Wave propagation in unbounded quasiperiodic media: the non-absorbing case

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We are interested in the Helmholtz equation with frequency $\omega \in \mathbb{R}$:

$$-\big(\mu_\theta\, u'\big)' - \rho_\theta\, \omega^2\, u = f \quad \text{in} \quad \mathbb{R}, \tag{1}$$

where $f \in L^2(\mathbb{R})$ has a compact support $(-a, a)$, $a > 0$, and where μ_θ and ρ_θ are ***quasiperiodic***, that is, there exists $\theta \in (0, \pi/2)$ and 1–periodic functions μ_p, $\rho_p \in \mathscr{C}^0(\mathbb{R}^2)$ such that

$$\mu_\theta(x) = \mu_p(x\,\vec{e}_\theta) \quad \text{and} \quad \rho_\theta(x) = \rho_p(x\,\vec{e}_\theta), \qquad \vec{e}_\theta := (\cos\theta, \sin\theta). \tag{2}$$

The definition of the good physical solution is delicate. In fact, one expects that this solution, if it exists, may not belong to $H^1(\mathbb{R})$, due to a lack of decay at infinity. It is usually defined by using the ***limiting absorption principle***, which consists in (i) assuming that $\mathfrak{Im}\,\omega^2 > 0$, in which case (1) admits a unique H^1 solution, and (ii) studying the limit of the solution u as $\mathfrak{Im}\,\omega^2 \to 0$. Then, this limit solution can be hopefully characterized via a so-called ***radiation condition*** which imposes its behaviour at infinity.

Understanding the limit process described above is closely related to the spectral analysis of the self-adjoint differential operator in $L^2(\mathbb{R}; \rho_\theta\, dx)$:

$$H_\theta u = -\frac{1}{\rho_\theta}\,\big(\mu_\theta\, u'\big)', \qquad D(H_\theta) = \{u \in H^1(\mathbb{R}),\ (\mu_\theta\, u')' \in L^2(\mathbb{R})\}.$$

When μ_θ and ρ_θ are periodic i.e. when $\tan\theta \in \mathbb{Q}$, Floquet theory shows that the spectrum $\sigma(H_\theta)$ is purely continuous with a band structure. When $\tan\theta$ is irrationnal, $\sigma(H_\theta)$ has an absolutely continuous part as in the periodic case, but may also have a point part, and even a singular continuous part that may contain a ***Cantor set*** (that is, a closed set with no isolated points and whose complement is dense, see [Eliasson, 1992] for related results). Concerning the limiting absorption principle, there is no problem when ω^2 is not in $\sigma(H_\theta)$, and of course, it cannot hold when ω^2 is an eigenvalue of H_θ. But for all the other cases, the question is still open.

Even if, from a theoretical point of view, the answer to the limiting absorption principle is not clear, we can propose a numerical procedure assuming that it holds. In fact, in the case where $\mathfrak{Im}\,\omega^2 > 0$, since the coefficients μ_θ and ρ_θ are traces along a particular line of periodic functions of higher dimensions, the first step is to ***interpret the solution of* (1) *as the trace along the same line of the solution of an augmented PDE in higher dimensions, with periodic coefficients***. This so-called ***lifting approach*** allows one to extend the ideas of the Dirichlet-to-Neumann methods developed for periodic media [Joly, Li, and Fliss, 2006]. In particular, the corresponding numerical method is based on the resolution of ***Dirichlet cell problems***, and the computation of a ***propagation operator***, solution of a ***constrained Riccati equation***.

The natural idea is then to pass to the limit in the above method when $\mathfrak{Im}\,\omega^2$ tends to 0. Doing so however raises several difficulties. The first difficulty is that we have shown that ***the Dirichlet cell problems are not well-posed for intervals of frequencies***. The solution is to solve Robin cell problems instead and extend our method to construct Robin-to-Robin boundary conditions. The second difficulty concerns the propagation operator, which needs an additional condition in order to be fully characterized. The additional condition we use is inspired by [Fliss, Joly, and Lescarret, 2021]. Numerical results will be provided to illustrate the efficiency of the method.

References

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