ON THE RECURSIVE GENERATION OF MARKOV CHAINS

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ABSTRACT

Using the fact that by a suitable recursive generation from independent random processes with arbitrary image spaces Markov chains with regular transition probabilities are obtained, an easy construction of Markovian record processes is presented where the after-record distributions may depend on the last record value as well as the last inter-record time.

INTRODUCTION

Let $\{\xi_n;\ n \geq 0\}$ be a sequence of independent random processes on a common probability space $(\Omega,\mathcal{A},P)$ with arbitrary image spaces $(\mathcal{X}_n,\mathcal{B}_n)$, $n \geq 0$, and let $\{f_n;\ n \geq 1\}$ be a sequence of measurable functions

$$f_{n+1} : (\mathcal{Y}_n \times \mathcal{X}_{n+1}, \mathcal{D}_n \otimes \mathcal{B}_{n+1}) \to (\mathcal{Y}_{n+1}, \mathcal{D}_{n+1})$$

where $(\mathcal{Y}_n,\mathcal{D}_n)$, $n \geq 0$ are arbitrary measurable spaces with $(\mathcal{Y}_0,\mathcal{D}_0) = (\mathcal{X}_0,\mathcal{B}_0)$. Define a sequence $\{X_n;\ n \geq 0\}$ of random processes on $(\Omega,\mathcal{A})$ recursively by

(1) $$X_0 = \xi_0;\ X_{n+1} = f_{n+1}(X_n,\xi_{n+1}),\ n \geq 0.$$

Then we can prove the following result.

Theorem. $\{X_n;\ n \geq 0\}$ is a Markov chain with regular transition probabilities

(2) $P(X_{n+1} \in D \mid X_n = x) = P(f_{n+1}(x,\xi_{n+1}) \in D),\ D \in \mathscr{D}_{n+1},\ x \in \mathscr{Y}_n.$

Proof of the Theorem. Let simply $X = (X_0,\dots,X_n)$, $Y = \xi_{n+1}$, $A = \mathscr{Y}_0 \times \dots \times \mathscr{Y}_{n-1} \times f_{n+1}^{-1}(D)$. Since X and Y are independent (note that X is a function of $\xi_0,\dots,\xi_n$ alone), we have (let $x = (x_0,\dots,x_n)$)

$$P(X_{n+1} \in D \mid X_0 = x_0,\dots,X_n = x_n) = P(f_{n+1}(X_n,\xi_{n+1}) \in D \mid X = x)$$

$$= P((X,Y) \in A \mid X = x) = P((x,Y) \in A) = P(f_{n+1}(x_n,\xi_{n+1}) \in D) \text{ a.s. } P^X.$$

Hence $\{X_n;\ n \geq 0\}$ is a Markov chain with regular transition probabilities.

APPLICATIONS

In this section, we present an extension of the record model which has recently been introduced by the author (1982), allowing the after-record distributions to depend on the last record value as well as the last inter-record time. The resulting two-dimensional record process of inter-record times and record values then turns out to be a Markov chain. In case that the after-record distribution depends on the last record value alone the record value sequence will also be a Markov chain. It is shown that this property does not hold in the general case.

Formally, let $P_{n,t}$ ($n \in \mathbb{N}$, $t \in \mathbb{R}^1$) be probability measures on $(\mathbb{R}^1, \mathscr{B}^1)$, measurable with respect to t. Then there exist independent processes

$$\xi_0 = (0, \xi_{00})$$

$$\xi_1 = \{t_1, \xi_{11}(t), \xi_{12}(t), \dots, \xi_{1\infty}(t) \equiv \infty \mid t \in \mathbb{R}^1\}$$

$$\vdots$$

$$\xi_n = \{t_n, \xi_{n1}(t), \xi_{n2}(t), \dots, \xi_{n\infty}(t) \equiv \infty \mid t \in \mathbb{R}^1\}$$

$$\vdots$$

on a suitable probability space $(\Omega,\mathcal{A},P)$ with independent components, measurable w.r.t. t, and

$$P^{\xi_{nk}(t)} = P_{n,t}\ ,\ k \in \mathbb{N}$$

where $t_1,t_2,\ldots$ are real-valued random variables on $(\overline{\mathbb{N}} \times \overline{\mathbb{R}}^1, \mathcal{P}(\overline{\mathbb{N}})\otimes\overline{\mathcal{B}}^1)$ $(\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\},\ \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\})$. Define a record process $\{(\Delta_n,R_n);\ n \geq 0\}$ by

$$(3)\qquad (\Delta_0,R_0) = \xi_0;\ (\Delta_{n+1},R_{n+1}) = f((\Delta_n,R_n);\ \xi_{n+1})$$
$$= \Big(\min\ \{k \in \mathbb{N} \mid \xi_{n+1,k}(t_{n+1}(\Delta_n,R_n)) > R_n\}$$
$$\equiv \Delta_{n+1},\xi_{n+1,\Delta_{n+1}}(t_{n+1}(\Delta_n,R_n))\Big)$$

where $\min(\emptyset) = \infty$.

This means that after the observation of an inter-record time $\Delta_n = k$ and a record value $R_n = x$ a strategy $t_{n+1}(k,x)$ is chosen such that the subsequent observations are following a distribution given by $P_{n+1,t_{n+1}(k,x)}$ (cf. also the example given in Pfeifer (1982)).

Now by the Theorem, $\{(\Delta_n,R_n);\ n \geq 0\}$ is a Markov chain with regular transition probabilities

$$P(\Delta_{n+1} = m,\ R_{n+1} \leq y \mid \Delta_n = k,\ R_n = x) = P(f((k,x);\ \xi_{n+1}) \in \{m\} \times (-\infty,y])$$
$$= P(\max_{1\leq j<m} \xi_{n+1,j}(t_{n+1}(k,x)) \leq x < \xi_{n+1,m}(t_{n+1}(k,x)) \leq y)$$
$$= \{P_{n+1,t_{n+1}(k,x)}((-\infty,x])\}^{m-1} P_{n+1,t_{n+1}(k,x)}((x,y]);\ m \in \mathbb{N},\ y \geq x.$$

Hence the transition functions for the record process are given by (let $Q_{n,k,x} = P_{n,t_n(k,x)}$)

$$(4)\qquad P_{n-1,n}(k,x \mid A \times B) = Q_{n,k,x}((x,\infty) \cap B) \sum_{j\in A} \{Q_{n,k,x}((-\infty,x])\}^{j-1},$$
$$A \subseteq \mathbb{N},\ B \in \mathcal{B}^1,\ k \in \mathbb{N},\ x \in \mathbb{R}^1.$$

Corollary.

a) Δ_{n+1} and R_{n+1} are conditionally independent given (Δ_n, R_n).

b) If t_n does not depend on the first argument, $\{R_n;\ n \geq 0\}$ is a Markov chain with transition functions

$$(5) \qquad P_{n-1,n}(x|B) = Q_{n,\cdot,x}(B|(x,\infty));\ B \in \mathscr{B}^1,\ x \in \mathbb{R}^1.$$

In this case, $\Delta_1,\dots,\Delta_{n+1}$ also are conditionally independent given $R_0,\dots,R_n$ with

$$(6) \qquad P(\bigcap_{i=1}^{n+1}\{\Delta_i = k_i\}|R_0 = x_0,\dots,R_n = x_n)$$

$$= \prod_{i=1}^{n+1} Q_{i,\cdot,x_{i-1}}((x_{i-1},\infty))\{Q_{i,\cdot,x_{i-1}}((-\infty,x_{i-1}]))\}^{k_i-1},$$

$$k_1,\dots,k_{n+1} \in \mathbb{N},\ x_0,\dots,x_n \in \mathbb{R}^1.$$

c) If t_n depends on the first argument, $\{R_n;\ n \geq 0\}$ will in general *not* be a Markov chain since

$$(7) \qquad P(R_2 \in B|R_1 = x,\ R_0 = y)$$

$$= \int P(R_2 \in B|R_1 = x,\ \Delta_1 = k,\ R_0 = y)P^{\Delta_1}(dk|R_1 = x,\ R_0 = y)$$

$$= \sum_{k=1}^{\infty} P(R_2 \in B|R_1 = x,\ \Delta_1 = k)P(\Delta_1 = k|R_0 = y)$$

by a) which is in general not independent of y; $B \in \mathscr{B}^1$, $x,y \in \mathbb{R}^1$.

Note that if $U_n = 1 + \sum_{k=1}^{n} \Delta_k$, $n \geq 0$ denotes the n-th record time, under the conditions of b) the process $\{(U_n,R_n);\ n \geq 0\}$ forms a Markov additive chain (see Çinlar (1972), Pfeifer (1982)). This fact provides another immediate proof of (5) and (6).

Remark.

The construction presented above is not only restricted to ordinary record values. For instance, let $S_n : \overline{\mathbb{R}}^1 \to \overline{\mathcal{B}}^1$, $n \geq 1$ be suitable mappings; define

$$(8)\qquad (\Delta_0, R_0) = \xi_0;\ (\Delta_{n+1}, R_{n+1}) = f_{n+1}((\Delta_n, R_n);\ \xi_{n+1})$$

$$= \Big(\min\{k \in \mathbb{N} \mid \xi_{n+1,k}(t_{n+1}(\Delta_n, R_n)) \in S_{n+1}(R_n)\}$$

$$\equiv \Delta_{n+1},\ \xi_{n+1,\Delta_{n+1}}(t_{n+1}(\Delta_n, R_n))\Big).$$

Then $\{(\Delta_n, R_n);\ n \geq 0\}$ is a Markov chain with transition probabilities (cf. (4))

$$(9)\qquad P_{n-1,n}(k,x|A \times B) = Q_{n,k,x}(S_n(x) \cap B) \sum_{j \in A} \{1 - Q_{n,k,x}(S_n(x))\}^{j-1},$$

and the Corollary holds with (x,∞) being replaced by $S_n(x)$. For instance, if additionally security margins $\varepsilon_n > 0$ are built in after every "record" event, take $S_n(x) = (x + \varepsilon_n, \infty)$.

REFERENCES

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