1}_A$ denotes the indicator random variable of the event *A*, as usual.

A simple interpretation of grid-type copulas is as follows: suppose that the discrete random vector $\mathbf{Z} = (Z_1,\cdots,Z_d)$ has support $N_n{}^d = \bigtimes_{j=1}^{d} N_n$, with $P(Z_1 = i_i,\cdots,Z_d = i_d) = a_{i_1,\cdots,i_d}(n)$. Further assume that the random vectors $\mathbf{X}_{i_1,\cdots,i_d}$ are uniformly distributed over the interval $I_{i_1,\cdots,i_d}(n)$ each, for $(i_1,\cdots,i_d) \in N_n{}^d$, and are independent of **Z**. Then the random vector $\mathbf{X}_\mathbf{Z}$ has the density f_n above. In other words, the distribution of $\mathbf{X}_\mathbf{Z}$ is a mixture of standard multivariate uniform distributions over the disjoint intervals $I_{i_1,\cdots,i_d}(n)$, with weights given by the $a_{i_1,\cdots,i_d}$, $i_1,\cdots,i_d \in N_n$.

It is easy to see that in case of an absolutely continuous *d*-dimensional copula *C*, with continuous density

$$c(u_1,\cdots,u_d) = \frac{\partial^d}{\partial u_1 \cdots \partial u_d} C(u_1,\cdots,u_d),\ (u_1,\cdots,u_d) \in (0,1)^d,$$

c can be approximated arbitrarily close by a density of a grid-type copula. We only have to choose

$$a_{i_1,\cdots,i_d}(n) := \int_{\frac{i_d-1}{n}}^{\frac{i_d}{n}} \cdots \int_{\frac{i_1-1}{n}}^{\frac{i_1}{n}} c(u_1,\cdots,u_d)\,du_1 \cdots du_d = \Delta C_{\boldsymbol{\alpha}_n}^{\boldsymbol{\beta}_n}, \quad i_1,\cdots,i_d \in N_n$$

with $\alpha_{nk} = \frac{i_k - 1}{n}$, $\beta_{nk} = \frac{i_k}{n}$, $k = 1,\cdots,d$. This follows e.g. from the classical multivariate mean-value-theorem of calculus. Moreover, a sequence of random vectors $\{\mathbf{X}_n\}_{n\in\mathbb{N}}$ with a grid-type copula density c_n of this type for each $\mathbf{X}_n$ converges weakly to a random vector **X** with the given copula *C*.

In this paper, we shall mainly concentrate on the case of multidimensional uniform risks as in section 3.2 of EMBRECHTS, HÖING AND PUCCETTI (2005), although the ideas developed here can easily be applied at least in the case of multidimensional risks with compact support.

Lemma 2.3. Let $U_1,\cdots,U_d$ be independent standard uniformly distributed random variables and let f_d and F_d denote the density and cumulative distribution function of $S_d := \sum_{i=1}^{d} U_i$, resp., for $d \in \mathbb{N}$. Then

$$\begin{aligned} f_d(x) &= \frac{1}{2(d-1)!}\sum_{k=0}^{d}(-1)^k \binom{d}{k}(x-k)^{d-1}\operatorname{sgn}(x-k)\ \mathbb{1}_{[0,d]}(x) \\ F_d(x) &= \frac{1}{2d!}\sum_{k=0}^{d}(-1)^k \binom{d}{k}\left((-k)^d + (x-k)^d \operatorname{sgn}(x-k)\right)\mathbb{1}_{[0,d]}(x) + \mathbb{1}_{(d,\infty]}(x) \end{aligned} \qquad \text{for } x \in \mathbb{R}.$$

This follows e.g. from USPENSKY (1937), Example 3, p.277, who attributes this result already to Laplace.

Corollary 2.4. Let $h > 0$ be a fixed real number and $V_1,\cdots,V_d$ be independent random variables such that V_i is uniformly distributed over the interval $[(j_i-1)h, j_i h]$ with some integer $j_i \in \mathbb{Z}^+$ for all $i \in \{1,\cdots,d\}$, and $h > 0$. Then $T_d := \sum_{i=1}^{d} V_i$ has density and cumulative distribution function $f_d(h;\bullet)$ and $F_d(h;\bullet)$, resp., given by

$$\begin{aligned} f_d(h;x) &= \frac{1}{h} f_d\left(\frac{x}{h} + d - \sum_{i=1}^{d} j_i\right) \\ F_d(h;x) &= F_d\left(\frac{x}{h} + d - \sum_{i=1}^{d} j_i\right) \end{aligned} \qquad \text{for } x \in \mathbb{R}.$$

Proof: Follows immediately from the fact that V_i can be represented as $V_i = (j_i - 1)h + hU_i$, where U_i has a standard uniform distribution over $[0,1]$, such that

$$T_d := h\sum_{i=1}^{d}(j_i - 1) + h\sum_{i=1}^{d} U_i = h\sum_{i=1}^{d} j_i - hd + hS_d =: m_d + hS_d,$$

say. Hence we have

$$F_d(h;x) = P(T_d \le x) = P\left(S_d \le \frac{x - m_d}{h}\right) = F_d\left(\frac{x - m_d}{h}\right)$$

and thus also $f_d(h;x) = \frac{d}{dx}F_d(h;x) = \frac{1}{h} f_d\left(\frac{x}{h} + d - \sum_{i=1}^{d} j_i\right)$ *a.e.*, for $x \in \mathbb{R}$.

The preceding results allows us to formulate the main result of this section.

Theorem 2.5. Let $(X_1,\cdots,X_d)$ be a random vector whose joint cumulative distribution function is given by a grid-type copula in the sense of Definition 2.2, with density $c_n := n^d \sum_{(i_1,\cdots,i_d)\in N_n^d} a_{i_1,\cdots,i_d}(n)\mathbb{1}_{I_{i_1,\cdots,i_d}(n)}$. Then the density and cumulative distribution function $\tilde{f}_d(n;\bullet)$ and $\tilde{F}_d(n;\bullet)$, resp., for the sum $S_d := \sum_{i=1}^{d} X_i$ is given by

$$\left.\begin{aligned}\tilde{f}_d(n;x) &= n \sum_{(i_1,\cdots,i_d)\in N_n^d} a_{i_1,\cdots,i_d}(n)\cdot f_d\left(nx+d-\sum_{j=1}^{d} i_j\right)\\ \tilde{F}_d(n;x) &= \sum_{(i_1,\cdots,i_d)\in N_n^d} a_{i_1,\cdots,i_d}(n)\cdot F_d\left(nx+d-\sum_{j=1}^{d} i_j\right)\end{aligned}\right\} \text{ for } x\in\mathbb{R},$$

with the functions f_d and F_d from Lemma 2.3.

It is easy to see that Theorem 2.5 extends readily in an approximate manner to the case of aggregated dependent risks in the situation where the joint distribution has a compact support $\mathcal{X}$ and a continuous density since it is always possible to find a sequence $\mathcal{X}_n$ of disjoint unions of closed non-empty symmetric hypercubes in *d* dimensions whose are close to $\mathcal{X}$, i.e. $\lim_{n\to\infty} \mathfrak{m}^d\left(\mathcal{X}_n \triangle \mathcal{X}\right)=0$, where $\mathfrak{m}^d$ denotes Lebesgue measure and $\triangle$ denotes the symmetric difference of sets. Likewise, the joint density can be approximated by step functions defined on $\mathcal{X}_n$ in the same way as for grid-type copulas. The corresponding details will be left to the reader; some examples will be given in Section 4.

Theorem 2.5 and its extensions allow for an explicit representation of the density and cumulative distribution function of several dependent (uniformly distributed) risks in terms of piecewise defined polynomials of degree *d*, and therefore also for explicit expressions for the Value-at-Risk and the Expected Shortfall of the aggregated risk. This might be a good alternative to simulation studies which otherwise must be performed in order to obtain such kind of information. Also, with this approach, the dependence of VaR and ES on parameters of the distribution can be studied on a theoretical basis.

3. SUMS OF DEPENDENT UNCORRELATED RISKS: SOME CASE STUDIES

In this section we want to show that even for uncorrelated risks[2], a broad range of different aggregate sum distributions and representations for VaR and ES are possible. We start with the most general case of a grid-type copula with 9 subsquares, i.e. we first consider the situation $d=2$ and $n=3$ in Definition 2.2. The weights $a_{ij}(n)$ for the copula density can then be described in matrix form as

[2] Although in the papers cited in the introduction it is pointed out several times that correlation is no appropriate measure of dependence, it still seems to be the “standard” dependence measure in practice.

$$A(n)=\left[a_{ij}(n)\right]=\begin{bmatrix} a & b & 1/3-a-b \\ c & d & 1/3-c-d \\ 1/3-a-c & 1/3-b-d & -1/3+a+b+c+d \end{bmatrix}$$

with suitable real numbers $a,b,c,d\in[0,1/3]$. It follows that the covariance of the corresponding random variables X_1, X_2 is given by

$$E\left(X_1X_2\right)-E\left(X_1\right)E\left(X_2\right)=\frac{1}{9}\sum_{i=1}^{3}\sum_{j=1}^{3}a_{ij}(n)\left(i-\frac{1}{2}\right)\left(j-\frac{1}{2}\right)-\frac{1}{4}=\frac{4a+2b+2c+d-1}{9}$$

vanishing in the case

$$d=1-4a-2b-2c.$$

The case of uncorrelated (but possibly dependent) risks hence corresponds to a three-parameter grid-type copula with parameter $\gamma=(a,b,c)$ given by

$$A(n)=\left[a_{ij}(n)\right]=\begin{bmatrix} a & b & 1/3-a-b \\ c & 1-4a-2b-2c & -2/3+4a+2b+c \\ 1/3-a-c & -2/3+4a+b+2c & 2/3-3a-b-c \end{bmatrix}.$$

The density and cumulative distribution function of the aggregated risk $S_2:=X_1+X_2$ are thus, by Theorem 2.5, given by

$$\tilde{f}_2(3;\gamma;x)=\begin{cases} 9ax, & 0\le x\le\frac{1}{3} \\ 3(2a-\{b+c\})+9(-a+\{b+c\})x, & \frac{1}{3}\le x\le\frac{2}{3} \\ 9(4a+3\{b+c\})-10+3(5-18a-12\{b+c\})x, & \frac{2}{3}\le x\le 1 \\ 32-9(16a+7\{b+c\})+9(14a+6\{b+c\}-3)x, & 1\le x\le\frac{4}{3} \\ -28+3(52a+19\{b+c\})+3(6-33a-12\{b+c\})x, & \frac{4}{3}\le x\le\frac{5}{3} \\ 6(2-9a-3\{b+c\})+3(-2+9a+3\{b+c\})x, & \frac{5}{3}\le x\le 2 \\ 0, & \text{otherwise;} \end{cases}$$

$$\tilde{F}_2(3;\gamma;x) = \begin{cases} 0, & x \le 0 \\ \frac{9a}{2}x^2, & 0 \le x \le \frac{1}{3} \\ \frac{9}{2}(-a+\{b+c\})x^2 + 3(2a-\{b+c\})x + \frac{1}{2}(-2a+\{b+c\}), & \frac{1}{3} \le x \le \frac{2}{3} \\ \frac{3}{2}(5-18a-12\{b+c\})x^2 + (-10+36a+27\{b+c\})x + \\ \qquad + \frac{1}{6}(20-66a-57\{b+c\}), & \frac{2}{3} \le x \le 1 \\ \frac{9}{2}(-3+14a+6\{b+c\})x^2 + (32-144a-63\{b+c\})x + \\ \qquad + \frac{1}{6}(-106+474a+213\{b+c\}), & 1 \le x \le \frac{4}{3} \\ \frac{9}{2}(2-11a-4\{b+c\})x^2 + (-28+156a+57\{b+c\})x + \\ \qquad + \frac{1}{6}(134-726a-267\{b+c\}), & \frac{4}{3} \le x \le \frac{5}{3} \\ \frac{3}{2}(-2+9a+3\{b+c\})x^2 + 3(4-18a-6\{b+c\})x + \\ \qquad + (-11+54a+18\{b+c\}), & \frac{5}{3} \le x \le 2 \\ 1, & x \ge 2. \end{cases}$$

Note that both functions only depend on the sum $\{b+c\}$ and not on b or c alone. The following graphs show five different densities $\tilde{f}_2(3;\gamma;\bullet)$ and cumulative distribution functions $\tilde{F}_2(3;\gamma;\bullet)$ for the sum $S_2 = X_1 + X_2$, for various choices of $\gamma = (a,b,c)$.

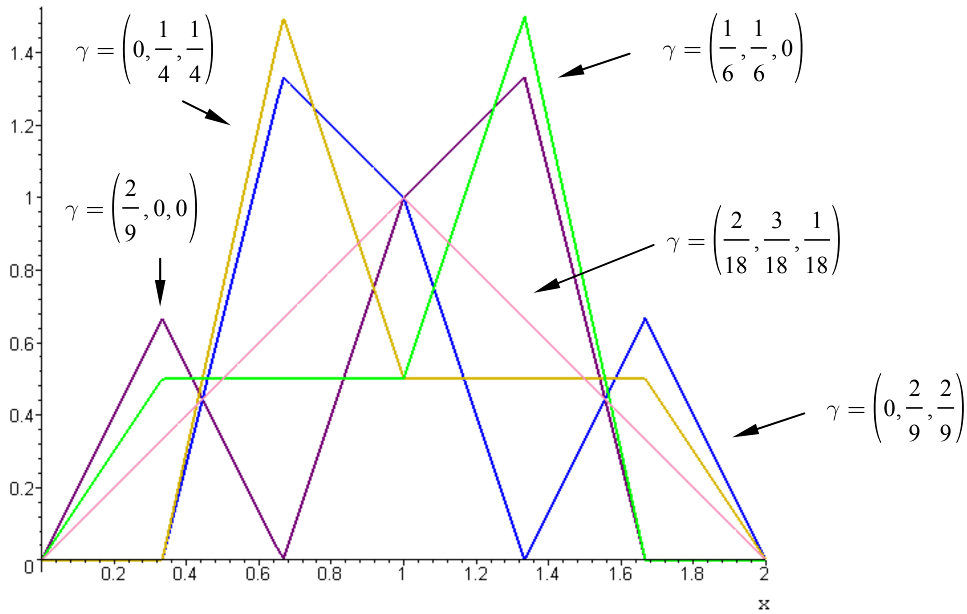

Fig. 1: densities $\tilde{f}_2(3;\gamma;x)$

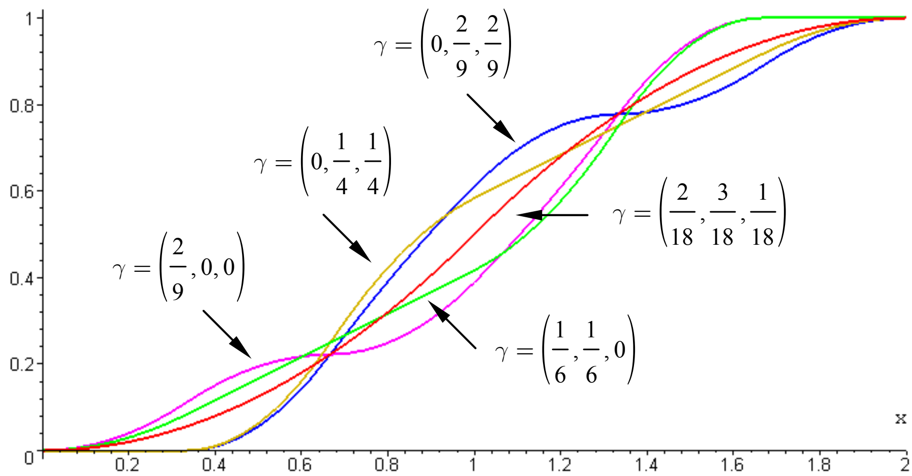

Fig 2: corresponding cdf's $\tilde{F}_2(3;\gamma;x)$

For finding the "worst" VaR scenario in this setup we have to minimize the cumulative distribution function

$$\tilde{F}_2(3;\gamma;x)=\frac{3}{2}(-6+9a+3\{b+c\})x^2+3(4-6a-18\{b+c\})x+(-11+54a+18\{b+c\})$$

at the point $x=\frac{5}{3}$, which is the solution of the following linear programming problem:

min! $6a+2b+2c$ under the conditions

$$\begin{array}{l} a+b\leq\frac{1}{3}\\ a+c\leq\frac{1}{3}\\ 3a+b+c\leq\frac{2}{3}\\ 4a+b+2c\geq\frac{2}{3}\\ 4a+2b+2c\leq 1\\ a,b,c\geq 0.\end{array}$$

This follows from the entries in the matrix $A(n)$ above and the fact that the marginal sums (rows and columns) add up to $1/3$ each.

The solution of this problem is given by all $\gamma=(0,b,c)$ fulfilling the condition $b+c=\frac{4}{9}$. In particular, $b=c=\frac{2}{9}$ is a feasible solution, which is among the cases shown in Fig. 1 and 2 above (blue line). In a similar way, we can determine the "best" VaR scenario here, which is uniquely determined by the parameter $\gamma = \left(\frac{2}{9}, 0, 0\right)$. This case is also shown above (pink line).

Naturally, it is also possible to express the quantile function $Q(3;\gamma;\bullet)$ for all choices of γ in an explicit way, by solving the appropriate quadratic equations in the representation of $\tilde{F}_2(3;\gamma;\bullet)$ above. For instance, for $\gamma \neq \left(\frac{2}{9}, 0, 0\right)$, we obtain

$$Q(3;\gamma;u) = \frac{6(2-3K) - \sqrt{6(2-3K)(1-u)}}{3(2-3K)} \quad \text{for } \tilde{F}_2\left(3;\gamma;\frac{5}{3}\right) \leq u \leq 1$$

with $K := 3a + \{b + c\}$. For the following numerical example, we shall, for simplicity, restrict our considerations to the range of $0,\overline{7} = \frac{7}{9} \leq u \leq 1$. For the three cases "worst" VaR scenario, independence and "best" VaR scenario in this setup, we obtain

$$Q(3;\gamma;u) = \begin{cases} \begin{cases} \frac{4}{3} + \frac{1}{3}\sqrt{9u-7}, & \frac{7}{9} \leq u \leq \frac{8}{9} \\ 2 - \sqrt{1-u}, & \frac{8}{9} \leq u \leq 1, \end{cases} & \gamma = \left(0, \frac{2}{9}, \frac{2}{9}\right) \\ 2 - \sqrt{2(1-u)}, \quad \frac{7}{9} \leq u \leq 1, & \gamma = \left(\frac{1}{9}, \frac{1}{9}, \frac{1}{9}\right) \\ \frac{5}{3} - \frac{1}{2}\sqrt{2(1-u)}, \quad \frac{7}{9} \leq u \leq 1, & \gamma = \left(\frac{2}{9}, 0, 0\right). \end{cases}$$

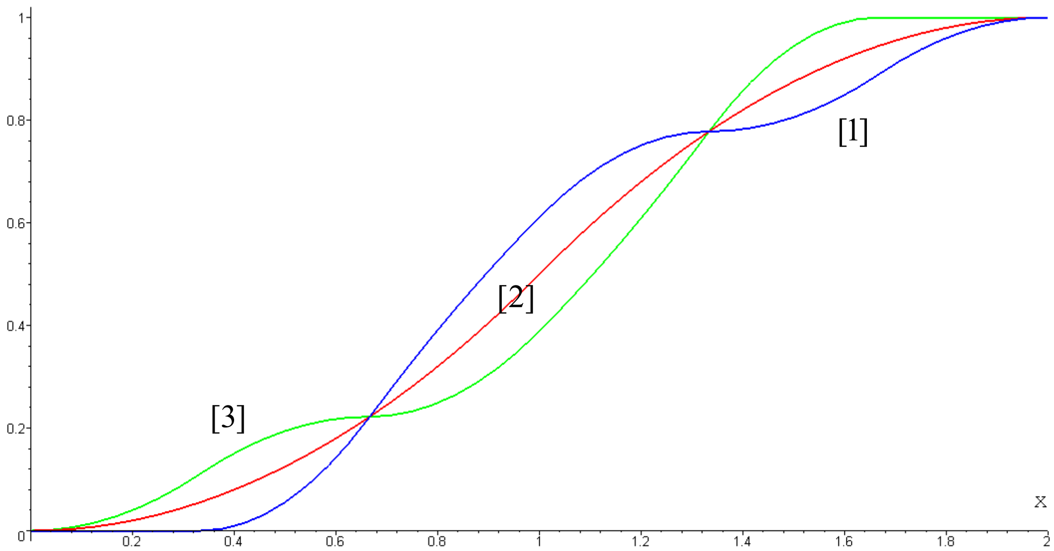

Fig. 3: cdf's for "worst" VaR [1], independence[2], and "best" VaR [3] scenario

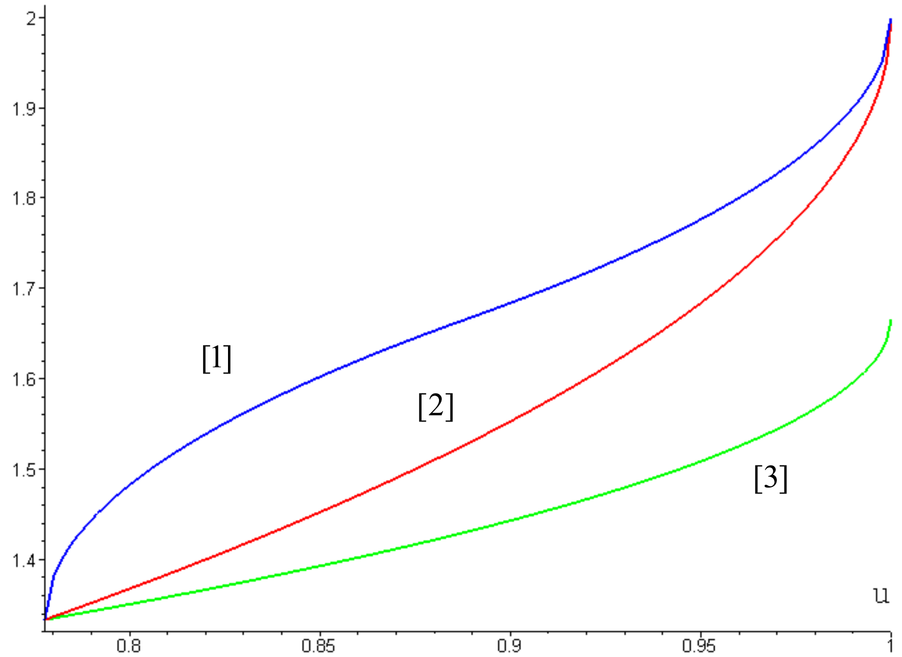

Fig. 4: quantile functions $Q(3;\gamma;u)$ for "worst" VaR [1], independence[2], and "best" VaR [3] scenario

Likewise, for the Expected Shortfall[3], we obtain

$$\mathrm{ES}(3;\gamma;u)=\frac{1}{1-u}\int_u^1 Q(3;\gamma;v)\,dv=\begin{cases}\begin{cases}\dfrac{\frac{38-36u}{27}-\frac{2}{81}\sqrt{9u-7}^{3}}{1-u}, & \frac{7}{9}\le u\le\frac{8}{9}\\ 2\left(1-\frac{1}{3}\sqrt{1-u}\right), & \frac{8}{9}\le u\le 1,\end{cases} & \gamma=\left(0,\frac{2}{9},\frac{2}{9}\right)\\ 2-\frac{1}{3}\sqrt{2(1-u)}, \quad \frac{7}{9}\le u\le 1, & \gamma=\left(\frac{1}{9},\frac{1}{9},\frac{1}{9}\right)\\ \frac{5}{3}-\frac{1}{3}\sqrt{2(1-u)}, \quad \frac{7}{9}\le u\le 1, & \gamma=\left(\frac{2}{9},0,0\right).\end{cases}$$

The following graph shows both the VaR and ES in the range $0,\overline{7}=\frac{7}{9}\le u\le 1,$ for the three cases considered.

[3] Here we denote $\mathrm{ES}_u=ES(n;\gamma;u)=E\left(S_d\,|\,S_d\ge VaR_u\right)$ with $VaR_u=Q(n;\gamma;u)$ for $0<u<1$. In the literature, one often finds the complementary notation with $\varepsilon=1-u$.

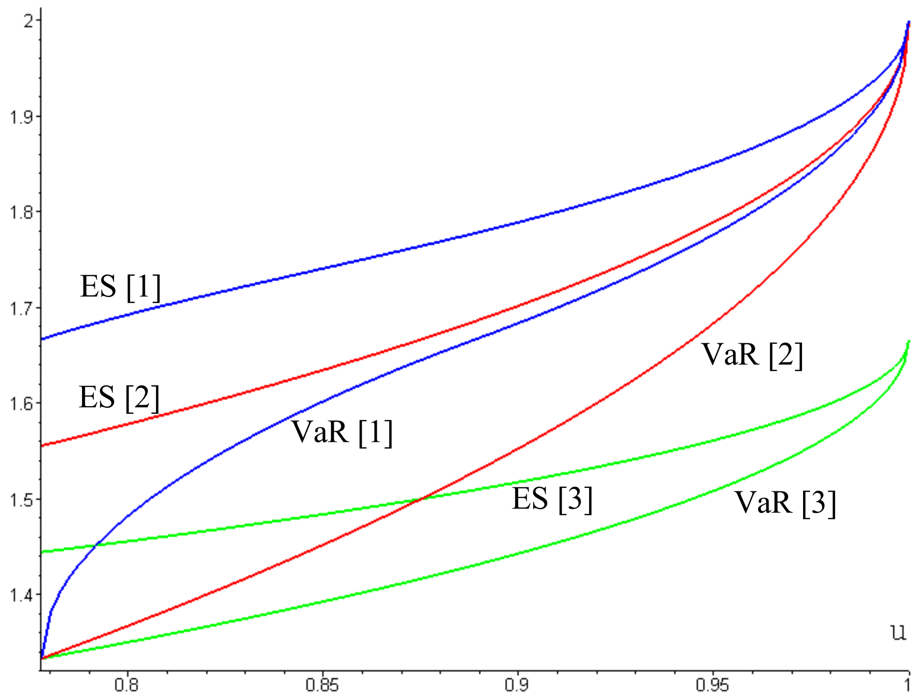

Fig. 5: VaR and Expected Shortfall for “worst” case [1], independence case [2], and “best” case [3]

For instance, we obtain

	worst case	independence	best case
$VaR_{0,9}$	1,6838	1,5528	1,4430
$ES_{0,9}$	1,7892	1,7019	1,5269
$VaR_{0,99}$	1,9000	1,8586	1,5960
$ES_{0,99}$	1,9333	1,9057	1,6225

Tab. 1

Interestingly, the VaR and ES values in the worst case are between 16% and 19% larger than in the best case, which shows that even in the case of uncorrelated risks, the range of values for the most popular risk measures for the aggregate risk is still enormous!

4. Sums of dependent risks: more general cases

In this section we shall show that the concept of grid-type copulas and their generalizations outlined above is numerically very attractive and easy to implement, for instance by use of computer algebra systems. This applies especially to situations where the dimension is larger than 2 (cf. Embrechts, Höing and Puccetti (2005)).

Example 4.1. Suppose that the risks X_1 and X_2 are each uniformly distributed and their joint cumulative distribution function is a copula of Clayton or Gumbel type, resp., i.e.

$$\begin{aligned} C_1(\theta;u,v) &= \left(u^{-\theta}+v^{-\theta}-1\right)^{-1/\theta} \quad \text{(Clayton)} \quad \text{or} \\ C_2(\theta;u,v) &= \exp\left(-\left\{(-\ln u)^{\theta}+(-\ln v)^{\theta}\right\}^{1/\theta}\right) \quad \text{(Gumbel)} \end{aligned} \quad \text{for } (u,v)\in(0,1)^2, \text{ with } \theta\geq 1.$$

We consider the distribution of the aggregated risk $S_2 := X_1 + X_2$. The following graph shows the calculated densities for S_2 under these copulas, for a grid-type copula approximation (see the remark after Definition 2.2), with 10000 subsquares of the unit interval of equal area each.

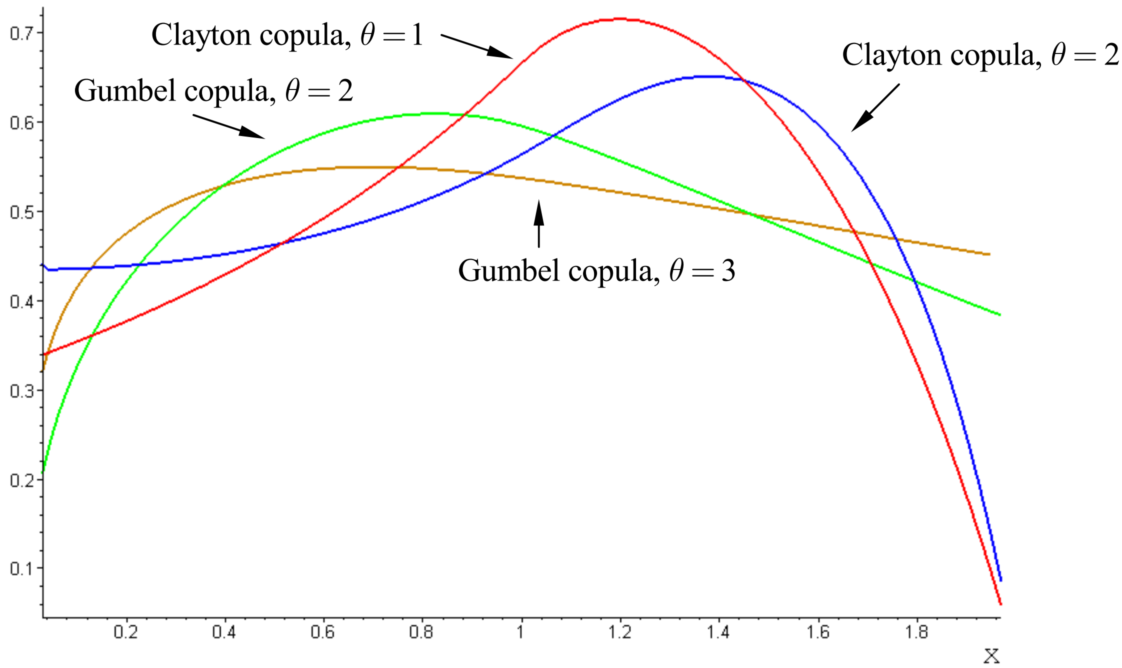

Fig. 6: approximation of densities for two aggregate dependent risks

Example 4.2. Suppose that the risks X_1, X_2 and X_3 are each uniformly distributed and their joint cumulative distribution function is a grid-type copula with $2^{3m} = 8^m$ subcubes of the unit cube, with equal volumes each, for some $m \in \mathbb{N}$, with weights

$$a_{ijk} := (-1)^{i+j+k} p + \frac{1}{8^m} \quad \text{for } i,j,k \in \{1,2\} \text{ and } p < \frac{1}{8^m}.$$

Then

$$f_3(x) = \begin{cases} \dfrac{x^2}{2}, & 0 \leq x \leq 1 \\ \dfrac{3}{4} - \left(x - \dfrac{3}{2}\right)^2, & 1 \leq x \leq 2 \\ \dfrac{(3-x)^2}{2}, & 2 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

with the density f_3 from Lemma 2.3, and the density $\tilde{f}_3(2^m;\bullet)$ of the aggregate risk $S_3 := X_1 + X_2 + X_3$ is given by

$$\tilde{f}_3(2^m;x) = 2^m \sum_{k=1}^{2^m}\sum_{j=1}^{2^m}\sum_{i=1}^{2^m} a_{ijk}\, f_3\left(2^m x + 3 - \{i+j+k\}\right) \text{ for } 0 \le x \le 3.$$

Note that due to

$$\sum_{i=1}^{2^m}(-1)^{i+j+k} p = (-1)^{j+k} p \sum_{i=1}^{2^m}(-1)^i = 0 = \sum_{j=1}^{2^m}(-1)^{i+j+k} p = \sum_{k=1}^{2^m}(-1)^{i+j+k} p$$

the a_{ijk} actually define a copula. The following graphs show the density of the aggregated risk $S_3 = X_1 + X_2 + X_3$ for $m \in \{1,2\}$ and $p = \dfrac{1}{8^m+1}$ (green line) together with the density resulting from independence (red line).

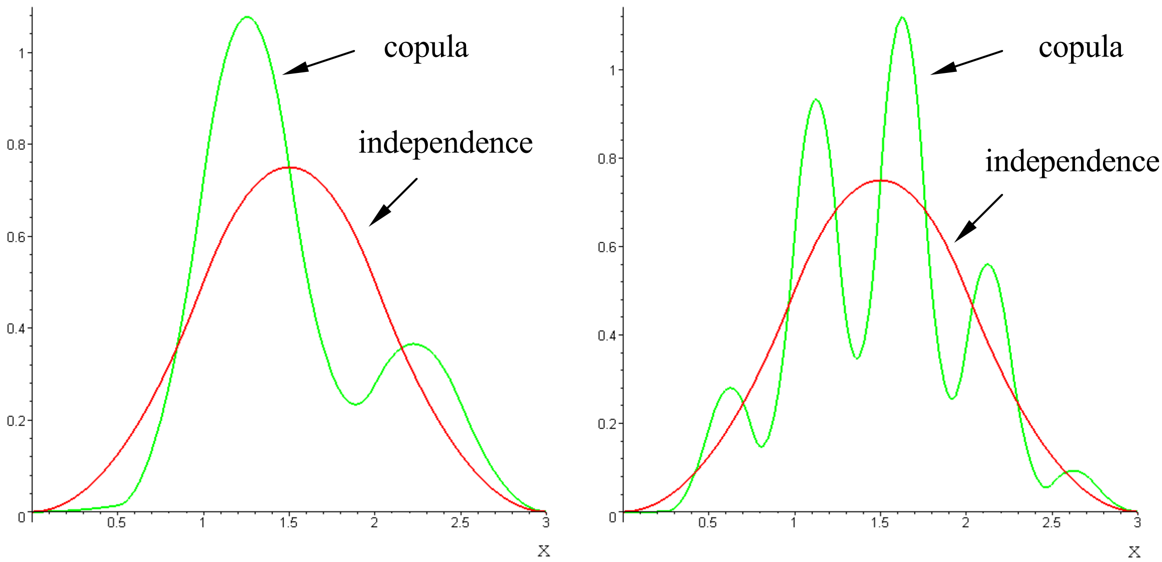

Fig. 7: densities of three aggregate dependent risks, left: $m=1$, right: $m=2$

Example 4.3. Here we consider five dependent risks $X_1,\cdots,X_5$ with different marginal distributions and joint density given by

$$f(x_1,\cdots,x_5) = \sum_{i=1}^{20}\sum_{j=1}^{20}\sum_{k=1}^{20}\sum_{l=1}^{20}\sum_{m=1}^{20} \alpha(i,j,k,l,m)\cdot \mathbb{1}_{I(i,j,k,l,m)}(x_1,\cdots,x_5)$$

with $I(i,j,k,l,m) := (i-1,i]\times(j-1,j]\times(k-1,k]\times(l-1,l]\times(m-1,m]$ and

$$\alpha(i,j,k,l,m) = \frac{1}{K}\cdot\frac{\frac{4}{3}+\sin(i+j+k)}{i+\sqrt{j}+k+l^2+m^3-4} \quad \text{for } i,j,k,l,m \in \{1,\cdots,20\},$$

with the normalizing constant

$$K=\sum_{i=1}^{20}\sum_{j=1}^{20}\sum_{k=1}^{20}\sum_{l=1}^{20}\sum_{m=1}^{20}\frac{\frac{4}{3}+\sin(i+j+k)}{i+\sqrt{j+k+l^2+m^3-4}}\approx 12198.$$

Note that the support of the joint distribution consists of $20^5=3\,200\,000$ disjoint hypercubes in $\mathbb{R}^5$ of equal Lebesgue measure here.

The following graph shows the resulting marginal densities g_r of these five risks.

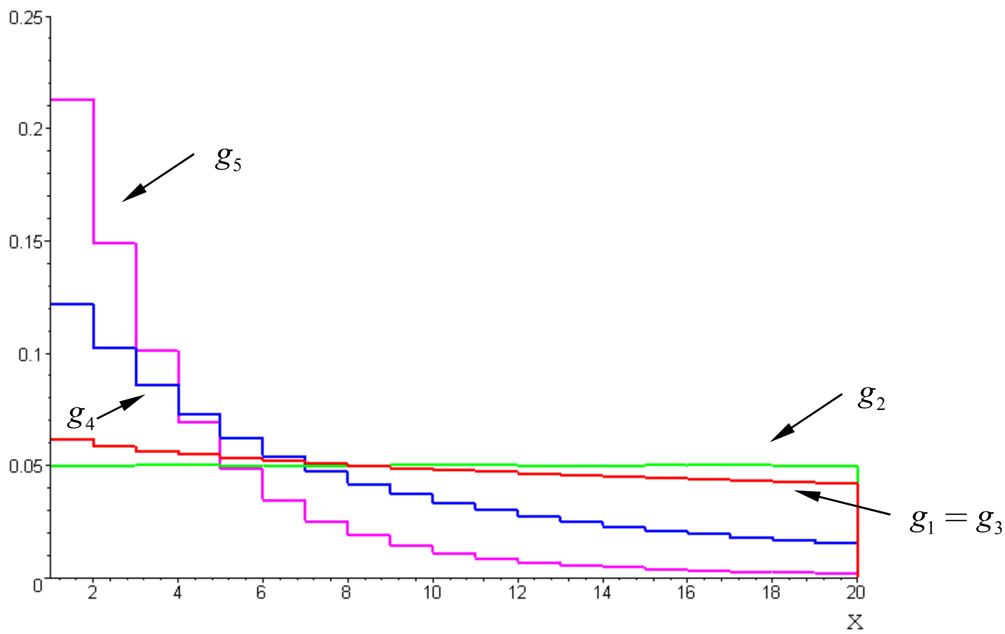

Fig. 8: marginal densities

The density of the aggregate risk is now given by

$$f_{\text{sum}}(x)=\sum_{i=1}^{20}\sum_{j=1}^{20}\sum_{k=1}^{20}\sum_{l=1}^{20}\sum_{m=1}^{20}\alpha(i,j,k,l,m)\cdot f_5\left(x+5-\{i+j+k+l+m\}\right)\text{ for }0\le x\le 100$$

with the density f_5 from Lemma 2.3. The following graph shows the result, in comparison with the corresponding case of independence.

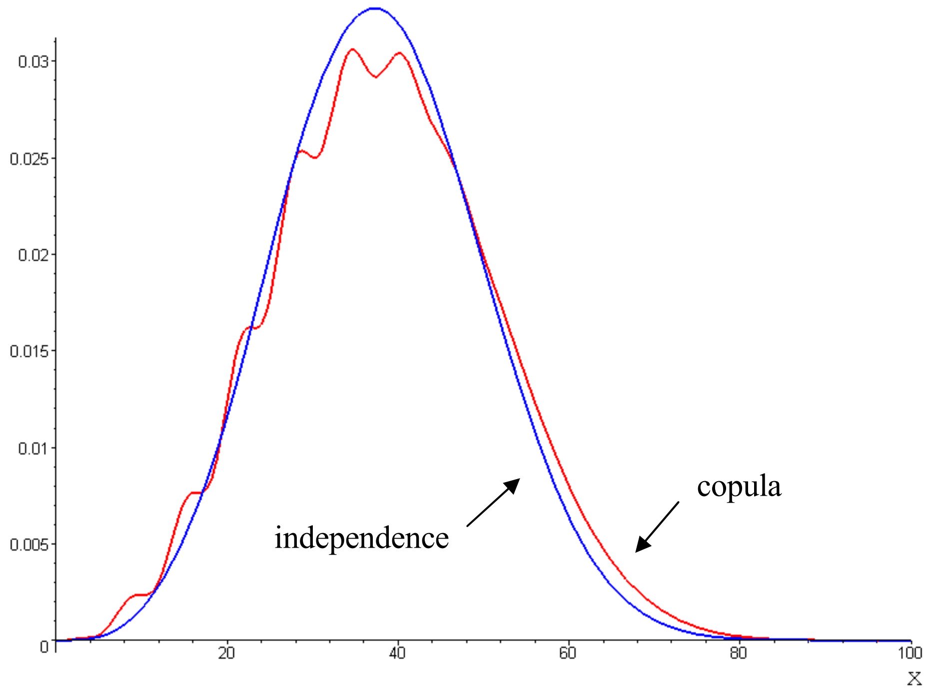

Fig. 9: density of five aggregated risks, dependent vs. independent case

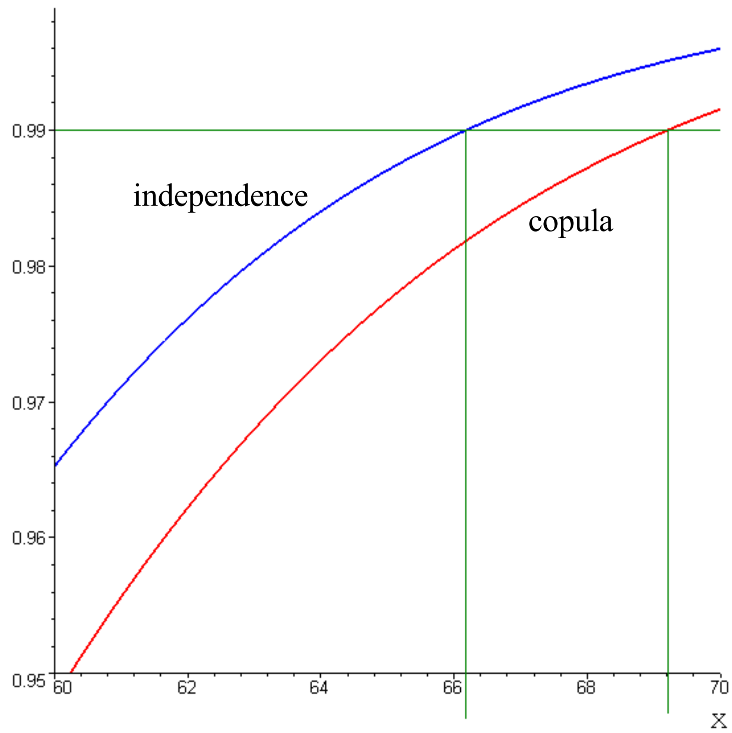

Fig. 10: cumulative distribution function, dependent vs. independent case

Obviously, in all of the above examples, the influence of the copula on the aggregate sum distribution is not negligible. In the last example (see Fig. 10), the VaR at 99% safety level is 66,15 for the independent case, but 69,22 in the dependent case which is about 4,6% larger.

5. Sums of dependent risks with heavy tails

In the preceding sections, all calculations were based on the fact that the risk distributions had in some way a compact support. In the actuarial practice, however, it is convenient to consider also unbounded risks, especially those with heavy tails. Such distributions are of relevance e.g. when natural perils have to be modeled. Although the approximation argument would be applicable here as well, it is interesting to see what the consequences of dependence between risks explicitly are in this area. It turns out that the existence of expectations is crucial here.

The following results sharpen in some sense Examples 6 and 7 in Embrechts, McNeil and Straumann (2002).

Lemma 5.1. Suppose that the risks X_1 and X_2 follow a Pareto distribution with density

$$f(x) = \frac{1}{2\sqrt{1+x}^3},\ x \geq 0$$

each. Then the density g and cumulative distribution function G of the aggregated risk $S_2 := X_1 + X_2$ can be explicitly calculated in the following cases:

Case 1: X_1 and X_2 are independent:

$$g(z) = \frac{z}{(2+z)^2\sqrt{1+z}} \approx \frac{1}{\sqrt{1+z}^3},\ G(z) = 1 - 2\frac{\sqrt{1+z}}{2+z},\ z \geq 0$$

Case 2: X_1 and X_2 are co-monotonic, i.e. the corresponding copula is the upper Fréchet bound $\mathcal{M}$:

$$g(z) = \frac{1}{4\sqrt{1+z/2}^3} \approx \frac{1}{\sqrt{2}\sqrt{1+z}^3},\ G(z) = 1 - \sqrt{\frac{2}{2+z}},\ z \geq 0$$

Case 3: X_1 and X_2 are counter-monotonic, i.e. the corresponding copula is the lower Fréchet bound $\mathcal{W}$:

$$g(z) = \frac{4+z-2\sqrt{3+z}}{\sqrt{z+6-4\sqrt{3+z}}\ \sqrt{3+z}\sqrt{2+z}^3} \approx \frac{1}{\sqrt{1+z}^3},$$

$$G(z) = \frac{\sqrt{z^2+8z+12-(8+4z)\sqrt{3+z}}}{2+z},\ z \geq 6.$$

Proof: In the independent case, we have

$$g(z) = \int_0^z f(x)f(z-x)\,dx = \frac{1}{4}\int_0^z \frac{1}{\sqrt{1+x}^3}\frac{1}{\sqrt{1+(z-x)}^3}\,dx \text{ for } z > 0.$$

Let

$$h_a(x) := \frac{2(1-a+2x)}{(1+a)^2\sqrt{(1+x)(a-x)}} \text{ for } 0 \le x < a \text{ with } a > 0.$$

Then

$$h_a'(x) := \frac{1}{\sqrt{(1+x)(a-x)}^3} \text{ for } 0 \le x < a.$$

Thus $h_{1+z}(x) := \dfrac{2(2x-z)}{(2+z)^2\sqrt{(1+x)(1+z-x)}}$ is the indefinite integral of $\dfrac{1}{\sqrt{1+x}^3}\dfrac{1}{\sqrt{1+(z-x)}^3}$ and hence we have

$$g(z) = \frac{1}{4}\int_0^z \frac{1}{\sqrt{1+x}^3}\frac{1}{\sqrt{1+(z-x)}^3}dx = \frac{1}{4}\left(h_{1+z}(z) - h_{1+z}(0)\right) = \frac{z}{(2+z)^2\sqrt{1+z}} \text{ for } z > 0,$$

with $g(z) = \dfrac{d}{dz}\left(1 - 2\dfrac{\sqrt{1+z}}{2+z}\right)$ from which the statement follows.

Case 2 is trivial since the distribution of S_2 is identical to that of $2X_1$. Case 3 follows from the observation that for the cumulative distribution function F, we have

$$F(x) = \int_0^x \frac{1}{2\sqrt{1+y}^3}dy = 1 - \frac{1}{\sqrt{1+x}},\ x \ge 0,$$

which in turn implies

$$F^{-1}(u) = \frac{1}{(1-u)^2} - 1,\ 0 < u < 1.$$

Hence $S_2 = X_1 + X_2$ is distributed as $F^{-1}(U) + F^{-1}(1-U) = \dfrac{1}{U^2} + \dfrac{1}{(1-U)^2} - 2$ for a standard uniformly distributed random variable U. Since the mapping $u \mapsto \dfrac{1}{u^2} + \dfrac{1}{(1-u)^2} - 2$ is strictly convex and symmetric w.r.t. the point $u = 1/2$ with its minimum attained there (with value 6) we see that the feasible set of real valued solutions u for the inequality

$$\frac{1}{u^2} + \frac{1}{(1-u)^2} - 2 \le z,\ z \ge 6$$

is given by the compact interval

$$\left[u_0 := \frac{2+z-\sqrt{z^2+8z+12-(8+4z)\sqrt{3+z}}}{2(2+z)}, \frac{2+z+\sqrt{z^2+8z+12-(8+4z)\sqrt{3+z}}}{2(2+z)} =: u_1\right].$$

It follows readily that

$$P(S_2 \le z) = P\left(\frac{1}{U^2} + \frac{1}{(1-U)^2} \le z+2\right) = u_1 - u_0 = \frac{\sqrt{z^2 + 8z + 12 - (8+4z)\sqrt{3+z}}}{2+z}, \ z \ge 6.$$

The density given above now follows by differentiation.

It is interesting to see that asymptotically, Case 1 and Case 3 are equal, and that in all three cases the density of the aggregate sum is – up to a constant factor – of the same Pareto type as the distribution of each summand.

The following graphs show the cumulative distribution functions and VaR's for S_2 in the three cases above.

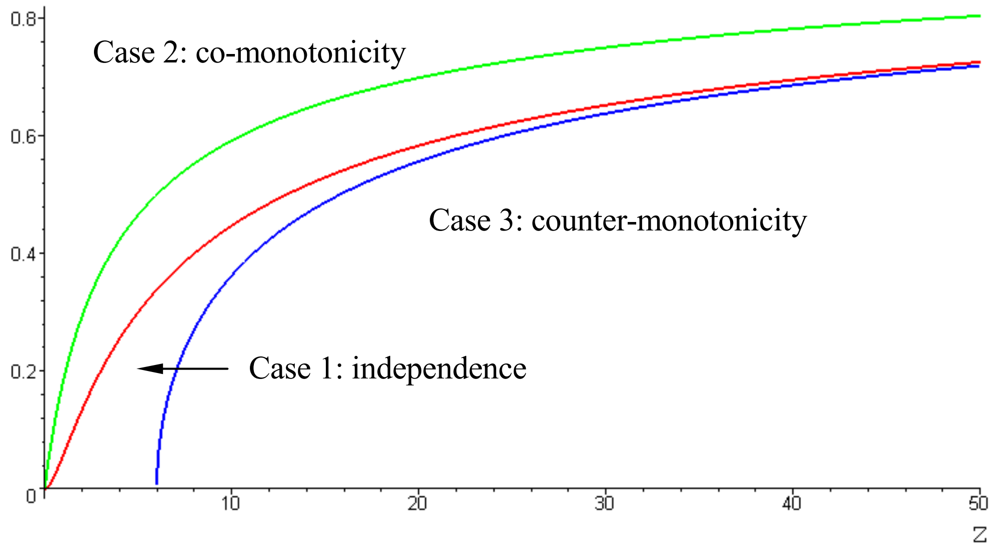

Fig. 11: cdf's for the three cases

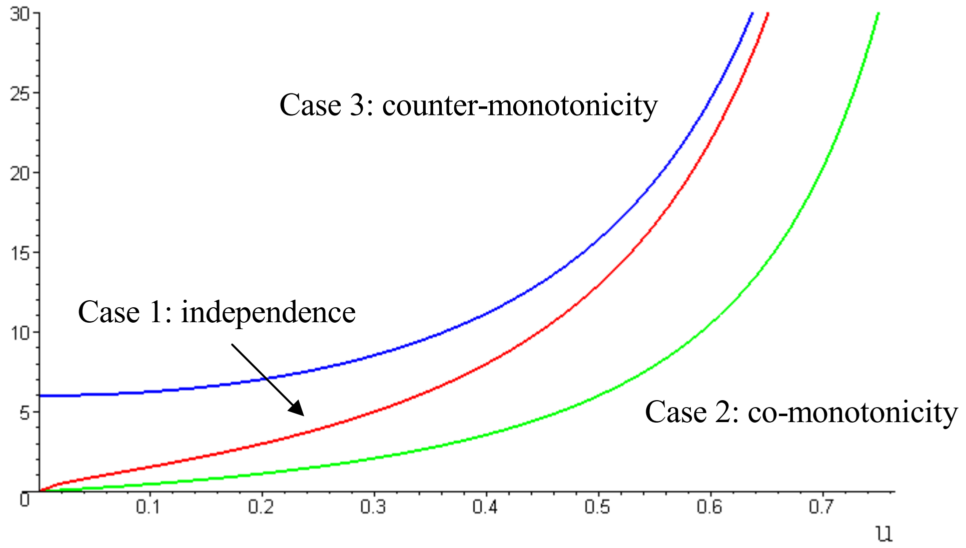

Fig. 12: VaR's for the three cases

Surprisingly, Case 3 with counter-monotonicity produces the "worst" VaR scenario here, while Case 2 with co-monotonicity corresponds to the "best" VaR scenario! This seems to contradict the intuition that counter-monotonicity creates a diversification effect since "large" risks are always coupled with "small" risks.

A little more calculus shows that we have indeed:

Case 1: $\mathrm{VaR}_u = \dfrac{4}{(1-u)^2} - 2 - \dfrac{2}{1+\sqrt{u(2-u)}} \sim \dfrac{4}{(1-u)^2} - 4 \ \ (u \to 1)$

Case 2: $\mathrm{VaR}_u = \dfrac{2}{(1-u)^2} - 2$

Case 3: $\mathrm{VaR}_u = \dfrac{4}{(1-u)^2} - 2 + \dfrac{4}{(1+u)^2} \sim \dfrac{4}{(1-u)^2} + 2 \ \ (u \to 1)$

for $0 < u < 1$.

The following result extends a modification of Example 2.4 in TASCHE (2002).

Lemma 5.2. Suppose that the risks X_1 and X_2 follow a Pareto distribution with density

$$f(x) = \frac{1}{(1+x)^2}, \ \ x \geq 0$$

each. Then the density g and cumulative distribution function G of the aggregated risk $S_2 := X_1 + X_2$ can be explicitly calculated in the following cases:

Case 1: X_1 and X_2 are independent:

$$g(z) = 4\frac{\ln(1+z)}{(2+z)^3} + \frac{2z}{(1+z)(2+z)^2} \approx \frac{2}{(1+z)^2}, \ \ G(z) = \frac{z^2+2z-2\ln(1+z)}{(2+z)^2}, \ \ z \geq 0$$

Case 2: X_1 and X_2 are co-monotonic, i.e. the corresponding copula is the upper Fréchet bound $\mathcal{M}$:

$$g(z) = \frac{1}{2(1+z/2)^2} = \frac{2}{(2+z)^2} \approx \frac{2}{(1+z)^2}, \ \ G(z) = 1 - \frac{2}{2+z}, \ z \geq 0$$

Case 3: X_1 and X_2 are counter-monotonic, i.e. the corresponding copula is the lower Fréchet bound $\mathcal{W}$:

$$g(z) = \frac{2}{\sqrt{z-2}(2+z)^{3/2}} \approx \frac{2}{(1+z)^2}, \ \ G(z) = \sqrt{\frac{z-2}{z+2}}, \ \ z \geq 2.$$

Proof: In the independent case, we have

$$g(z) = \int_0^z f(x)f(z-x)\,dx = \int_0^z \frac{1}{(1+x)^2}\frac{1}{(1+z-x)^2}\,dx \ \text{ for } z > 0.$$

Let

$$h_a(x) := \frac{2}{(1+a)^3}\ln\left(\frac{1+x}{a-x}\right) + \frac{2x-a+1}{(1+a)^2(1+x)(a-x)} \text{ for } 0 \le x < a \text{ with } a > 0.$$

Then

$$h_a'(x) := \frac{1}{(1+x)^2(a-x)^2} \text{ for } 0 \le x < a.$$

Thus $h_{1+z}(x) := \frac{2}{(2+z)^3}\ln\left(\frac{1+x}{1+z-x}\right) + \frac{2x-z}{(2+z)^2(1+x)(1+z-x)}$ is the indefinite integral of $\frac{1}{(1+x)^2}\frac{1}{(1+z-x)^2}$ and hence we have

$$g(z) = \int_0^z \frac{1}{(1+x)^2}\frac{1}{(1+z-x)^2}dx = h_{1+z}(z) - h_{1+z}(0) = 4\frac{\ln(1+z)}{(2+z)^3} + \frac{2z}{(1+z)(2+z)^2} \text{ for } z > 0,$$

with $g(z) = \frac{d}{dz}\left(\frac{z^2+2z-2\ln(1+z)}{(2+z)^2}\right)$ from which the statement follows.

Case 2 is again trivial since the distribution of S_2 is identical to that of $2X_1$. Case 3 follows from the observation that for the cumulative distribution function *F*, we have

$$F(x) = \int_0^x \frac{1}{(1+y)^2}dy = 1 - \frac{1}{1+x},\ x \ge 0,$$

which in turn implies

$$F^{-1}(u) = \frac{1}{1-u} - 1,\ 0 < u < 1.$$

Hence $S_2 = X_1 + X_2$ is distributed as $F^{-1}(U) + F^{-1}(1-U) = \frac{1}{U} + \frac{1}{1-U} - 2$ for a standard uniformly distributed random variable *U*. Since the mapping $u \mapsto \frac{1}{u} + \frac{1}{1-u} - 2$ is again strictly convex and symmetric w.r.t. the point $u = 1/2$ with its minimum attained there (with value 2) we see that the feasible set of real valued solutions *u* for the inequality

$$\frac{1}{u} + \frac{1}{1-u} - 2 \le z,\ z \ge 2$$

is given by the compact interval

$$\left[u_0 := \frac{1}{2} - \frac{1}{2}\sqrt{\frac{z-2}{z+2}}, \frac{1}{2} + \frac{1}{2}\sqrt{\frac{z-2}{z+2}} =: u_1\right].$$

It follows again readily that

$$P(S_2 \le z) = P\left(\frac{1}{U} + \frac{1}{1-U} \le z+2\right) = u_1 - u_0 = \sqrt{\frac{z-2}{z+2}},\ z \ge 2.$$

The density given above again follows by differentiation.

The following graphs show the cumulative distribution functions and VaR's for S_2 in the three cases above.

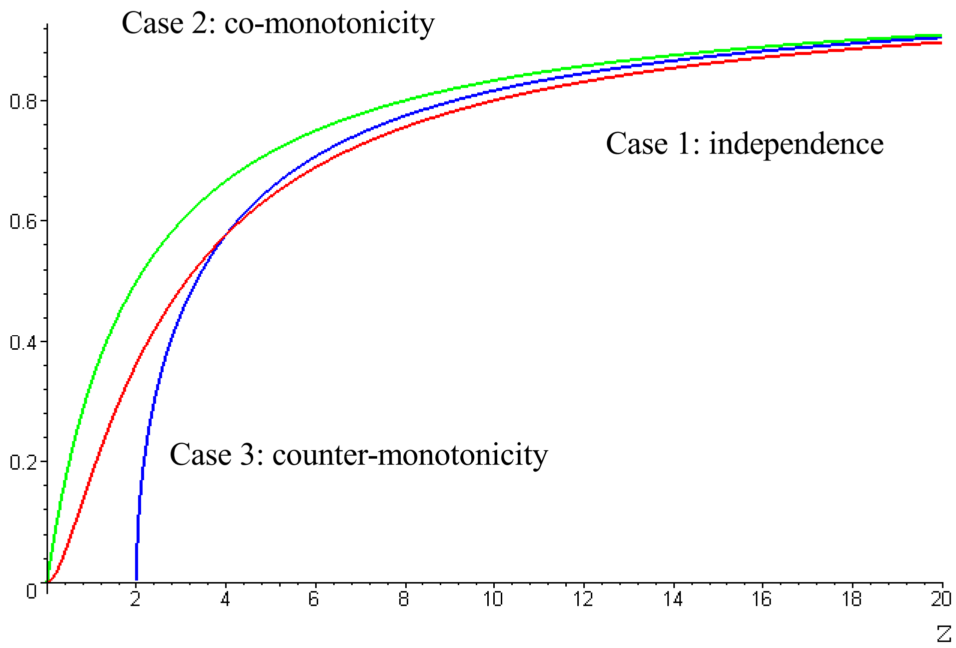

Fig. 13: cdf's for the three cases

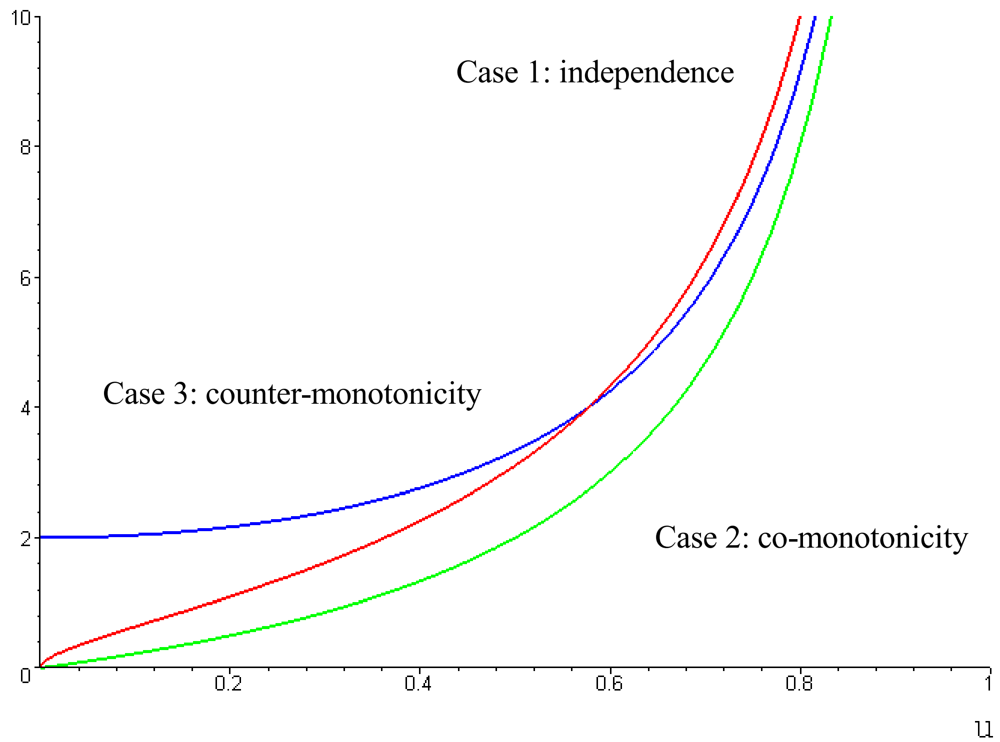

Fig. 14: VaR's for the three cases

Unlike above, Case 1 with independence produces the "worst" VaR scenario for large values of *u*, while Case 2 with co-monotonicity again corresponds to the "best" VaR scenario. Note, however, that there is exactly one intersection point between Case 1 and Case 3, and that all VaR scenarios are asymptotically equivalent. This will be discussed later in more detail.

Note also that in Case 1, no explicit representation of the VaR is possible. For the other two cases, we obtain:

Case 2: $\mathrm{VaR}_u = \dfrac{2u}{1-u} \sim \dfrac{2}{1-u} \quad (u \to 1)$

Case 3: $\mathrm{VaR}_u = \dfrac{1+u^2}{1+u} \cdot \dfrac{2}{1-u} \sim \dfrac{2}{1-u} \quad (u \to 1)$

for $0 < u < 1$.

Lemma 5.3. Suppose that the risks X_1 and X_2 follow a Pareto distribution with density

$$f(x) = \frac{2}{(1+x)^3}, \; x \ge 0$$

each. Then the density g and cumulative distribution function G of the aggregated risk $S_2 := X_1 + X_2$ can be explicitly calculated in the following cases:

Case 1: X_1 and X_2 are independent:

$$g(z) = \frac{48\ln(1+z)}{(2+z)^5} + \frac{4z\left(10+10z+z^2\right)}{(2+z)^4(1+z)^2} \approx \frac{4}{(1+z)^3},$$
$$G(z) = z\frac{z^3+7z^2+16z+6}{(2+z)^3(1+z)} - \frac{12}{(2+z)^4}\ln(1+z), \; z \ge 0$$

Case 2: X_1 and X_2 are co-monotonic, i.e. the corresponding copula is the upper Fréchet bound $\mathcal{M}$:

$$g(z) = \frac{8}{(2+z)^3} \approx \frac{8}{(1+z)^3}, \; G(z) = 1 - \frac{4}{(2+z)^2}, \; z \ge 0$$

Case 3: X_1 and X_2 are counter-monotonic, i.e. the corresponding copula is the lower Fréchet bound $\mathcal{W}$:

$$g(z) = \frac{4(2+z)}{\sqrt{z^2+4z+2-2\sqrt{z^2+4z+5}}\left(\sqrt{z^2+4z+5}-1\right)^2\sqrt{z^2+4z+5}} \approx \frac{4}{(1+z)^3},$$
$$G(z) = \frac{1}{(2+z)^2}\sqrt{(2+z)^4 - 4(2+z)^2 - 8 - 8\sqrt{(2+z)^2+1}}, \; z \ge 2.$$

Proof: In the independent case, we have

$$g(z)=\int_0^z f(x)f(z-x)\,dx=4\int_0^z \frac{1}{(1+x)^3}\frac{1}{(1+z-x)^3}\,dx \text{ for } z>0.$$

Let

$$h_a(x):=\frac{24}{(1+a)^5}\ln\left(\frac{1+x}{a-x}\right)+12\frac{2x-a+1}{(1+a)^4(1+x)(a-x)}+2\frac{2x-a+1}{(1+a)^2(1+x)^2(a-x)^2}$$

for $0\le x<a$ with $a>0$. Then

$$h_a'(x):=\frac{4}{(1+x)^3(a-x)^3} \text{ for } 0\le x<a.$$

Thus

$$h_{1+z}(x):=\frac{24}{(2+z)^5}\ln\left(\frac{1+x}{1+z-x}\right)+12\frac{2x-z}{(2+z)^4(1+x)(1+z-x)}+2\frac{2x-z}{(2+z)^2(1+x)^2(1+z-x)^2}$$

is the indefinite integral of $\frac{2}{(1+x)^3}\cdot\frac{2}{(1+z-x)^3}$ and hence we have

$$g(z)=4\int_0^z \frac{1}{(1+x)^3}\frac{1}{(1+z-x)^3}\,dx=h_{1+z}(z)-h_{1+z}(0)=48\frac{\ln(1+z)}{(2+z)^5}+\frac{24z}{(1+z)(2+z)^4}+\frac{4z}{(1+z)^2(2+z)^2}$$

$$=48\frac{\ln(1+z)}{(2+z)^5}+4z\frac{10+10z+z^2}{(1+z)^2(2+z)^4}$$

for $z>0$,

with $g(z)=\frac{d}{dz}\left(z\frac{z^3+7z^2+16z+6}{(2+z)^3(1+z)}-\frac{12}{(2+z)^4}\ln(1+z)\right)$ from which the statement follows.

Case 2 is again trivial since the distribution of S_2 is identical to that of $2X_1$. Case 3 follows from the observation that for the cumulative distribution function *F*, we have

$$F(x)=\int_0^x \frac{2}{(1+y)^3}\,dy=1-\frac{1}{(1+x)^2},\ x\ge 0,$$

which in turn implies

$$F^{-1}(u)=\frac{1}{\sqrt{1-u}}-1,\ 0<u<1.$$

Hence $S_2=X_1+X_2$ is distributed as $F^{-1}(U)+F^{-1}(1-U)=\frac{1}{\sqrt{U}}+\frac{1}{\sqrt{1-U}}-2$ for a standard uniformly distributed random variable *U*. Since the mapping $u\mapsto\frac{1}{\sqrt{u}}+\frac{1}{\sqrt{1-u}}-2$ is again strictly convex and symmetric w.r.t. the point $u=1/2$ with its minimum attained there (with value $2\sqrt{2}-2$) we see that the feasible set of real valued solutions *u* for the inequality

$$\frac{1}{\sqrt{u}}+\frac{1}{\sqrt{1-u}}-2\le z,\ \ z\ge 2\sqrt{2}-2$$

or, equivalently, by taking squares on both sides,

$$\frac{1}{u(1-u)}+\frac{2}{\sqrt{u(1-u)}}\le (2+z)^2,\ \ z\ge 2\sqrt{2}-2$$

is given by the compact interval $[u_0, u_1]$ with (substitute $u(1-u)=v$)

$$\begin{aligned}u_0 &:= \frac{1}{2}-\frac{1}{2(2+z)^2}\sqrt{(2+z)^4-4(2+z)^2-8-8\sqrt{(2+z)^2+1}},\\ u_1 &:= \frac{1}{2}+\frac{1}{2(2+z)^2}\sqrt{(2+z)^4-4(2+z)^2-8-8\sqrt{(2+z)^2+1}}.\end{aligned}$$

It follows again readily that

$$\begin{aligned}P(S_2\le z) &= P\left(\frac{1}{\sqrt{U}}+\frac{1}{\sqrt{1-U}}\le z+2\right)=u_1-u_0\\ &= \frac{1}{(2+z)^2}\sqrt{(2+z)^4-4(2+z)^2-8-8\sqrt{(2+z)^2+1}},\ \ z\ge 2.\end{aligned}$$

The density given above again follows by differentiation.

The following graphs show the cumulative distribution functions and VaR's for S_2 in the three cases above.

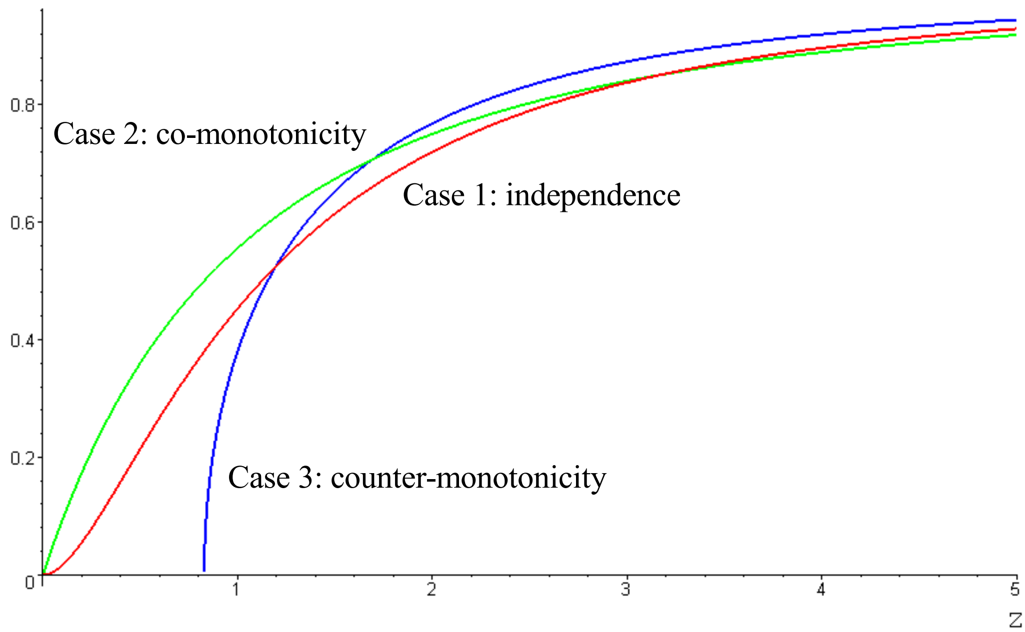

Fig. 15: cdf's for the three cases

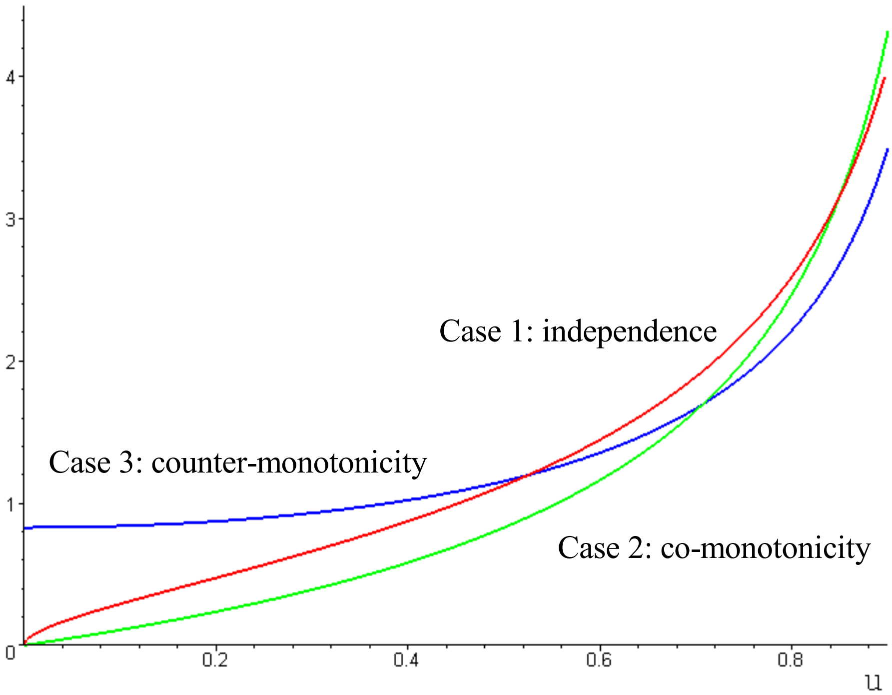

Fig. 16: VaR's for the three cases

Note that here Case 2 with co-monotonicity produces the "worst" VaR scenario for large values of u, while Case 3 with counter-monotonicity corresponds to the "best" VaR scenario. This is in accordance with intuition. Also, there is exactly one intersection point between Case 2 and Case 3 with Case 1.

Note also that in Case 1, again no explicit representation of the VaR is possible. For the other two cases, we obtain:

Case 2: $\text{VaR}_u = \dfrac{2}{\sqrt{1-u}} - 2$

Case 3: $\text{VaR}_u = \dfrac{2}{\sqrt{1-u}}\sqrt{\dfrac{1+\sqrt{1-u^2}}{1+u}} - 2 \sim \dfrac{\sqrt{2}}{\sqrt{1-u}} - 2 \ (u \to 1)$

for $0 < u < 1$.

The fact that pairwise intersections between the cumulative distribution functions and the quantile functions, resp. for the dependent cases and the independent case occur in Lemma 5.3 is due to the fact that the expectation of the risks exists here. Namely, if F and G are *different* cumulative distribution functions for non-negative risks with the same expectation, then

$$\int_0^\infty (1-F(x))\,dx = \int_0^\infty (1-G(x))\,dx.$$

Hence it is not possible that we can have $F(x) < G(x)$ or $F(x) > G(x)$ for all $x \in \mathbb{R}$, such that at least one intersection point between F and G must exist. [The examples in sections 3 and 4 show that we can even have arbitrarily many.] This implies also that a uniformly "worst" or "best" VaR scenario cannot exist in case of finite expectation, as was also pointed out in EMBRECHTS, HÖING AND PUCCETTI (2005). Note however, that in Lemma 5.2 this is possible, since all risks have infinite expectation here.

5. IMPLICATIONS FOR DFA AND SOLVENCY II

It should have become clear from the preceding discussion that neglecting dependencies between risks in an insurer's portfolio can lead to a substantial misspecification of the target or solvency capital, which is strongly related to the overall (aggregated) risk of the company. For example, in the Swiss Solvency Test (see e.g. KELLER AND LUDER (2004)), all typical insurance risks are considered to be independent, as becomes clear form the use of the word "convolution" everywhere.

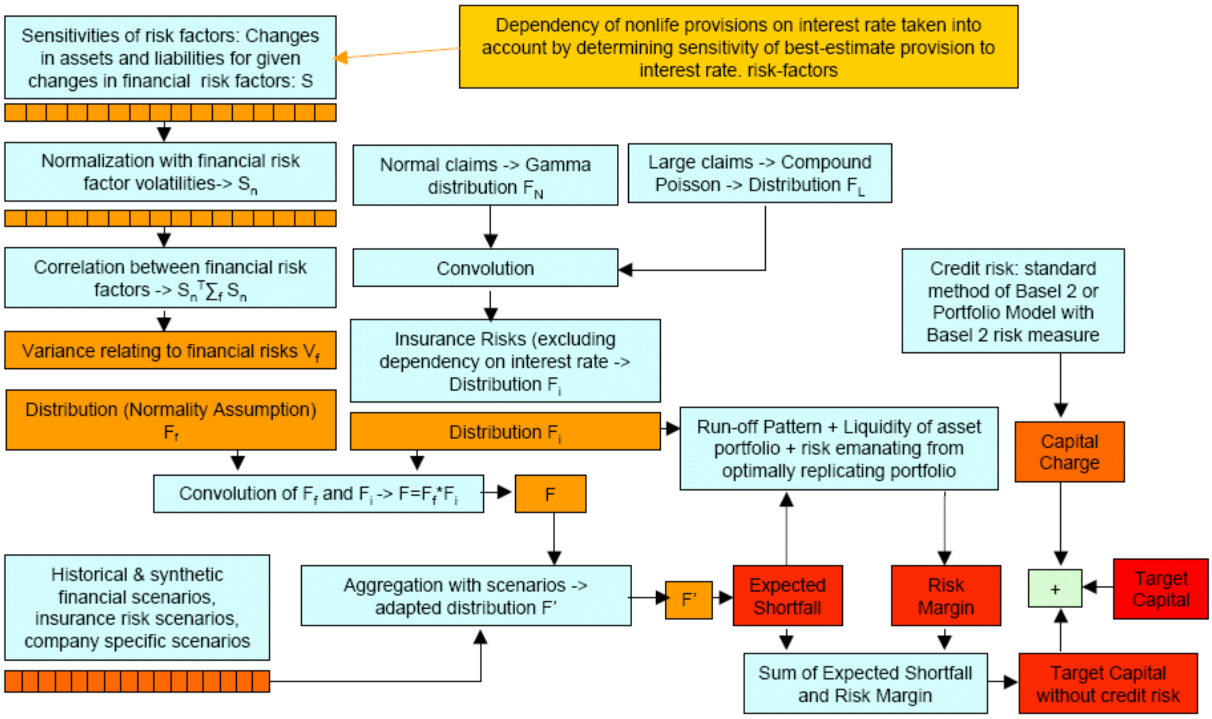

Fig. 17: scheme of the Swiss Solvency Test; source: Swiss Federal Office of Private Insurance, Bern, Switzerland

The present discussion about which risk measure should be used to calculate the target or solvency capital concentrates on VaR and Expected Shortfall mainly. However, both measures are heavily influenced by the underlying dependence structure, even in the case of uncorrelated risks, as has been shown in section 3. This is particularly crucial when natural perils such as windstorm, hailstorm, flooding, earthquakes and others are considered. The first mentioned hazards have sometime joint climatic triggers, which leads to dependencies especially of the larger losses, due to spatial or temporal dependence. The following graph shows a scatterplot pertaining to an empirical copula derived form a portfolio consisting of windstorm (U) and flooding (V) losses. It can be clearly seen that the corresponding copula is not symmetric, and that – due to a lack of data – there is not necessarily a tail dependence visible in the right upper part of the square.

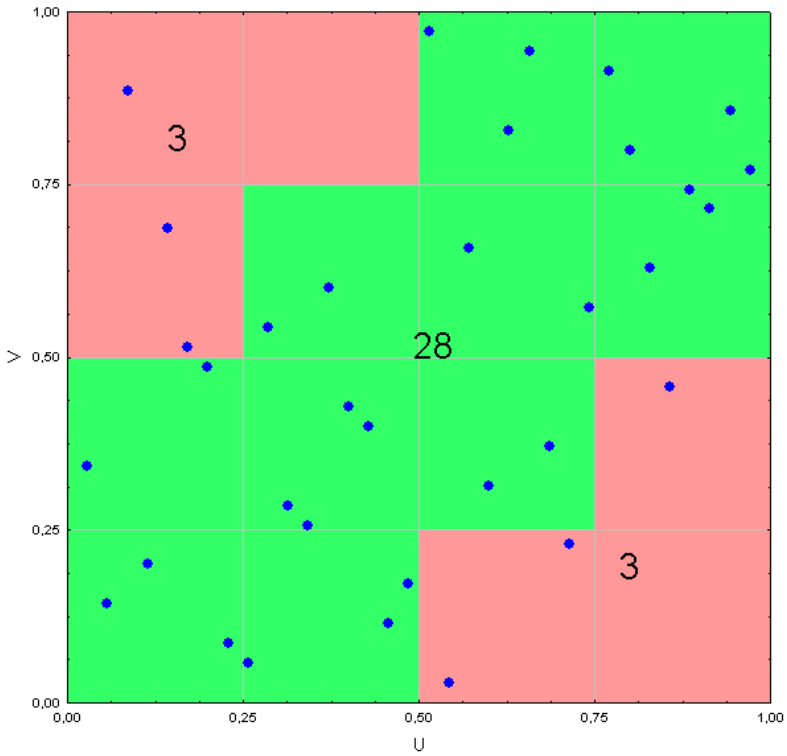

Fig. 17: empirical copula for windstorm vs. flooding

Performing a χ^2-test here with three cells (one marked in green and two in red), and hence 2 degrees of freedom, gives a test statistic of $T = 5,7176$ corresponding to a *p*-value of 0,0574. So it is reasonable to assume that there is some dependence between these risks. The data can be well fitted to a 4×4 grid-type copula represented by the following weight matrix (see section 3):

$$A = \frac{1}{136}\begin{bmatrix} 13 & 8 & 8 & 5 \\ 12 & 15 & 7 & 0 \\ 8 & 17 & 7 & 12 \\ 1 & 4 & 12 & 17 \end{bmatrix}$$

Note that if we relate the weights a_{ij} to the physical cells in the scatterplot above, we obtain the following picture:

a_{14}	a_{24}	a_{34}	a_{44}
a_{13}	a_{23}	a_{33}	a_{43}
a_{12}	a_{22}	a_{32}	a_{42}
a_{11}	a_{21}	a_{31}	a_{41}

To illustrate the usefulness of grid-type copulas, we assume for simplicity and purposes of comparison that the marginal distributions of windstorm and flooding are of the same Pareto type as in Lemma 5.3 above. The following graph shows the empirical quantile function for the aggregate risk from a Monte Carlo study with 100 000 simulations using this copula:

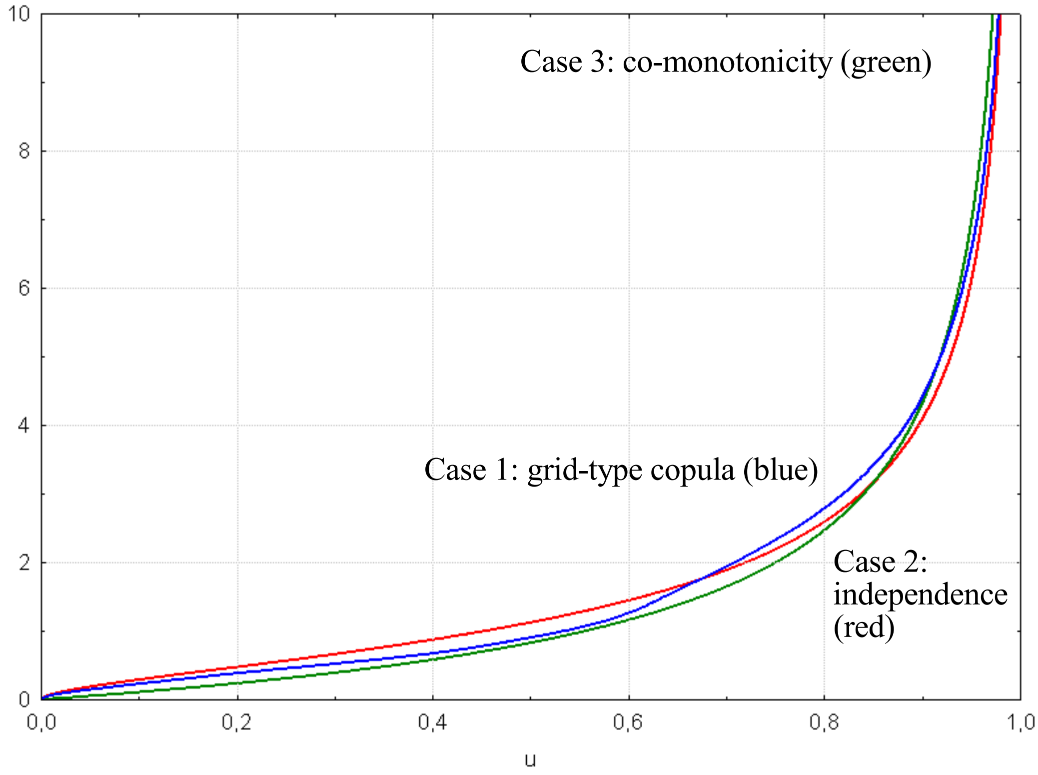

Fig. 18: VaR's for aggregate risk

It is clearly seen that the VaR for the aggregate risk under the grid-type copula is strictly below the VaR under independence for values of $u < 0,6$ but larger than the VaR under independence for values of $u > 0,7$. It is comparable to the VaR under co-monotonicity for values of u around 0,9.

Using grid-type copulas or related concepts of dependence can thus improve very much the reliability of estimations of the target or solvency capital in the Solvency II process.

Finally, it should be pointed out that most DFA tools such as commercial geophysical modeling software (see e.g. DONG (2001), GROSSI AND KUNREUTHER (2005), KATHER AND KUZAK (2002)) do not properly implement dependence structures but rely rather on correlation, which is crucial as was shown in section 2. The use of grid-type copulas or related dependence concepts could likewise improve the performance of such products. The mathematical background of the typical modeling approach used here is explicitly described in PFEIFER (2004). This approach is based on a special case of the classical collective risk model, consisting of Poisson distributed claim numbers and random or deterministic claim sizes. This is also an element of the Swiss Solvency Test (cf. Fig. 17). Possible constructions of dependencies within such type of structures have recently been described in PFEIFER AND NESLEHOVA (2004).

REFERENCES

[1] Blum, P., Dias, A. and Embrechts, P.: The ART of dependence modelling: the latest advances in correlation analysis. In: M. Lane (ed.): *Alternative Risk Strategies*, Risk Books, London (2002), 339 – 356.

[2] Dong, W. (2001): *Building a More Profitable Portfolio. Modern Portfolio Theory with Application to Catastrophe Insurance*. Reactions Publishing Group, London.

[3] Embrechts, P., Straumann, D. and McNeil, A.J.: Correlation: pitfalls and alternatives. In: P. Embrechts (ed.): *Extremes and Integrated Risk Management*. Risk Books, London (2000), 71 – 76.

[4] Embrechts, P., McNeil, A.J. and Straumann, D.: Correlation and dependence in risk management: properties and pitfalls. In: M.A.H. Dempster (ed.): *Risk Management: Value at Risk and Beyond*. Cambridge Univ. Press, Cambridge (2002), 176 – 223.

[5] Embrechts, P., Höing, A. and Juri, A.: Using Copulae to bound the Value-at-Risk for functions of dependent risks. Finance Stoch. 7 (2) (2003), 145 – 167.

[6] Embrechts, P., Höing, A. and Puccetti, G.: Worst VaR Scenarios. Insurance: Mathematics and Economics 37(1) (2005), 115-134.

[7] Cherubini, U., Luciano, E. and Vecchiato, W.: *Copula Methods in Finance*. Wiley, N.Y.

[8] Grossi, P. and Kunreuther, H. (ed).: *Catastrophe Modeling: A New Approach to Managing Risk*. Springer, N.Y. (2005).

[9] Khater, M. and Kuzak, D.E.: Natural catastrophe loss modelling. In: *Alternative Risk Strategies*, M. Lane, Ed.(2002). RISK Books, London, 271 – 299.

[10] Keller, P. and Luder. T.: White Paper of the Swiss Solvency Test. Swiss Federal Office of Private Insurance, Bern, Switzerland (2004).

[11] Nelsen, R.B.: *An Introduction to Copulas*. Lecture Notes in Statistics 139, Springer, N.Y. (1999).

[12] Pfeifer, D.: Solvency II: neue Herausforderungen an Schadenmodellierung und Risikomanagement? In: *Risikoforschung und Versicherung* - Festschrift für Elmar Helten zum 65. Geburtstag, Verlag Versicherungswirtschaft (2004), 467 – 481.

[13] Pfeifer, D. and Neslehova, J.: Modeling and generating dependent risk processes for IRM and DFA. ASTIN Bulletin Vol. 34, No. 2 (2004), 333 – 360.

[14] Tasche, D.: Expected Shortfall and beyond. Journal of banking and finance 26(7), 1519 – 1533.

[15] Uspensky, J.V.: *Introduction to Mathematical Probability*. McGraw-Hill, N.Y. (1937).

[16] Wüthrich, M.V.: Asymptotic Value-at-Risk estimates for sums of dependent random variables. ASTIN Bulletin 33 (1) (2003), 75 – 92.

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