Model Validation with Q-Q-Plots under Solvency II

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Agenda

1. Motivation

Commission Delegated Regulation (EU) Article 306:
Own-Risk and **S**olvency **A**ssessment Supervisory Report

The ORSA supervisory report shall present all of the following:

(a) the qualitative and quantitative results of the own risk and solvency assessment and the conclusions drawn by the insurance or reinsurance undertaking from those results;

(b) the methods and main assumptions used in the own risk and solvency assessment;

(c) information on the undertaking's overall solvency needs and a comparison between those solvency needs, the regulatory capital requirements and the undertaking's own funds;

(d) qualitative information on, and where significant deviations have been identified, a quantification of the extent to which quantifiable risks of the undertakings are not reflected in the calculation of the Solvency Capital Requirement.

1. Motivation

With this presentation, we refer in particular to aspects (b) and (d) in the Delegated Regulation in order to make a visual check about the distributional assumptions in Pillar I of Solvency II via Q-Q-plots mathematically more rigorous.

The concrete background for this investigation is a controversial correspondence with the German regulator BaFin concerning the reliability of visual Q-Q-plots in the ORSA. So we will here mainly concentrate on Q-Q-plots for normally distributed random variables, since lognormal risks (as assumed in pillar I of Solvency II for combined ratios) can be transformed to normal risks by applying logarithms.

2. Q-Q-Plots in Location-Scale Families

Formal framework: We consider risks X of the form $X = \mu + \sigma Z$ where Z denotes a standard risk with a continuous, strictly increasing cdf F_Z (location-scale family), with $\mu \in \mathbb{R}$ and $\sigma > 0$. $Q_Z = F_Z^{-1}$ defines the corresponding standard quantile function. Then $Q_X = \mu + \sigma Q_Z$.

Let in general $Y_{(k)}$ denote the k-largest order statistic in a series of n i.i.d. observations $Y_1, \cdots, Y_n$. A quantile-quantile-plot in a location-scale family (Q-Q-plot) is a scatterplot of the points $\left(Q_Z(u_k), X_{(k)}\right)$, $k = 1, \cdots, n$ for n i.i.d. observations $X_1, \cdots, X_n$ distributed as X where $0 < u_1 < \cdots < u_n < 1$ are chosen appropriately (so called "plotting positions").

2. Q-Q-Plots in Location-Scale Families

In case that the data are actually replicates of X, the Q-Q-plot should approximately show a linear structure. This is due to the fact that if we generate the replicates of the risk X via $X_k := \mu + \sigma Q_Z(U_k)$, $k = 1, \cdots, n$ where $U_1, \cdots, U_n$ are standard uniform random numbers, then the points $\left(Q_Z(U_{(k)}), X_{(k)}\right)$ are laying precisely on a straight line with slope σ and intercept μ.

This is historically the basis for a visual test ("probability paper") whether the distributional assumptions on the risk X are justified or not (see Gumbel (1958) or David and Nagaraja (2003) for a survey).

If the points $\left(Q_Z(u_k), X_{(k)}\right)$ deviate „significantly" by eye from a straight line the distributional assumptions on the risk X should be rejected.

2. Q-Q-Plots in Location-Scale Families

The empirical slope $\hat{\sigma}$ and intercept $\hat{\mu}$ for the corresponding regression line are given by

$$\hat{\sigma} = \frac{\frac{1}{n}\sum_{k=1}^{n} X_{(k)} Q_Z(u_k) - \left(\frac{1}{n}\sum_{k=1}^{n} X_{(k)}\right)\cdot\left(\frac{1}{n}\sum_{k=1}^{n} Q_Z(u_k)\right)}{\frac{1}{n}\sum_{k=1}^{n} Q_Z^2(u_k) - \left(\frac{1}{n}\sum_{k=1}^{n} Q_Z(u_k)\right)^2}$$

and

$$\hat{\mu} = \frac{1}{n}\sum_{k=1}^{n} X_{(k)} - \frac{\hat{\sigma}}{n}\sum_{k=1}^{n} Q_Z(u_k).$$

2. Q-Q-Plots in Location-Scale Families

Proposition: $\hat{\sigma}$ and $\hat{\mu}$ from the Q-Q-plot are unbiased estimators for the scale and location parameters σ and μ if we choose $u_k = F_Z\left(E\left(Z_{(k)}\right)\right)$ for $k = 1,\cdots,n$.

Formally, we have

$$E\left(Z_{(k)}\right) = k\binom{n}{k}\int_{-\infty}^{\infty} x\,F_Z^{k-1}(x)\left(1-F_Z(x)\right)^{n-k}\,dF_Z(x) \text{ for } k = 1,\cdots,n$$

which in general cannot be calculated explicitly. For the particular case of a normally distributed standard risk Z, Harter (1961) has published extensive numerical tables.

2. Q-Q-Plots in Location-Scale Families

Historically, the plotting positions u_k for a normal Q-Q-plot have been described in the form $u_k = \frac{k-a}{n+b}$, $k = 1, \cdots, n$ with $a, b > 0$ which is independent of *n*. This is - in general - only a crude approximation. Here is a historical list of suggestions:

2. Q-Q-Plots in Location-Scale Families

Hazen (1914)	$u_k = \dfrac{k-0{,}5}{n+1}$
Weibull (1939)	$u_k = \dfrac{k}{n+1}$
Beard (1943)	$u_k = \dfrac{k-0{,}31}{n+0{,}38}$
Benard and Bos-Levenbach (1953)	$u_k = \dfrac{k-0{,}30}{n+0{,}20}$
Blom (1958)	$u_k = \dfrac{k-0{,}375}{n+0{,}25}$
Tukey (1962)	$u_k = \dfrac{k-0{,}333}{n+0{,}333}$
Gringorten (1963)	$u_k = \dfrac{k-0{,}44}{n+0{,}12}$

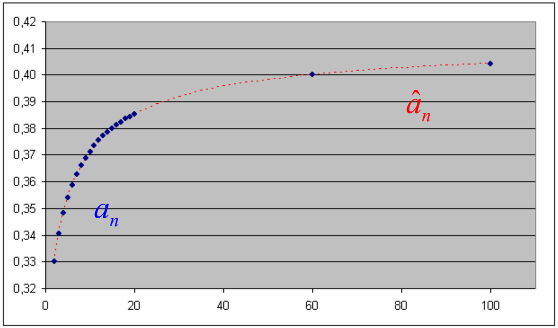
2. Q-Q-Plots in Location-Scale Families

As has been shown by Pfeifer (2019) a much better approximation is given by the choice $u_k = \dfrac{k-\hat{a}_n}{n+\hat{b}_n}$, $k = 1,\cdots,n$ with $\hat{a}_n, \hat{b}_n > 0$ where

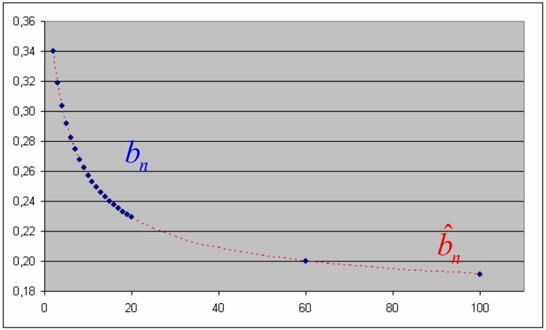
$$\hat{a}_n = 0{,}27950585 + \frac{0{,}04684273}{0{,}34986981 + n^{-0{,}79499457}},$$
$$\hat{b}_n = 0{,}44480354 - \frac{0{,}09890767}{0{,}36353365 + n^{-0{,}78493983}},\ k = 1,\ldots,n,\ n \le 100.$$

The following graph shows a comparison of the approximate $\hat{a}_n, \hat{b}_n > 0$ with the true values of $a_n, b_n > 0$.

2. Q-Q-Plots in Location-Scale Families



a_n vs. $\hat{a}_n$



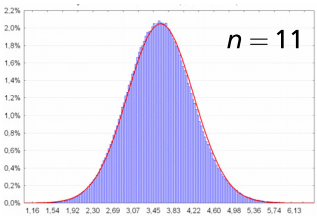
b_n vs. $\hat{b}_n$

Asymptotically, we have $\lim_{n\to\infty} \hat{a}_n = 0{,}413392$ and $\lim_{n\to\infty} \hat{b}_n = 0{,}172730$ which is close to Gringorten's (1963) suggestion.

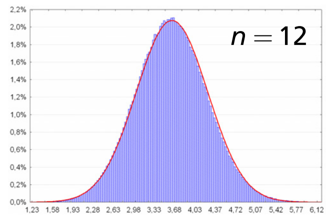
3. A Correlation-based GoF Test with Q-Q-Plots

Here we use a slight modification of a test procedure suggested by Lockhard and Stephens (1998). As a test statistic T for the simple significance goodness-of-fit test we use $T = -\ln(1-\rho)$ where ρ is the empirical correlation coefficient between the $Q_Z(u_k)$ and the $X_{(k)}$, with the plotting positions u_k above. Note that the correlation coefficient is independent of the true location and scale parameters μ and σ. By a larger Monte Carlo study (Pfeifer (2019)) is has turned out that the distribution of T in the normal model is itself approximately normally distributed under the null hypothesis with mean μ_T and standard deviation σ_T, for $n \geq 10$. The following graphs show some histograms of simulated T-values in comparison with a normal density. The simulation size was 1 Mio. observations.

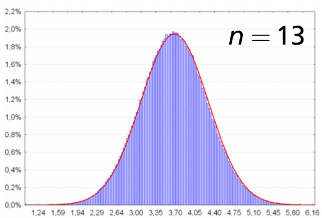
3. A Correlation-based GoF Test with Q-Q-Plots



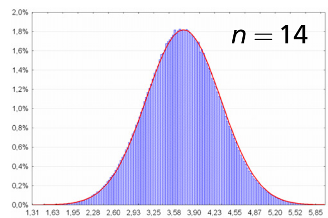
$\mu_T = 3{,}5727;\ \sigma_T = 0{,}6201$



$\mu_T = 3{,}6219;\ \sigma_T = 0{,}6103$

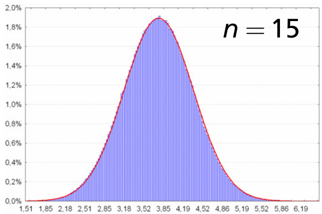


$\mu = 3{,}6696;\ \sigma = 0{,}6005$

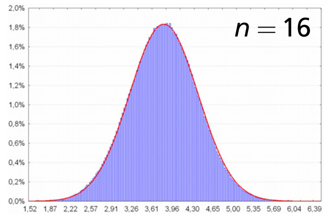


$\mu = 3{,}7152;\ \sigma = 0{,}5935$

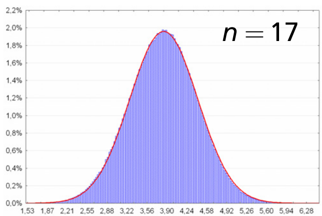
3. A Correlation-based GoF Test with Q-Q-Plots



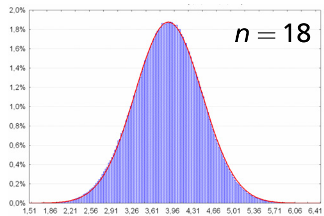
$\mu_T = 3{,}7584;\ \sigma_T = 0{,}5873$



$\mu_T = 3{,}7998;\ \sigma_T = 0{,}5813$



$\mu_T = 3{,}8385;\ \sigma_T = 0{,}5767$



$\mu_T = 3{,}8773;\ \sigma_T = 0{,}5730$

3. A Correlation-based GoF Test with Q-Q-Plots

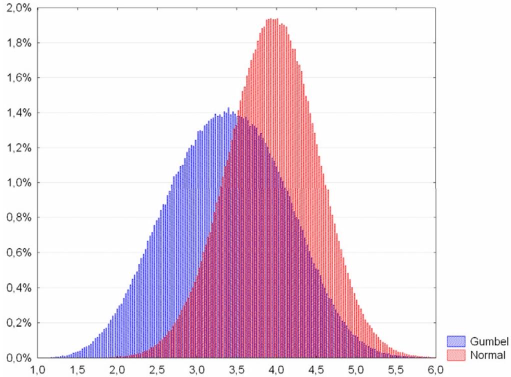
A good numerical approximation to μ_T and σ_T in the range $n = 10, \cdots, 50$ is

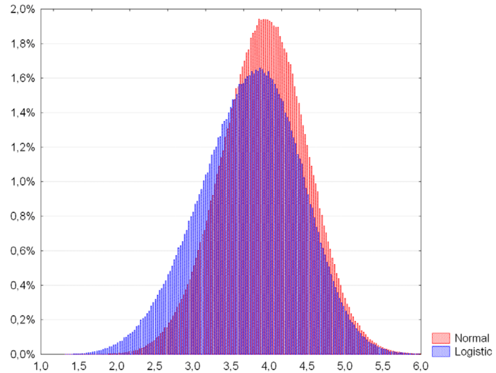
$$\hat{\mu}_T = \frac{5{,}87383n + 101{,}011}{n + 35{,}3404} \quad \text{and} \quad \hat{\sigma}_T = \frac{0{,}477812n + 3{,}25495}{n + 2{,}72721}.$$

The p-value for the correlation test can hence be calculated approximately by the formula $\Phi\left(\dfrac{T - \hat{\mu}_T}{\hat{\sigma}_T}\right)$ with the standard normal cdf Φ.

The following graphs show the test selectivity between a normal and a Gumbel, and a normal and a logistic distribution model, for $n = 20$. The plots represent histograms of the T distribution under the three different models with the same normal $Q_Z(u_k)$. We also show a table with corresponding errors of second kind (β) versus errors of first kind (α).

3. A Correlation-based GoF Test with Q-Q-Plots



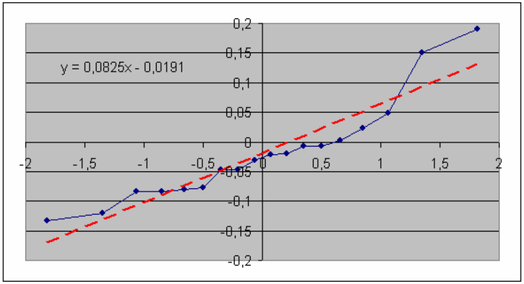


α	1%	5%	10%
critical T-value	2,6180	3,0045	3,2159
β Gumbel	84,70%	69,31%	59,11%
β logistic	94,92%	86,73%	78,84%

4. A Case Study from the German Insurance Market

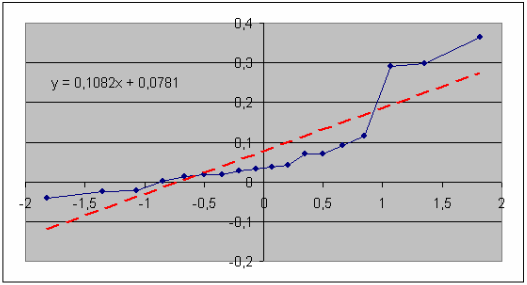
The following analysis is based on statistical data for gross combined ratios from past years from the German insurance market published by the German Insurance Federation GDV in 2018. We show the corresponding Q-Q-plots for the logarithmic combined ratios together with the approximate p-values of the correlation test.

4. A Case Study from the German Insurance Market



property

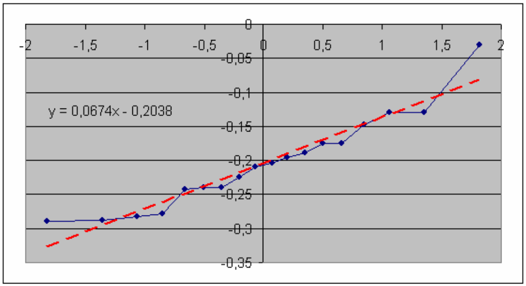
$T = 2{,}8831 \quad p = 4{,}32\%$



building

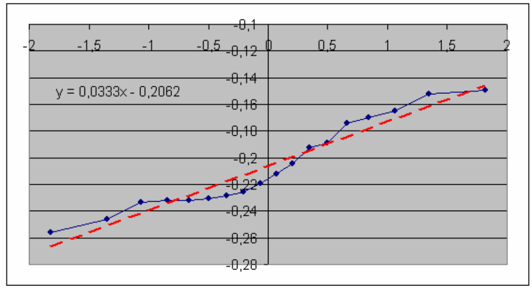
$T = 2{,}1378 \quad p = 0{,}11\%$

4. A Case Study from the German Insurance Market



content

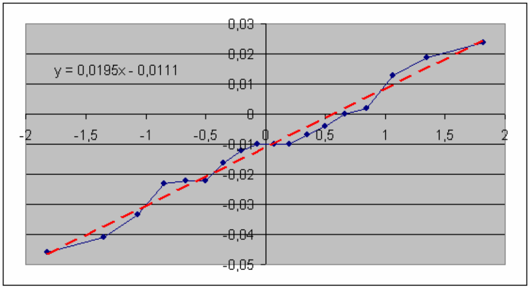
$T = 3{,}3515 \quad p = 17{,}95\%$



accident

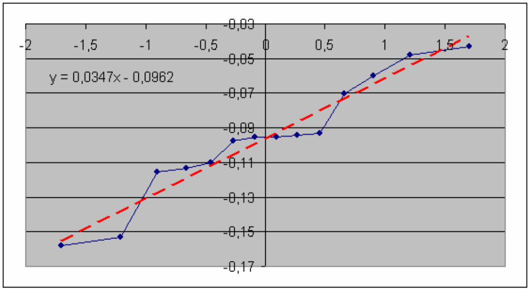
$T = 3{,}5621 \quad p = 26{,}76\%$

4. A Case Study from the German Insurance Market



legal expenses

$T = 4{,}6539 \quad p = 91{,}30\%$



liability

$T = 3{,}9443 \quad p = 64{,}96\%$

5. References

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