VaR vs. Expected Shortfall

Risk Measures under Solvency II

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- *Risk measures and premium principles – a comparison*
- *VaR vs. Expected Shortfall*
- *Dependence and its implications for risk measures*
- *A (frightening?) example from theory*
- *A (frightening?) example from the real world*
- *Conclusions*

A *premium principle H* resp. a *risk measure R* is a non-negative mapping on $\mathcal{Z}$, the set of non-negative *risks*, with the property

$$P^X = P^Y \;\Rightarrow\; H(X) = H(Y) \quad \text{for all } X, Y \in \mathcal{Z},$$

i.e. the premium resp. the risk measure depends only on the *distribution* of the risk.

Example: $H(X) = R(X) = E(X)$

[Expected risk; *net risk premium; average risk*]

A premium principle H is called

positively loaded (pl), if:

$$H(X) \geq E(X) \text{ for all } X \in \mathcal{Z};$$

positively homogeneous (ph), if:

$$H(cX) = c\,H(X) \text{ for all } c \geq 0 \text{ and } X \in \mathcal{Z};$$

additive (ad), if:

$$H(X+Y) = H(X) + H(Y) \text{ for all } X, Y \in \mathcal{Z},$$

with X, Y being *stochastically independent*;

total loss bounded (tb), if:

$$H(X) \le \varpi_X := \sup\{x \in \mathbb{R} \mid F_X(x) < 1\} \text{ for all } X \in \mathcal{Z},$$

where F_X denotes the *cumulative distribution function* of *X*;

stochastically increasing (si), if:

$$H(X) \le H(Y) \text{ for all } X, Y \in \mathcal{Z}$$

being *stochastically ordered*, i.e. $F_X(x) \ge F_Y(x)$ for all $x \in \mathbb{R}$.

Remark: Each *additive* premium principle *H* also is *translation invariant (ti)*, i.e.

$$H(X + c) = H(X) + c \text{ for all } X \in \mathcal{Z} \text{ and } c \in \mathbb{R}.$$

Let $\delta \geq 0$. Then the premium principle H with

$$H(X) = (1+\delta)E(X), \ X \in \mathcal{Z}$$

is called *expectation principle (ExP)* with safety loading factor δ.

Let $\delta \geq 0$. If the variance $Var(X)$ exists, then the premium principle H with

$$H(X) = E(X) + \delta Var(X), \ X \in \mathcal{Z}$$

is called *variance principle (VaP)* with safety loading factor δ.

Let $\delta \geq 0$. If the variance $Var(X)$ exists, then the premium principle H with

$$H(X) = E(X) + \delta \sqrt{Var(X)}, \ X \in \mathcal{Z}$$

is called *standard deviation principle (StP)* with safety loading factor δ.

Let $g:\mathbb{R}^+ \to \mathbb{R}^+$ be strictly increasing and convex. Then the premium principle H with

$$H(X)=g^{-1}\big(E\big(g(X)\big)\big),\ \ X\in\mathcal{Z}$$

is called *mean value principle (MvP)* w.r.t. g.

Let $g:\mathbb{R}^+ \to \mathbb{R}^+$ be strictly increasing. Then the premium principle H with

$$H(X)=\frac{E\big[X\cdot g(X)\big]}{E\big[g(X)\big]},\ \ X\in\mathcal{Z}$$

is called *Esscher principle (EsP)* w.r.t g.

Let $\alpha \in (0,1)$. Then the premium principle H with

$$H(X) = F_X^{-1}(1-\alpha) = \inf\left\{x \in \mathbb{R}^+ \mid F_X(x) \geq 1-\alpha\right\}, \quad X \in \mathcal{Z}$$

is called *percentile principle (PcP)* at risk level α.

This premium principle is also known as

Value at Risk at risk level α

resp. as VaR_α $(\rightarrow PML)$.

1. Risk measures and premium principles – a comparison

property / principle	*pl*	*ph*	*ad*	*tb*	*si*
ExP	yes	yes	yes	no	yes
VaP	yes	no	yes	no	no
StP	yes	yes	no	no	no
MvP	yes	no	no	yes	yes
EsP	yes	no	no	yes	no
PcP (VaR)	no	yes	no	yes	yes

Each *premium principle* can in principle also be considered as a *risk measure*. However, more specific attributes of *risk measures* have been developed in the context of the solvency capital for banking and insurance companies (→ Basel II, Solvency II).

A *risk measure* R is called *coherent* according to ARTZNER, DELBAEN, EBER und HEATH, if it possesses the following properties:

- R is *positively homogeneous (ph)*, i.e.

$$R(cX) = c\,R(X) \text{ for all } c \geq 0 \text{ and } X \in \mathcal{Z};$$

- R is *translation invariant (ti)* , i.e.

$$R(X+c) = R(X) + c \text{ for all } X \in \mathcal{Z} \text{ and } c \in \mathbb{R};$$

- R ist *sub-additive (sa)*, i.e.

$$R(X+Y) \leq R(X) + R(Y) \text{ for all } X, Y \in \mathcal{Z};$$

- R is *increasing (in)*, i.e.

$$R(X) \leq R(Y) \text{ for all } X, Y \in \mathcal{Z} \text{ with } X \leq Y.$$

2. VaR vs. Expected Shortfall

The international discussion of risk measures as a basis for the determination of the target capital for Solvency II (IAA, DAV, SST) concerning the total risk $S=\sum_{i=1}^{n} X_i$ (from assets and liabilities) clearly focuses on

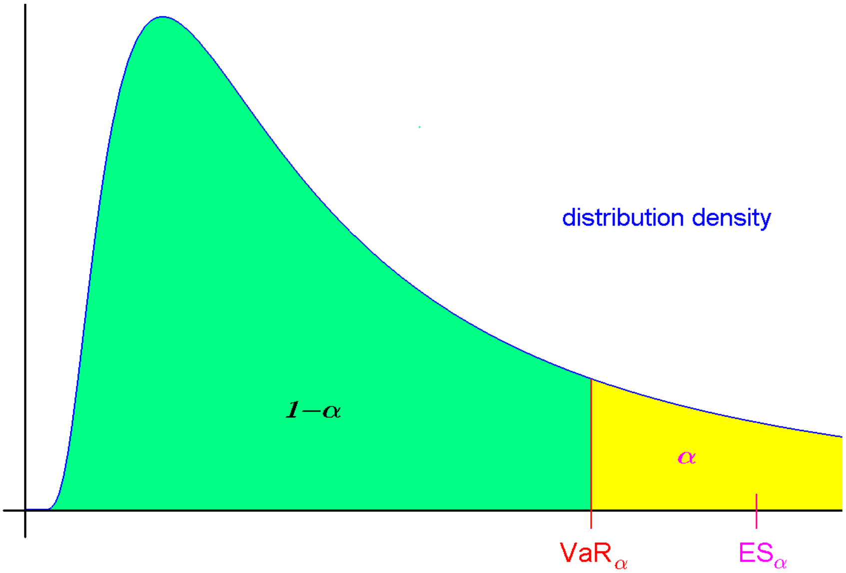
$$\textbf{Value at Risk: } VaR_\alpha = F_S^{-1}(1-\alpha) = \inf\left\{x \in \mathbb{R}^+ \mid F_S(x) \geq 1-\alpha\right\}$$

(→ *Life Insurance*) and

$$\textbf{Expected Shortfall: } ES_\alpha = E\left(S \mid S > VaR_\alpha\right)$$

(→ *Nonlife Insurance*).

2. VaR vs. Expected Shortfall



2. VaR vs. Expected Shortfall

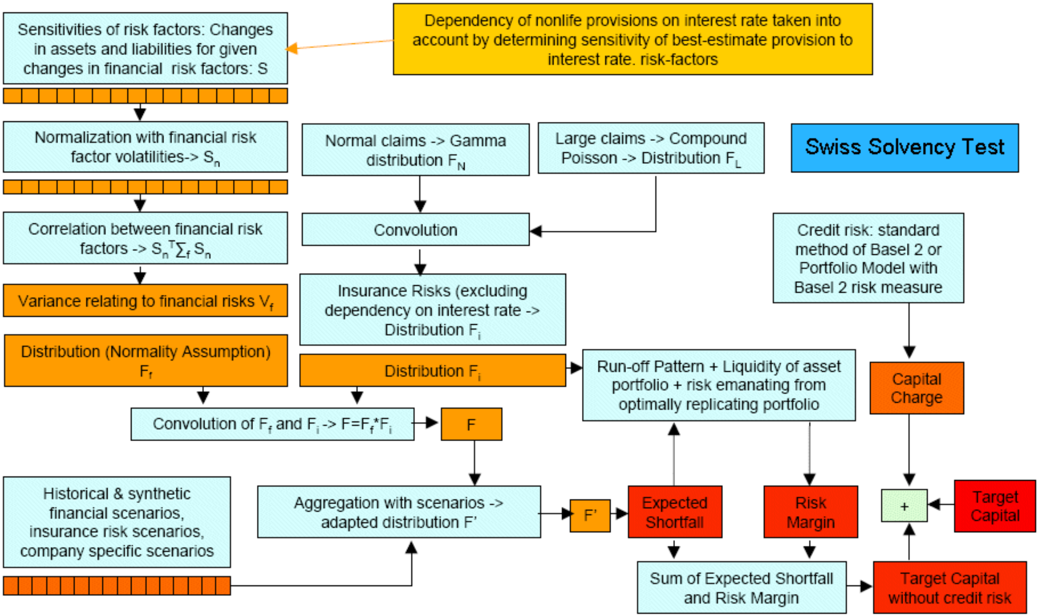
Common Pro's in favour of *Expected Shortfall* against *Value at Risk*:

- ES_α is a *coherent* risk measure, VaR_α is not ($\rightarrow$ *sub-additivity*)
- ES_α provides a *quantification* of the potential (high) loss, VaR_α does not
- ES_α enables a risk-adjusted *additive* capital allocation through

$$EX_{i:\alpha} = E\left(X_i \,|\, S > VaR_\alpha\right)$$

Theoretically o.k., but does it work in practice?

2. VaR vs. Expected Shortfall



A function C of n variables on the unit n-cube $[0,1]^n$ is called a *copula* if it is a *multivariate distribution function* that has *continuous uniform margins*.

Fréchet-Hoeffding bounds:

$$\max(u_1 + \cdots + u_n - n + 1, 0) \leq C(u_1, \cdots, u_n) \leq \min(u_1, \cdots, u_n)$$

Theorem (Sklar). Let H denote a n-dimensional distribution function with margins $F_1,\cdots,F_n$. Then there exists a copula C such that for all real $(x_1,\cdots,x_n)$,

$$H(x_1,\cdots,x_n)=C\left(F_1(x_1),\cdots,F_n(x_n)\right).$$

If all the margins are continuous, then the copula is unique, and is determined uniquely on the ranges of the marginal distribution functions otherwise. Moreover, if we denote by $F_1^{-1},\cdots,F_n^{-1}$ the generalized inverses of the marginal distribution functions, then for every $(u_1,\cdots,u_n)$ in the unit n-cube,

$$C(u_1,\cdots,u_n)=H\left(F_1^{-1}(u_1),\cdots F_n^{-1}(u_n)\right).$$

Familiar examples of copulas:

Gauß:

$$C_\Phi(u_1,\cdots,u_n) = \int_{-\infty}^{\Phi^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi^{-1}(u_n)} \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} \exp\left(-\frac{1}{2}(\mathbf{v}-\boldsymbol{\mu})^{\mathrm{tr}} \Sigma^{-1} (\mathbf{v}-\boldsymbol{\mu})\right) dv_1 \cdots dv_n,$$

Σ positive-definite

Student's t:

$$C_t(u_1,\cdots,u_n) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \cdots \int_{-\infty}^{t_\nu^{-1}(u_n)} \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\pi\nu)^n \det(\Sigma)}} \left(1+\frac{1}{\nu}(\mathbf{v}-\boldsymbol{\mu})^{\mathrm{tr}} \Sigma^{-1} (\mathbf{v}-\boldsymbol{\mu})\right)^{\left(-\frac{\nu+n}{2}\right)} dv_1 \cdots dv_n,$$

Σ positive-definite, $\nu \in \mathbb{N}$

Familiar examples of copulas (cont.):

Clayton:

$$C_{Cl}(u_1,\cdots,u_n)=\left[\sum_{i=1}^{n}u_i^{-\theta}-n+1\right]^{-1/\theta},\ \ \theta>0$$

Gumbel:

$$C_{Gu}(u_1,\cdots,u_n)=\exp\left(-\left\{\sum_{i=1}^{n}\left(-\ln(u_i)\right)^{\theta}\right\}^{1/\theta}\right),\ \ \theta\geq 1$$

Frank:

$$C_{Fr}(u_1,\cdots,u_n)=-\frac{1}{\theta}\ln\left(1+\left(e^{-\theta}-1\right)\prod_{i=1}^{n}\left\{\frac{e^{-\theta u_i}-1}{e^{-\theta}-1}\right\}\right),\ \ \theta>0$$

General message: dependence resp. copula has *essential influence* on risk measures:

Example: *heavy-tailed* risk distributions, Pareto-type with shape parameter $\lambda = \frac{1}{2}$:

density $$f(x) = \frac{1}{2\sqrt{1+x}^{\,3}}, \quad x \geq 0$$

cumulative distribution function $$F(x) = 1 - \frac{1}{\sqrt{1+x}}, \quad x \geq 0$$

Case 1: two *independent* risks X, Y of the same type:

$$f_{X+Y}(z) = \frac{z}{(2+z)^2\sqrt{1+z}} \approx \frac{1}{\sqrt{1+z}^3}$$

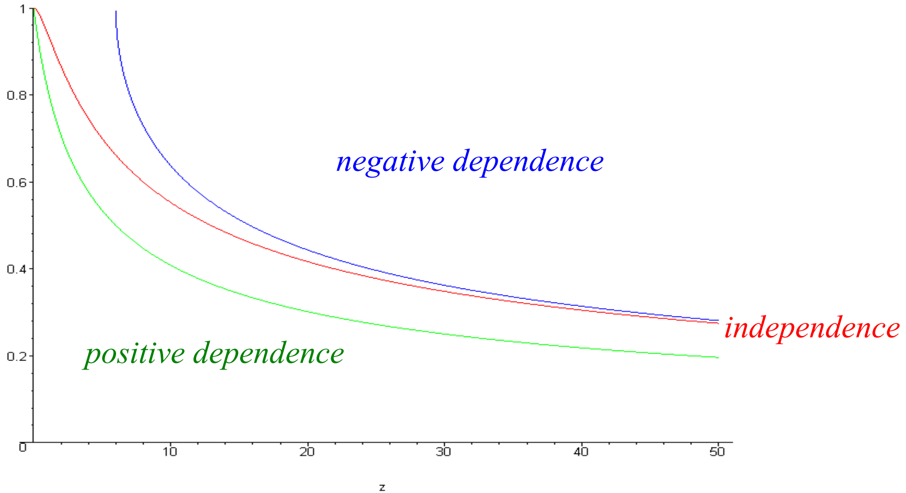
Case 2: two maximally *positively dependent* risks X, Y of the same type:

$$f_{X+Y}(z) = \frac{1}{4\sqrt{1+z/2}^3} \approx \frac{1}{\sqrt{2}\sqrt{1+z}^3}$$

Case 3: two maximally *negatively dependent* risks X, Y of the same type:

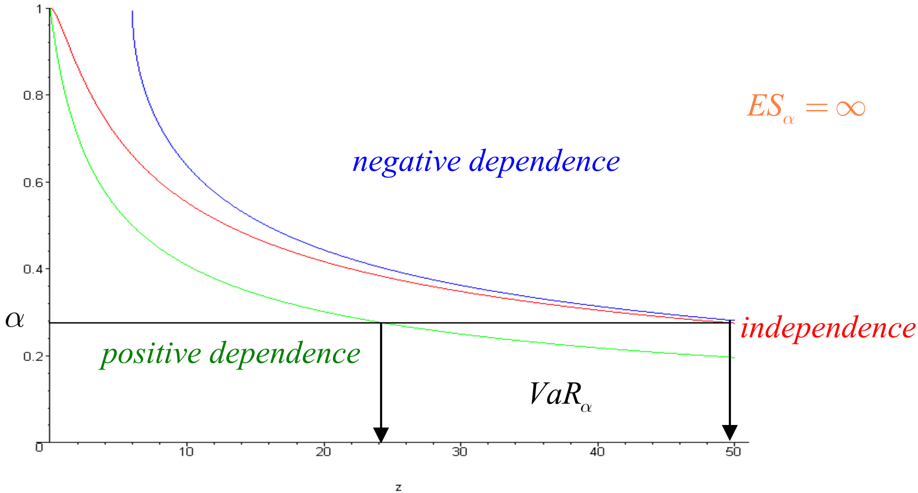
$$f_{X+Y}(z) = \frac{4+z-2\sqrt{3+z}}{\sqrt{(6+z)\sqrt{3+z}-4z-12}\,\sqrt{2+z}^3} \approx \frac{1}{\sqrt{1+z}^3}$$

3. Dependence and its implications for risk measures



plot of survival functions for **cases 1,2,3**

3. Dependence and its implications for risk measures



plot of survival functions for **cases 1,2,3**

Exact calculation of Value at Risk:

Case 1: two *independent* risks X,Y of the same type:

$$VaR_\alpha = \frac{4}{\alpha^2} - 2 - \frac{2}{1+\sqrt{1-\alpha^2}}$$

Case 2: two maximally *positively dependent* risks X,Y of the same type:

$$VaR_\alpha = \frac{2}{\alpha^2} - 2$$

Case 3: two maximally *negatively dependent* risks X,Y of the same type:

$$VaR_\alpha = \frac{4}{\alpha^2} - 2 - \frac{4}{(2-\alpha)^2}$$

Consequence:

1. Solvency capital for

   one portfolio consisting of two *independent* risks of the same type

   is *strictly larger* than the

   sum of the solvency capitals for two portfolios, each consisting of a single risk!

→ no diversification effect!

2. Solvency capital for

one portfolio consisting of two *independent* risks of the same type

is *asymptotically equivalent* (for large return periods) to the solvency capital for

one portfolio consisting of two *negatively dependent* risks of the same type!

→ independence close to worst case!

Example (cont.):

heavy-tailed risk distributions, Pareto-type with shape parameter $\lambda = 2$:

density $$f(x) = \frac{2}{(1+x)^3}, \ x \geq 0$$

cumulative distribution function $$F(x) = 1 - \frac{1}{(1+x)^2}, \ x \geq 0$$

here: $E(X+Y) = 2$

Case 1: two *independent* risks X,Y of the same type:

$$f_{X+Y}(z) = \frac{48\ln(1+z)}{(2+z)^5} + \frac{4z\left(10+10z+z^2\right)}{(2+z)^4(1+z)^2} \approx \frac{4}{(1+z)^3}$$

Case 2: two maximally *positively dependent* risks X,Y of the same type:

$$f_{X+Y}(z) = \frac{8}{(2+z)^3} \approx \frac{8}{(1+z)^3}$$

Case 3: two maximally *negatively dependent* risks X,Y of the same type:

no closed form available

Calculation of *Value at Risk*, special case $\alpha = 0,99$ (i.e. 100 year return period):

Case 1: two *independent* risks X, Y of the same type:

$VaR_\alpha = 14,14 \quad ES_\alpha = 28,72$ (numerical evaluation / simulation)

Case 2: two maximally *positively dependent* risks X, Y of the same type:

$VaR_\alpha = 18 \quad ES_\alpha = 38$ (exact calculation)

Case 3: two maximally *negatively dependent* risks X, Y of the same type:

$VaR_\alpha = 13,15 \quad ES_\alpha = 27,24$ (estimated by simulation)

$\rightarrow$ **rule of thumb:** for $\lambda \approx 2: \quad \dfrac{ES_{0,99}}{VaR_{0,99}} \approx 2!$

4. A (frightening?) example from theory

We consider a portfolio with two risks X and Y and their distributions given by

x	1	3	100
$P(X=x)$	0,90	0,09	0,01

y	1	5
$P(Y=y)$	0,20	0,80

(amounts in Mio. €)

Risk level: $\alpha = 0,01$ corresponding to a return period T of 100 years

Risk X is "dangerous", risk Y is "harmless"

Distribution of *total risk* $S = X + Y$ under *independence*:

s	2	4	6	**8**	101	105
$P(S = s)$	0,180	0,018	0,720	**0,072**	0,002	0,008
$P(S \leq s)$	0,180	0,198	0,918	**0,990**	0,992	1

This implies:

$$VaR_\alpha = 8 \quad \text{and} \quad ES_\alpha = \frac{101 \times 0,002 + 105 \times 0,008}{0,01} = 104,2$$

x	1	3	100
$P(X = x)$	0,90	0,09	0,01

y	1	5
$P(Y = y)$	0,20	0,80

Risk-based capital allocation with ES_α :

s	2	4	6	**8**	101	105
$P(S=s)$	0,180	0,018	0,720	**0,072**	0,002	0,008
$P(S \leq s)$	0,180	0,198	0,918	**0,990**	0,992	1

$$EX_\alpha = E\left(X \mid S > 8\right) = 100$$

$$EY_\alpha = E\left(Y \mid S > 8\right) = 4,2$$

x	1	3	100
$P(X = x \mid S > 8)$	0	0	1

y	1	5
$P(Y = y \mid S > 8) = P(Y = y)$	0,20	0,80

consequence: insufficient capital in 80% of all cases!

What is the return period T corresponding to a risk (loss) of $ES_\alpha = 104,2$?

s	2	4	6	8	**101**	105
$P(S=s)$	0,180	0,018	0,720	0,072	**0,002**	0,008
$P(S \leq s)$	0,180	0,198	0,918	0,990	**0,992**	1

$$T = \frac{1}{1-0,992} = \frac{1}{0,008} = 125$$

Consequence: a risk-based capital allocation with $ES_\alpha = 104,2$ increases the former return period of 100 years by only 25% to 125 years, while the capital requirement is **13-times** as much as with a capital allocation based on $VaR_\alpha = 8$!!

Proportional risk-based capital allocation with VaR_α :

s	2	4	6	**8**	101	105
$P(S=s)$	0,180	0,018	0,720	**0,072**	0,002	0,008
$P(S\leq s)$	0,180	0,198	0,918	**0,990**	0,992	1

$$EX_\alpha = \frac{E(X\mid S\leq 8)}{E(S\mid S\leq 8)}\times 8 = \frac{1,182}{5,382}\times 8 = 1,757 \qquad EY_\alpha = \frac{E(Y\mid S\leq 8)}{E(S\mid S\leq 8)}\times 8 = \frac{4,2}{5,382}\times 8 = 6,243$$

x	1	3	100
$P(X=x\mid S\leq 8)$	0,909	0,091	0

y	1	5
$P(Y=y\mid S\leq 8) = P(Y=y)$	0,20	0,80

consequence: insufficient capital in 9,1% of all cases!

Optimal risk-based capital allocation:

s	2	4	6	**8**	101	105
$P(S=s)$	0,180	0,018	0,720	**0,072**	0,002	0,008
$P(S\leq s)$	0,180	0,198	0,918	**0,990**	0,992	1

$EX_\alpha = 3$

x	1	3	100
$P(X=x\mid S\leq 8)$	0,909	0,091	0

$EY_\alpha = 5$

y	1	5
$P(Y=y\mid S\leq 8)=$ $P(Y=y)$	0,20	0,80

consequence: 8 Mio. € capital cover both risks at $\alpha=0,01$ optimally!

Influence of *dependencies* (→ copulas) on ES_α and VaR_α :

$P(X=x,Y=y)$	$x=1$	$x=3$	$x=100$	
$y=1$	a	$-a+b+0{,}19$	$0{,}01-b$	0,2
$y=5$	$0{,}9-a$	$a-b-0{,}1$	b	0,8
	0,90	0,09	0,01	

with side conditions

$$0 < b < 0{,}01$$

$$0{,}1+b < a < 0{,}19+b$$

4. A (frightening?) example from theory

Distribution of *total risk* $S = X + Y$ under *dependence*:

s	2	4	6	**8**	101	105
$P(S = s)$	a	$-a+b+0{,}19$	$0{,}9-a$	$a-b-0{,}1$	$0{,}01-b$	b
$P(S \leq s)$	a	$b+0{,}19$	$1{,}09-a+b$	**0,99**	$1-b$	1

implying

$$VaR_\alpha = 8 \quad \text{and} \quad ES_\alpha = \frac{101 \times (0{,}01 - b) + 105 \times b}{0{,}01} = 101 + 400b$$

Consequence: VaR_α remains *unchanged*, ES_α *varies* between 101 and 105!

Possible reduction of ES_α by *reinsurance* with *priority* VaR_α :

(net) reinsurance premium:

$$RV_\alpha = \alpha\left(ES_a - VaR_\alpha\right) = E\left((S - VaR_\alpha)^+\right) = 0{,}93 + 4b \in \left(0{,}93 \mid 0{,}97\right)$$

[*reinsurance premium*: add safety loading]

Consequence: target capital reduces to roughly

10 Mio €

only!

Example company portfolio:

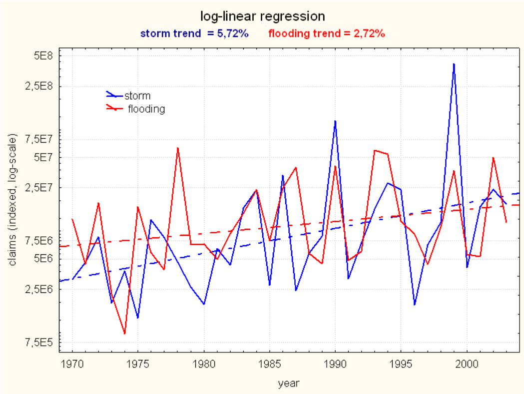
location 1	*location 2*
34 years of data	*18 years of data*
windstorm	windstorm
hailstorm	
flooding	

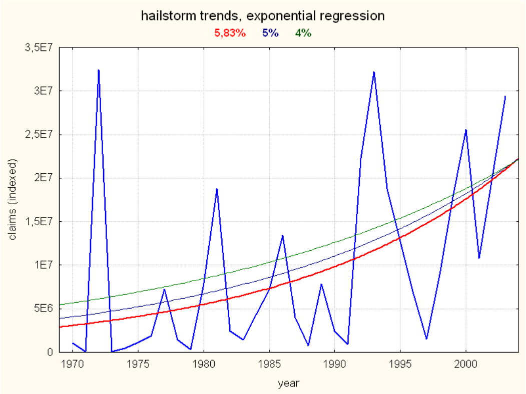
climatic dependence (?)

spatial dependence

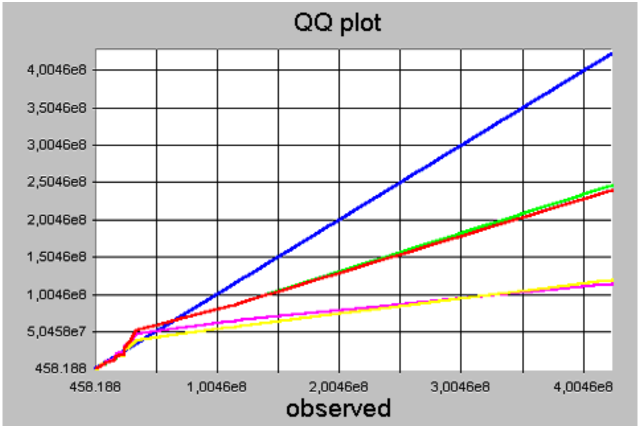
Marginal analysis location 1:

- Indexing
- Detrending





Distribution fitting location 1, windstorm:



Fréchet distribution, max. deviation: **0,36637**	
Loglogistic distribution, max. deviation: 0,64749	
Pearson type V distribution, max. deviation: 0,37747	
Lognormal distribution, max. deviation: 0,63799	

Anderson-Darling test:

Distribution type: Fréchet; test statistic: **0,18444**

α	0,25	0,1	0,05	0,025	0,01
critical value	0,458	0,616	0,732	0,848	1,004

Distribution type: Pearson type V; test statistic: **0,17911**

α	0,25	0,1	0,05	0,025	0,01
critical value	0,485	0,655	0,783	0,913	1,078

Distribution type: Loglogistic; test statistic: 0,34706

α	0,25	0,1	0,05	0,025	0,01
critical value	0,423	0,559	0,655	0,763	0,899

Distribution type: Lognormal; test statistic: 0,52042

α	0,25	0,1	0,05	0,025	0,01
critical value	0,459	0,616	0,734	0,853	1,011

χ^2 test:

Distribution type: Fréchet; test statistic: **0,58824**

d.f.	α	0,25	0,15	0,1	0,05	0,01
3	critical value	4,108	5,317	6,251	7,815	11,345
5	critical value	6,626	8,115	9,236	11,070	15,086

Distribution type: Pearson Type V; test statistic: 0,94118

d.f.	α	0,25	0,15	0,1	0,05	0,01
3	critical value	4,108	5,317	6,251	7,815	11,345
5	critical value	6,626	8,115	9,236	11,070	15,086

Distribution type: Loglogistic; test statistic: 3,41176

d.f.	α	0,25	0,15	0,1	0,05	0,01
3	critical value	4,108	5,317	6,251	7,815	11,345
5	critical value	6,626	8,115	9,236	11,070	15,086

Distribution type: Lognormal; test statistic: 3,41176

d.f.	α	0,25	0,15	0,1	0,05	0,01
3	critical value	4,108	5,317	6,251	7,815	11,345
5	critical value	6,626	8,115	9,236	11,070	15,086

Summary of *marginal statistical analysis*:

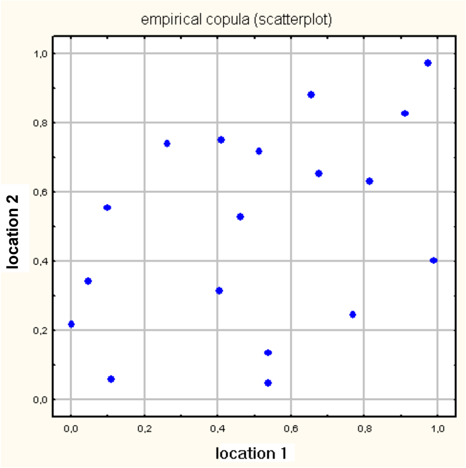
location 1	*location 2*
34 years of data	*18 years of data*
windstorm: *Fréchet*	windstorm: *Fréchet*
hailstorm: *Lognormal*	
flooding: *Lognormal*	

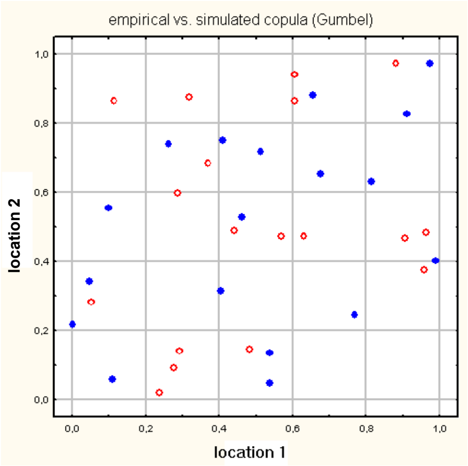
climatic dependence (?)

spatial dependence

Dependence analysis:

Spatial dependence location 1 / location 2 (windstorm):





Estimation methods for bivariate Gumbel copula $C_\lambda(u,v)$:

$$C_\lambda(u,v)=\exp\left\{\left((-\ln u)^\lambda+(-\ln v)^\lambda\right)^{1/\lambda}\right\},\quad 0<u,v\le 1,$$

with a structural parameter $\lambda \ge 1$. The corresponding density is given by

$$c_\lambda(u,v)=\frac{\partial^2}{\partial u\,\partial v}C_\lambda(u,v)=C_\lambda(u,v)\frac{(-\ln u)^{\lambda-1}(-\ln v)^{\lambda-1}}{uv}k(u,v)^{1/\lambda-2}\left[\lambda-1+k(u,v)^{1/\lambda}\right]$$

with $k(u,v)=(-\ln u)^\lambda+(-\ln v)^\lambda,\ 0<u,v\le 1.$

For $\lambda=1$, the *independence copula* is obtained.

Method I:

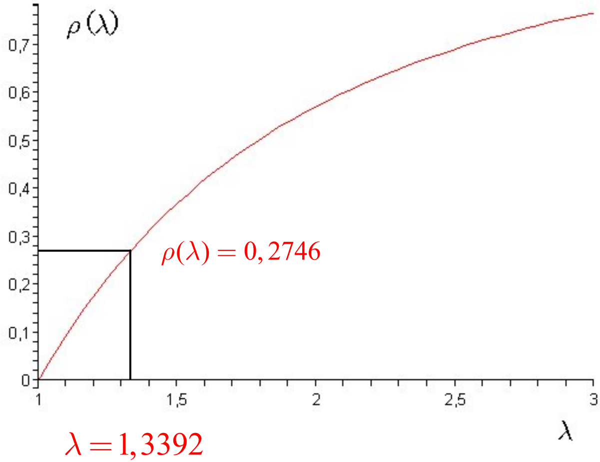
Use a functional relationship between the correlation of suitably transformed data and λ. This procedure is documented in REISS AND THOMAS (2001), p. 240f. and relies on the fact that, when the original distributions are of *negative exponential* type, i.e. the marginal c.d.f.'s of X and Y are

$$F_X(x) = F_Y(x) = \begin{cases} e^x, & x \le 0 \\ 1, & x > 0 \end{cases} \text{ for } x \in \mathbb{R},$$

then the correlation $\rho(\lambda)$ between X and Y is given by

$$\rho(\lambda) = 2\frac{\Gamma^2\left(+\frac{1}{\lambda}\right)}{\Gamma\left(+\frac{2}{\lambda}\right)} - 1 \text{ for } \lambda \ge 1.$$

5. A (frightening?) example from the real world



Method II:

Use the method of maximum-likelihood. For this purpose, consider the function

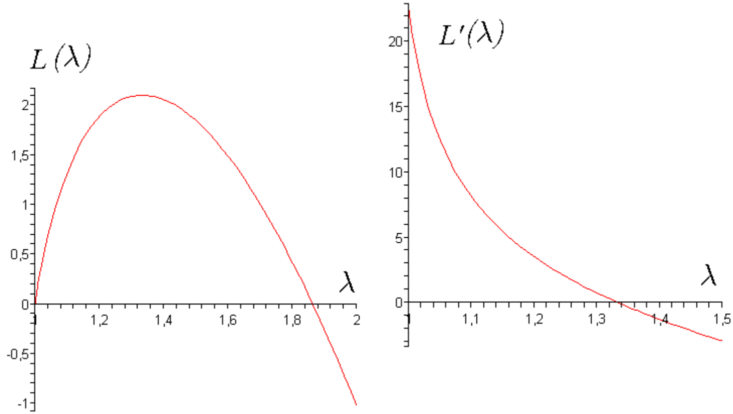
$$L\left((u_1,v_1),\cdots,(u_n,v_n);\lambda\right) := \ln\left(\prod_{i=1}^{n} c_\lambda(u_i,v_i)\right) =$$

$$= -\sum_{i=1}^{n} k(u_i,v_i)^{1/\lambda} + (\lambda-1)\sum_{i=1}^{n}\{\ln(-\ln u_i)+\ln(-\ln v_i)\} + \left(\frac{1}{\lambda}-2\right)\sum_{k=1}^{n}\ln\left(k(u_i,v_i)\right) +$$

$$+\sum_{i=1}^{n}\ln\left[\lambda-1+k(u_i,v_i)^{1/\lambda}\right] - \sum_{i=1}^{n}(\ln u_i+\ln v_i)$$

as a function of λ and find its argmax, i.e. the value of λ that maximizes $L\left((u_1,v_1),\cdots,(u_n,v_n);\lambda\right)$ given the data $(u_1,v_1),\cdots,(u_n,v_n)$.

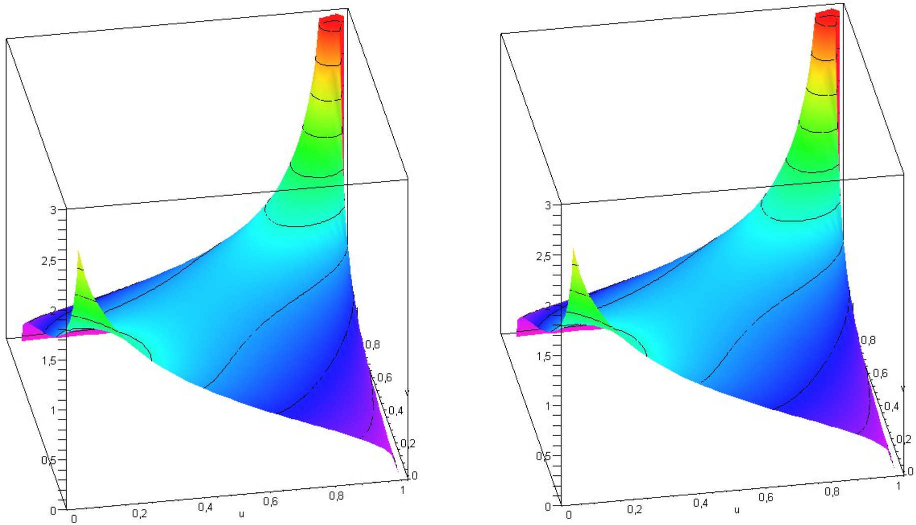
5. A (frightening?) example from the real world



log-likelihood-function and its derivative for storm data

$$\lambda = 1{,}33347$$

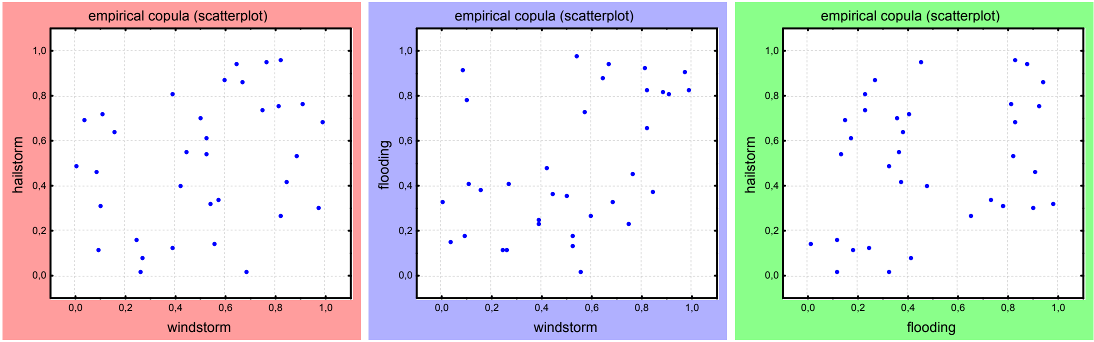
5. A (frightening?) example from the real world



copula densities $c(u,v;1,3392)$ und $c(u,v;1,33347)$ [truncated above]

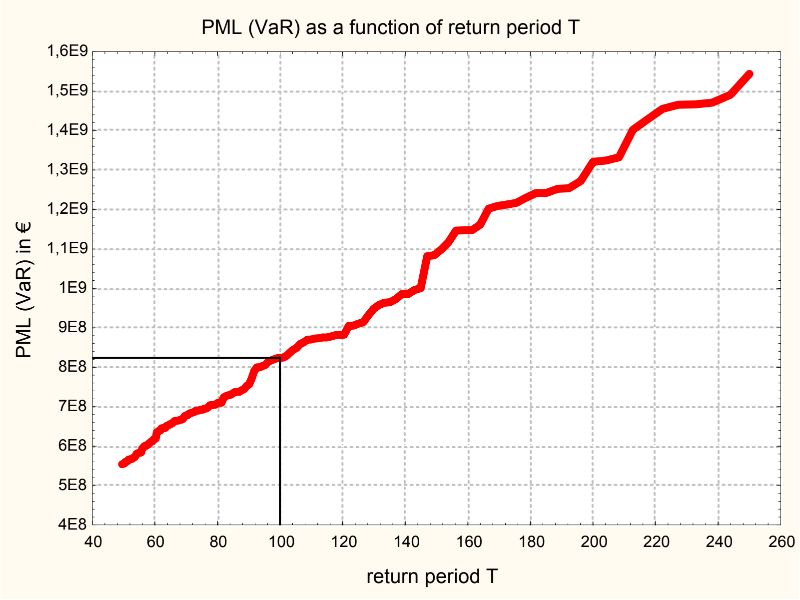
Dependence analysis:

Other dependencies windstorm / hailstorm / flooding:



$$\textit{Gauß copula}:\ \Sigma = \begin{bmatrix} 1 & 0{,}2226 & 0{,}3782 \\ 0{,}2226 & 1 & 0{,}3341 \\ 0{,}3782 & 0{,}3341 & 1 \end{bmatrix}$$

5. A (frightening?) example from the real world



$PML_{100} = VaR_{0,01} =$ 823 Mio € $ES_{0,01} = 1880$ Mio € **(!)**

Some Con's against Expected Shortfall:

- ES_α is based on the *average* of losses above VaR_α and can thus be rigorously motivated only by the *Law of Large Numbers*. However, this is not very meaningful from an economic point of view since defaults are just *single events*.

- ES_α may thus lead to *economically meaningless* risk-based capital allocations, which in particular do not provide the correct allocations of risks in the „normal" situation (i.e. in $(1-\alpha)\times 100\%$ of the years).

- Compared with VaR_α, ES_α does not increase the default return period significantly, although the capital requirement might be significantly higher.

- ES_α enforces insurance companies to buy *reinsurance* to a significantly higher extend than today.

- ES_α is definitely not appropriate for portfolios with *rare*, but potentially *very large* losses (e.g. natural perils: windstorm, flooding, earthquake, …)

- ES_α is extremely sensitive to the statistical estimation of *marginal distributions*.

- ES_α is sensitive to *dependence structures* in the risks.

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