The 3D scalar transmission problem in the presence of a conical tip of negative material

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In this work, we are interested in the study of the 3D scalar transmission problem between a positive material Ω^+ and a negative one Ω^-. The interface that separates Ω^+ and Ω^- (which will be denoted by Σ) is assumed to be smooth except near a certain point O where it has a conical singularity. More precisely, the problem that we are interested in is the following:

$$\text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that } -\mathrm{div}(\sigma\nabla u) = f \in \mathrm{H}^{-1}(\Omega),$$

where $\Omega = \Omega^+ \cup \Omega^- \cup \Sigma$, $\sigma_{|\Omega^+} = \sigma^+ \in \mathbb{R}^*_+$ and $\sigma_{|\Omega^-} = \sigma^- \in \mathbb{R}^*_-$. Since the function σ changes sign, the previous problem can be ill-posed even in the Fredholm sense (i.e. lost of ellipticity). By applying the Mellin transform to the problem, we can determine whether it is well-posed by studying the spectral properties of its Mellin symbol. By combining the T-coercivity approach and some localization techniques, we are able to show that the spectrum of the Mellin symbol is discrete, composed by the normal eigenvalues and localized inside a double sector of the complex plane. With this in mind and by using the classical Kondratiev theory, we show that the problem is well-posed in the classical Sobolev space H^1 if and only if the contrast (i.e. σ^-/σ^+) does not belong to a critical set called the critical interval.
This critical interval corresponds to the set of contrasts for which the problem has propagating singularities (singularities whose singular exponent is of the form $\lambda = -1/2 + i\eta$ with $\eta \in \mathbb{R}$). Note that for the particular case of the circular conical tip, an explicit expression for the critical interval can be found.
In order to recover the well-posedness for the case of critical contrasts, we need to study the asymptotic behavior of the spectrum and the associated eigenvectors/generalized eigenvectors of the Mellin symbol with respect to small perturbations. In this way, we are able to construct a new functional framework that accounts for some of the propagating singularities (i.e., the outgoing ones) in which the problem is again well-posed and that is consistent with physical reality (i.e. the limiting absorption principle).