A GEOMETRIC APPROACH TO CONSTRUCTING ELEMENTS OF K_2 OF CURVES

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ABSTRACT. We present a framework for constructing examples of smooth projective curves over number fields with explicitly given elements in their second K-group using elementary algebraic geometry. This leads to new examples for hyperelliptic curves and smooth plane quartics. Moreover, we show that most previously known constructions can be reinterpreted using our framework.

1. Introduction

The Beilinson conjectures are of fundamental importance in algebraic K-theory and arithmetic geometry, predicting a relation between special values of L-functions and regulators of certain higher K-groups of smooth projective varieties defined over number fields. See [Sch88] for an introduction.

Let C denote a smooth projective geometrically irreducible curve of genus g defined over a number field K with ring of integers $\mathcal{O}_K$. We denote by $K_2^T(C)$ the tame second K-group of C, defined in Section 2. A special case of Beilinson's conjecture predicts that that a certain subgroup $K_2(C;\mathcal{O}_K)$ of $K_2^T(C)/\text{torsion}$ is free of rank $g\cdot[K:\mathbb{Q}]$. In order to prove this conjecture or at least test it numerically in examples, one needs a method to come up with enough independent elements of $K_2(C;\mathcal{O}_K)$. In general it is quite difficult to construct elements of $K_2^T(C)$ (not to mention $K_2(C;\mathcal{O}_K)$) for a given curve C. Apart from the work of Beilinson [Bei85] (for modular curves over abelian number fields) and Deninger [Den89] (for elliptic curves with complex multiplication) no systematic constructions are known to date. Instead, a number of ad hoc approaches have been developed, see for instance [BG86], [RS98], [DdJZ06] and [LdJ15]. These produce certain families of curves for which it is known that many elements of $K_2^T(C)$ exist.

In this note we present a geometric approach to constructing algebraic curves C together with elements in $K_2^T(C)$ using elementary algebraic geometry. Our idea is as follows: We first choose plane curves

$$C_1,\dots,C_m\subset\mathbb{P}^2_K$$

of respective degrees $d_1,\dots,d_m$; then we consider functions f_{kl} on $\mathbb{P}^2_K$ such that

$$\operatorname{div}(f_{kl})=d_l\cdot C_k-d_k\cdot C_l.$$

These are determined by the equations of the curves $C_1,\dots,C_m$ up to scaling. We then construct a plane curve C/K such that many of the classes

$$\left\{f_{kl}\big|_C, f_{k'l'}\big|_C\right\}\in K_2(K(C)) \tag{1}$$

represent elements in $K_2^T(C)/\text{torsion}$ after a suitable scaling of the functions f_{kl} and $f_{k'l'}$. We require that the curves C_i intersect the curve C in very few points and that the intersection

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multiplicities between C and C_i at these points satisfy certain requirements. This yields a system of equations for the coefficients of C that in many cases has a solution. In these cases, we then change our point of view and treat the coefficients of the curves C_i and the coordinates of the points of intersection as indeterminates. Using this approach, we finally get a parametrization of a family of curves such that every curve in this family has a number of known representatives of elements in $K_2^T(C)/\text{torsion}$.

In fact, we show that we can reinterpret most known constructions using our method and we can also come up with some new families of smooth plane quartics and hyperelliptic curves. We focus on the geometric aspects, so we only discuss integrality of our elements in some examples (for some hyperelliptic curves and smooth plane quartics) and we touch upon independence of the elements only briefly. It would be an interesting project to check independence in the families constructed in Sections 5, 6 and 7, for instance using a limit argument as in [dJ05] or [LdJ15].

Remark 1.1. In recent work [LdJ15] de Jeu and Liu manage to construct g independent elements of $K_2^T(C)$ for certain families of curves, including the hyperelliptic examples of [DdJZ06] discussed in Section 4, but also non-hyperelliptic ones. Furthermore, they show that in some of these families, all of the elements are in fact integral. We would like to point out that their construction also fits into our general framework: They begin with a number of lines in $\mathbb{A}^2$; their curves are then given as the smooth projective model of a certain singular affine curve constructed using these lines.

The organization of the paper is as follows: In Section 2 we give a rather gentle and elementary introduction to K_2 of curves, closely following [DdJZ06]. The constructions in [BG86] and [DdJZ06] work for (hyper-) elliptic curves and use torsion divisors, following an approach which goes back to work of Bloch and which we recall in Section 3. Then we show in Section 4 that our method gives a simple way to obtain many examples of hyperelliptic curves with many explicitly given elements, including the examples of [DdJZ06]. Still using torsion divisors, we apply our approach to smooth plane quartics in Section 5, where the curves C_i are lines, and in Section 6, where the curves C_i are conics or lines. Finally, we show in Section 7 that we are not restricted to torsion divisors by generalizing a construction for elliptic curves discussed in [RS98], where it is attributed to Nekovář, to certain curves of higher genus.

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2. K_2 of curves and Beilinson's conjecure

We give a brief down-to-earth introduction to Beilinson's conjecture on K_2 of curves over number fields, following the discussion of [DdJZ06, §3]. The definition of higher K-groups of C is due to Quillen [Qui73] and is rather involved. However, Beilinson's original conjecture can be formulated in terms of the tame second K-group $K_2^T(C)$ of C, whose definition is quite simple in comparison, and we follow this approach here.

Let C be a smooth, projective, geometrically irreducible curve of genus g defined over a number field K with ring of integers $\mathcal{O}_K$ and fixed algebraic closure $\overline{K}$. By Matsumoto's Theorem [Mil71, Theorem 11.1], the second K-group of the field $K(C)$ is given by

$$K_2(K(C)) = K(C)^\times \otimes_{\mathbb{Z}} K(C)^\times / \langle f \otimes (1-f),\ f \in K(C)^\times \setminus \{1\}\rangle.$$

If $f, h \in K(C)^\times$, then we write $\{f, h\}$ for the class of $f \otimes h$ in $K_2(K(C))$. Hence $K_2(K(C))$ is the abelian group with generators $\{f, h\}$ and relations

$$\begin{aligned}\{f_1 f_2, h\} &= \{f_1, h\} + \{f_2, h\}\\ \{f, h_1, h_2\} &= \{f, h_1\} + \{f, h_2\}\\ \{f, 1-f\} &= 0 \text{ for } f \in K(C)^\times \setminus \{1\}.\end{aligned}$$

For P in the set $C^{(1)}$ of closed points of C and $\{f, h\} \in K_2(K(C))$, we define the *tame symbol*

$$T_P(\{f, h\}) = (-1)^{\mathrm{ord}_P(f)\,\mathrm{ord}_P(h)} \frac{f^{\mathrm{ord}_P(h)}}{h^{\mathrm{ord}_P(f)}}(P) \in K(P)^\times,$$

and we extend this to $K_2(K(C))$ by linearity. Then we have the product formula

$$\prod_{P \in C^{(1)}} N_{K(P)/K}\left(T_P(\alpha)\right) = 1 \quad \text{for all } \alpha \in K_2(K(C)), \tag{2}$$

which generalizes Weil's reciprocity law, see [Bas68, Theorem 8.2]. Setting

$$T = \prod_{P \in C^{(1)}} T_P,$$

we define the *tame second K-group*

$$K_2^T(C) = \ker\left(T : K_2(K(C)) \to \bigoplus_{P \in C^{(1)}} K(P)^\times\right)$$

of C. Note that in general $K_2^T(C)$ differs from the second K-group $K_2(C)$ associated to C by Quillen in [Qui73]. However, we get an exact sequence

$$\bigoplus\nolimits_{P \in C^{(1)}} K_2(K(P)) \longrightarrow K_2(C) \longrightarrow K_2(K(C)) \xrightarrow{T} \bigoplus\nolimits_{P \in C^{(1)}} K(P)^\times$$

from [Qui73] and since K_2 of a number field is torsion [Gar71], this implies

$$K_2^T(C)/\text{torsion} \cong K_2(C)/\text{torsion}.$$

As our motivation is Beilinson's conjecture on $K_2(C)$, which does not depend on the torsion subgroup of $K_2(C)$ at all, we only discuss $K_2^T(C)$.

By [BG86], the correct K-group for the statement of Beilinson's conjecture is not the quotient $K_2^T(C)/\text{torsion}$, but a subgroup $K_2(C; \mathcal{O}_K)$ of $K_2^T(C)/\text{torsion}$ defined by certain integrality conditions. Let $\mathcal{C} \to \mathrm{Spec}(\mathcal{O}_K)$ be a proper regular model of C. For an irreducible component Γ of a special fiber $\mathcal{C}_\mathfrak{p}$ of $\mathcal{C}$ and $\{f, h\} \in K_2^T(C)$ we define

$$T_\Gamma(\{f, h\}) = (-1)^{\mathrm{ord}_\Gamma(f)\,\mathrm{ord}_\Gamma(h)} \frac{f^{\mathrm{ord}_\Gamma(h)}}{h^{\mathrm{ord}_\Gamma(f)}}(\Gamma) \in k_\mathfrak{p}(\Gamma)^\times,$$

where $k_\mathfrak{p}(\Gamma)$ is the function field of Γ and ord_Γ is the normalized discrete valuation on the local ring $\mathcal{O}_{\mathcal{C},\Gamma}$, extended to its field of fractions $K(C)$. We extend T_Γ to $K_2^T(C)$ by linearity and set

$$K_2^T(\mathcal{C}) = \ker\left(K_2^T(C) \to \bigoplus_{\mathfrak{p} \subset \mathcal{O}_K} \bigoplus_{\Gamma \subset \mathcal{C}_\mathfrak{p}} k_\mathfrak{p}(\Gamma)^\times\right).$$

Here the Γ-component of the map is given by T_Γ and the sums run through the finite primes $\mathfrak{p}$ of $\mathcal{O}_K$ and the irreducibe components of $\mathcal{C}_\mathfrak{p}$, respectively.

One can show (see [LdJ15, Proposition 4.1]) that $K_2^T(\mathcal{C})$ does not depend on the choice of $\mathcal{C}$, so that

$$K_2(C;\mathcal{O}_K) := K_2^T(\mathcal{C})/\text{torsion}$$

is well-defined. We call an element of $K_2^T(C)/\text{torsion}$ *integral* if it lies in $K_2(C;\mathcal{O}_K)$.

Finally, we can state a weak version of Beilinson's conjecture (as modified by Bloch).

Conjecture 2.1. *Let C be a smooth, projective, geometrically irreducible curve of genus g defined over a number field K. Then the group $K_2(C;\mathcal{O}_K)$ is free of rank $g\cdot[K:\mathbb{Q}]$.*

Finite generation of $K_2(C;\mathcal{O}_K)$ was already conjectured by Bass. The full version of Beilinson's conjecture on K_2 can be found in [DdJZ06]. It relates the special value of the Hasse-Weil L-function of C at $s=2$ to the determinant of the matrix of values of a certain regulator pairing

$$H_1(C(\mathbb{C}),\mathbb{Z})\times K_2^T(C)/\text{torsion}\to\mathbb{R}$$

at the elements of a basis of the anti-invariants of $H_1(C(\mathbb{C}),\mathbb{Z})$ under complex conjugation and at the elements of a basis of $K_2(C,\mathcal{O}_K)$.

However, even Conjecture 2.1 is still wide open in general, largely due to the fact that it is rather difficult to construct elements of K_2. Therefore new methods for such constructions are needed.

3. Torsion construction

Most of the known constructions of elements of K_2 of curves use *torsion divisors*, in other words, divisors on the curve of degree zero whose divisor classes in the Jacobian of the curve are torsion. For simplicity, we restrict to the case $K=\mathbb{Q}$; all constructions can be extended to arbitrary number fields using straightforward modifications.

We first recall the following construction from [DdJZ06], originally due to Bloch:

Construction 3.1. ([DdJZ06, Construction 4.1]) Let $C/\mathbb{Q}$ be a smooth, projective, geometrically irreducible curve, let $h_1,h_2,h_3\in\mathbb{Q}(C)^*$ and let $P_1,P_2,P_3\in C(\mathbb{Q})$ be such that

$$\operatorname{div}(h_i)=m_i(P_{i+1})-m_i(P_{i-1}),\quad i\in\mathbb{Z}/3\mathbb{Z},$$

where $m_i\in\mathbb{N}$ is the order of the class of $(P_{i+1})-(P_{i-1})$ in $\operatorname{Pic}^0(C)$. Then we define symbols

$$S_i=\left\{\frac{h_{i+1}}{h_{i+1}(P_{i+1})},\frac{h_{i-1}}{h_{i-1}(P_{i-1})}\right\}\in K_2(\mathbb{Q}(C)),\quad i\in\mathbb{Z}/3\mathbb{Z}.$$

We summarize the most important facts about Construction 3.1:

Proposition 3.2. *Keep the notation of Construction 3.1.*

(i) The symbols S_i are elements of $K_2^T(C)$.

(ii) There is a unique element $\{P_1,P_2,P_3\}\in K_2^T(C)/\text{torsion}$ such that

$$S_i=\frac{\operatorname{lcm}(m_1,m_2,m_3)}{m_i}\{P_1,P_2,P_3\},\quad i\in\{1,2,3\}$$

in $K_2^T(C)/\text{torsion}$.

(iii) If $\sigma\in\mathfrak{S}_n$ is an even permutation, then we have $\{P_{\sigma(1)},P_{\sigma(2)},P_{\sigma(3)}\}=\{P_1,P_2,P_3\}$. If $\sigma\in\mathfrak{S}_n$ is odd, then $\{P_{\sigma(1)},P_{\sigma(2)},P_{\sigma(3)}\}=-\{P_1,P_2,P_3\}$.

(iv) Suppose that there exists, in addition, a point $P_4 \in C(\mathbb{Q})$ such that all differences $(P_i)-(P_j)$ are torsion divisors for $i, j \in \{1, \ldots, 4\}$. Then the following elements are linearly dependent in $K_2^T(C)/\mathrm{torsion}$:

$$\{P_1, P_2, P_3\},\ \{P_1, P_2, P_4\},\ \{P_1, P_3, P_4\},\ \{P_2, P_3, P_4\}$$

Proof. All assertions are stated and proved in [DdJZ06, §4]. □

Remark 3.3. We stress that there are curves which have no rational torsion points in their Jacobian, so we cannot expect that the torsion construction will always suffice to construct enough elements of $K_2(C, \mathbb{Z})$.

We will use the following notation:

Definition 3.4. If C, $D \subset \mathbb{P}^2(K)$ are plane curves defined over a field K and m is a positive integer, then we call a point $P \in C(\overline{K}) \cap D(\overline{K})$ an *m-contact point* between C and D if the intersection multiplicity $I_P(C, D)$ of C and D at P is equal to m. If C has degree d_C and D has degree d_D, then we call $P \in C(\overline{K})$ a *maximal contact point* between C and D if P is a $d_C d_D$-contact point between C and D.

Remark 3.5. It is possible for a point to be an m_i-contact point for several distinct positive integers m_i.

Construction 3.6. Now we describe the use of Construction 3.1 in our setup. We work in the projective plane $\mathbb{P}^2$ throughout. We fix a rational point $\infty \in \mathbb{P}^2$ and a line L_∞ through this point. We consider those smooth plane curves C such that L_∞ meets C precisely in ∞.

Then we choose $m \geq 2$ rational points $P_1, \ldots, P_m$ distinct from ∞ and require that C meets the lines L_{P_i} through P_i and ∞ precisely in these two points. By abuse of notation, the defining polynomial of a line L will also be denoted by L. Then each quotient L_{P_i}/L_∞ determines a function $h_i \in k(C)$ with divisor

$$\mathrm{div}(h_i) = m_i(P_i) - m_i(\infty) \in Z^1(C)$$

for some positive integer m_i. It follows that for $j \neq i$ the quotients L_{P_i}/L_{P_j} also define torsion divisors. Hence all functions obtained in this way are as in Construction 3.1, thus we get elements

$$\{\infty, P_i, P_j\} \in K_2^T(C)/\mathrm{torsion}.$$

In some cases we cannot get as many non-trivial elements as we want using just one point of maximal contact because of obvious relations, see for instance [DdJZ06, Example 5.2] or the discussion following Lemma 5.1 below. To remedy this, we can require that there is an additional rational point $O \neq \infty$ on C such that there is a curve D which meets C in precisely this point. Then, with abuse of notation as above, the quotient $D/L_\infty^{\deg(D)}$ determines a function $h \in k(C)$ with divisor

$$\mathrm{div}(h) = \deg(D)\deg(C)\,((O) - (\infty)) \in Z^1(C).$$

As above, we find that the divisors $(O) - (P_i)$ are torsion divisors as well. Therefore, again using Construction 3.1, we get further elements $\{\infty, O, P_i\} \in K_2^T(C)/\mathrm{torsion}$ and these might be nontrivial (in this case it suffices to have $m \geq 1$).

Moreover, we can add the requirement that there are rational points Q_j such that the line through Q_j and O intersects C precisely in these two points, leading to the further elements $\{\infty, Q_i, Q_j\}$ and $\{\infty, O, Q_j\}$ as before and, finally, $\{\infty, P_i, Q_j\}$.

Remark 3.7. We can also use plane curves with a singularity of a particular kind. Namely, let C' be a plane curve which is smooth outside a rational point ∞' of maximal contact between C' and a line $L_{\infty'}$, with the property that there is a unique point ∞ above ∞' in the normalization C of C'. Then there is a bijection between the closed points on C and those on C'. It follows that we can work on the singular curve C' as described in Construction 3.6 to obtain elements in $K_2^T(C)/\mathrm{torsion}$ as above.

4. Hyperelliptic curves

The Beilinson conjecture on K_2 of hyperelliptic curves was studied by Dokchitser, de Jeu and Zagier in [DdJZ06]. They considered several families of hyperelliptic curves $C/\mathbb{Q}$ which possess at least $g(C)$ elements of $K_2(C;\mathbb{Z})$ using Construction 3.1. We first recall their approach; then we describe how it can be viewed in terms of our geometric interpretation. This leads to a much simpler way of constructing their families (and others). Finally, we discuss integrality of the elements obtained in this manner.

Consider a hyperelliptic curve $C/\mathbb{Q}$ of degree $d \in \{2g+1, 2g+2\}$ with a $\mathbb{Q}$-rational Weierstrass point ∞ and a point $O \in C(\mathbb{Q})$ such that $(O) - (\infty)$ is a d-torsion divisor. Then, if P is another $\mathbb{Q}$-rational Weierstrass point, the divisor $(P) - (\infty)$ is 2-torsion and we can apply Construction 3.1 to find an element $\{\infty, O, P\} \in K_2^T(C)/\mathrm{torsion}$.

Using Riemann-Roch one can show [DdJZ06, Examples 5.3, 5.6] that such a curve C has an affine model

$$F(x,y) = y^2 + f_1(x)y + x^d = 0, \tag{3}$$

such that ∞ is the unique point at infinity on C and $O = (0,0)$. Here

$$f_1(x) = b_0 + b_1x + \ldots + b_gx^g + \delta x^{2g+1} \in \mathbb{Q}[x], \quad \delta = \begin{cases} 0, & d = 2g+1 \\ 2, & d = 2g+2 \end{cases} \tag{4}$$

and $b_g \neq 0$ if $d = 2g+1$; moreover, we have $\mathrm{disc}(-4x^d + f_1(x)^2) \neq 0$.

The results of [DdJZ06] rely on the following result, see [DdJZ06, §6]:

Proposition 4.1. *(Dokchitser-de Jeu-Zagier) Let C be the hyperelliptic curve associated to an affine equation (3). Let $\alpha \in \mathbb{Q}$ be a root of $t(x) = -x^d + f_1(x)^2/4$ and let $P = (\alpha, -f_1(\alpha)/2) \in C(\mathbb{Q})$. Then ∞, O and P define an element $\{\infty, O, P\}$ of $K_2^T(C)/\mathrm{torsion}$.*

Proof. Via the coordinate change $(x,y) \mapsto (x, y - f_1(x)/2)$, the curve C is isomorphic to the hyperelliptic curve with affine equation

$$y^2 - t(x) = 0, \tag{5}$$

where

$$t(x) = -x^d + f_1(x)^2/4.$$

On this model, the affine Weierstrass points are of the form $(\alpha, 0)$, where $t(\alpha) = 0$. So if $\alpha \in \mathbb{Q}$ is a root of t, then $P = (\alpha, -f_1(\alpha)/2) \in C(\mathbb{Q})$ is a rational Weierstrass point and the result follows from the discussion preceding the theorem. □

Remark 4.2. In [DdJZ06, §7,10] suitable polynomials were constructed using clever manipulation of polynomials and brute force computer searches. More precisely, Dokchitser-de Jeu-Zagier show that every $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-orbit $\{P_\sigma\}_\sigma$ of Weierstrass points, and therefore every $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-orbit $\{\alpha_\sigma\}_\sigma$ of roots of $t(x)$, defines an element of $K_2^T(C)/\mathrm{torsion}$. Hence the irreducible factors of $t(x)$ determine the number of elements in $K_2^T(C)/\mathrm{torsion}$ one can obtain in this way. The crucial, and most difficult, step is therefore the construction of polynomials $f_1(x)$ as above such that $t(x)$ has many rational factors.

The results from [DdJZ06] as stated in Proposition 4.1 fit into Construction 3.6, refined according to Remark 3.7: Because a hyperelliptic curve C of genus at least 2 cannot be embedded smoothly into $\mathbb{P}^2$, we consider the projective closure C' in $\mathbb{P}^2$ of the affine curve given by (3). The curve C' has a unique point ∞' at infinity; since this is the only singular point on C', we can identify affine points on C with their images on C' under the normalization morphism $C \to C'$ and Remark 3.7 applies. Then the points ∞' and $O = (0,0) \in C'(\mathbb{Q})$ are points of maximal contact for the line $L_{\infty'}$ at infinity and the tangent line $L_O : y = 0$, respectively, in the sense of Definition 3.4. An affine Weierstrass point P as in Theorem 4.1, viewed as a point on C', has the property that the tangent line L_P to C' at P only intersects C' in P and ∞'. The respective intersection multiplicities are $I_P(L_P, C') = 2$ and $I_{\infty'}(L_P, C') = d-2$. Therefore we recover the elements from Proposition 4.1 by applying Construction 3.6, see Remark 3.7. See Figure 1 on page 9 for an example of this geometric configuration.

Conversely, we now consider a curve C' given as the projective closure of an affine curve given by an equation (3) and a vertical line L_α given by $x - \alpha$, where $\alpha \in \mathbb{Q}$. This line intersects C' in ∞' with multiplicity $d-2$. If it intersects C' in another rational point P with multiplicity 2 (in other words, if L_α is tangent to C' in P), then we are in the situation of Construction 3.6, see Remark 3.7.

Lemma 4.3. *The vertical line L_α is a tangent line to C' in the point $P = (\alpha, \beta) \in C'(\mathbb{Q})$ if and only if*

$$\beta^2 = \alpha^d \ \text{ and } \ f_1(\alpha) = -2\beta. \tag{6}$$

Proof. It is easy to see that (6) is equivalent to

$$F(\alpha, y) = (y - \beta)^2. \tag{7}$$

The line L_α intersects C' only in P and ∞' if and only if (7) holds, because the zeros of the left hand side are precisely the y-coordinates of the affine intersection points between C' and L_α. □

Remark 4.4. A point $P = (\alpha, \beta)$ with tangent line L_α as in Lemma 4.3 is a rational affine Weierstrass point on the normalization C of C' and hence corresponds to a point P as in Proposition 4.1.

We can use Lemma 4.3 as a recipe for *constructing* examples of hyperelliptic curves with a number of given representatives of elements of $K_2^T(C)$/torsion. If we fix x_i and $y_i \in \mathbb{Q}$ such that $y_i^2 = x_i^d$ for $i = 1, \dots, m$ and treat the coefficients $b_0, \dots, b_g$ of f_1 as indeterminates, then the crucial condition (6) $f_1(x_i) = -2y_i$ translates into a system of m linear equations. Generically, the set of solutions to this system is parametrized by $g + 1 - m$ parameters. Therefore the maximal number of representatives of elements of $K_2^T(C)$/torsion that we can construct generically using this approach is $g + 1$. This leads to a unique solution of the system of linear equations, hence to a $g+1$-parameter family of examples.

We first carry this out for the case $d = 2g + 1$.

Lemma 4.5. *Let $g \geq 1$, $a_1, \dots, a_{g+1} \in \mathbb{Q}^\times$ such that $a_1^2, \dots, a_{g+1}^2$ are pairwise distinct. Let $V \in \mathrm{GL}_{g+1}(\mathbb{Q})$ denote the Vandermonde matrix $\left(a_i^{2j}\right)_{\substack{1\leq i\leq g+1\\0\leq j\leq g}}$ and let $w = \left(-2a_i^{2g+1}\right)_{1\leq i\leq g+1} \in$*

$\mathbb{Q}^{g+1}$. *Define*

$$b = (b_0, \dots, b_g) = V^{-1} \cdot w \in \mathbb{Q}^{g+1}$$

and let

$$f_1(x) = b_0 + b_1 x + \ldots + b_g x^g.$$

Then we have

$$f_1(a_i^2) = -2a_i^{2g+1}, \quad i \in \{1, \dots, g+1\}.$$

Proof. This is a consequence of classical interpolation properties of the Vandermonde matrix. □

Proposition 4.6. *Let C denote the hyperelliptic curve associated to the affine model*

$$y^2 + f_1(x)y + x^{2g+1} = 0,$$

where f_1 is as in Lemma 4.5. Then, for every $i \in \{1, \dots, g+1\}$, we get an element $\{\infty, O, P_i\} \in K_2^T(C)/\text{torsion}$, where $P_i = (a_i^2, a_i^{2g+1}) \in C(\mathbb{Q})$.

Proof. The result follows immediately from Lemma 4.5 using Lemma 4.3. □

Note that curves C as in Proposition 4.6 provide examples of the construction in [DdJZ06] such that the two-torsion-polynomial $t(x)$ has at least $g+2$ rational factors. This seems to be the easiest and most natural approach when $d = 2g+1$.

Let us write down the family we get from applying Proposition 4.6 for $g = 2$. This recovers [DdJZ06, Example 7.3], see also [DdJZ06, Remark 7.4].

Example 4.7. Let $a_1, a_2, a_3 \in \mathbb{Q}^\times$ such that a_1^2, a_2^2, a_3^2 are pairwise distinct and set

$$\begin{aligned}
\gamma &= (a_1 + a_2)(a_1 + a_3)(a_2 + a_3) \\
b_0 &= -\frac{2}{\gamma}(a_1a_2 + a_1a_3 + a_2a_3)a_3^2a_2^2a_1^2 \\
b_1 &= \frac{2}{\gamma}(a_1^3a_2^3 + a_1^3a_2^2a_3 + a_1^3a_2a_3^2 + a_1^3a_3^3 + a_1^2a_2^3a_3 + a_1^2a_2^2a_3^2 + a_1^2a_2a_3^3 + a_1a_2^3a_3^2 + a_1a_2^2a_3^3 + a_2^3a_3^3) \\
b_2 &= -\frac{2}{\gamma}(a_1^3a_2 + a_1^3a_3 + a_1^2a_2^2 + 2a_1^2a_2a_3 + a_1^2a_3^2 + a_1a_2^3 + 2a_1a_2^2a_3 + 2a_1a_2a_3^2 + a_1a_3^3 + a_2^3a_3 \\
&\quad + a_2^2a_3^2 + a_2a_3^3).
\end{aligned}$$

By Proposition 4.6, the genus 2 curve

$$C : y^2 + (b_0 + b_1x + b_2x^2)y + x^5 = 0$$

satisfies $\{\infty, O, P_i\} \in K_2^T(C)/\text{torsion}$ for $i = 1, 2, 3$, where $P_i = (a_i^2, a_i^5) \in C(\mathbb{Q})$. See Figure 1 on page 9 for the example $a_1 = 1, a_2 = 1/2, a_3 = 1/4$.

We could also use Lemma 4.3 to find examples for $d = 2g+2, m = g+1$ in a completely analogous way. However, here we have more flexibility. Namely, when $d = 2g+1$ is odd, then we have to make sure that the x-coordinate of the points $P = (\alpha, \beta)$ are rational squares in order to satisfy (6). In contrast, when $d = 2g+2$ is even, then P needs to satisfy

$$\beta = \pm\alpha^{g+1},$$

without any a priori conditions on α.

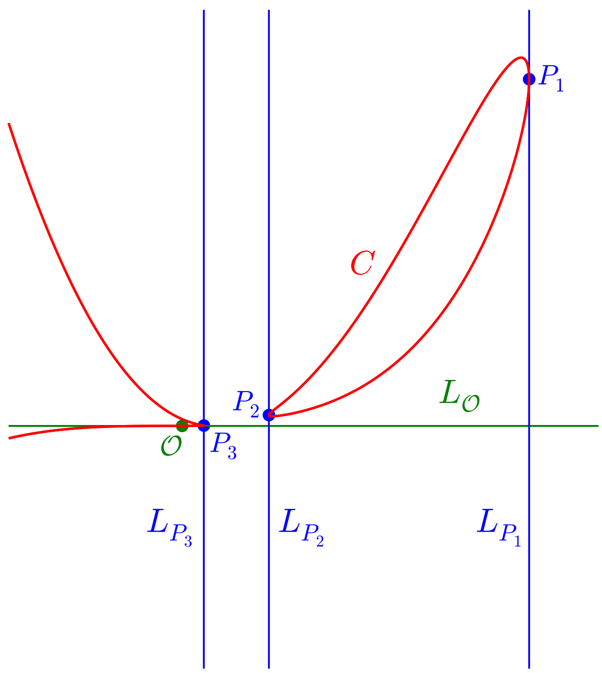

FIGURE 1. The curve C from Example 4.7 with $a_1 = 1, a_2 = 1/2, a_3 = 1/4$

Lemma 4.8. *Let $g \geq 1$ and let $a_1, \ldots, a_{g+1} \in \mathbb{Q}^\times$ be pairwise distinct. Let $V \in \mathrm{GL}_{g+1}(\mathbb{Q})$ denote the Vandermonde matrix $\left(a_i^j\right)_{\substack{1\leq i\leq g+1\\ 0\leq j\leq g}}$ and let $w = \left(-2a_i^{g+1} - 2\epsilon_i a_i^{g+1}\right)_{1\leq i\leq g+1} \in \mathbb{Q}^{g+1}$, where $\epsilon_1, \ldots, \epsilon_{g+1} \in \{\pm 1\}$. Define*

$$b = (b_0, \ldots, b_g) = V^{-1} \cdot w \in \mathbb{Q}^{g+1}$$

and let

$$f_1(x) = b_0 + b_1 x + \ldots + b_g x^g + 2x^{g+1}.$$

Then we have

$$f_1(a_i) = -2\epsilon_i a_i^{g+1}, \quad i \in \{1, \ldots, g+1\}.$$

Proof. As in Lemma 4.8, this follows from classical interpolation properties of the Vandermonde matrix. $\square$

Proposition 4.9. *Let C denote the hyperelliptic curve associated to the affine model*

$$y^2 + f_1(x)y + x^{2g+1} = 0,$$

where f_1 is as in Lemma 4.8 and $\epsilon_1, \ldots, \epsilon_{g+1} \in \{\pm 1\}$ are not all equal to -1. Then, for every $i \in \{1, \ldots, g+1\}$, we get an element $\{\infty, O, P_i\} \in K_2^T(C)/\mathrm{torsion}$, where $P_i = (a_i, \epsilon_i a_i^{g+1}) \in C(\mathbb{Q})$.

Proof. This follows from Lemma 4.8 using Lemma 4.3. We need that not all ϵ_i are equal to -1, as otherwise f_1 is the zero polynomial. $\square$

Remark 4.10. By [DdJZ06, Proposition 6.14] there is a universal relation between the classes of the elements from Proposition 4.9 in $K_2^T(C)/\text{torsion}$.

Curves C as in Proposition 4.9 provide examples of the construction in [DdJZ06] such that the two-torsion-polynomial $t(x)$ has at least $g+2$ rational factors. In the language of [DdJZ06], the indices i such that $\epsilon_i = -1$ correspond to prescribed rational roots of the first factor in the factorisation

$$t(x) = \frac{f_1(x)^2}{4} - x^{2g+2} = \left(\frac{f_1(x)}{2} - x^{g+1}\right)\left(\frac{f_1(x)}{2} + x^{g+1}\right) \tag{8}$$

of $t(x)$, whereas the indices i such that $\epsilon_i = 1$ correspond to prescribed rational roots of the second factor. From this point of view it is clear that we cannot prescribe $g+1$ points P_i having $\epsilon_i = -1$, because the first factor $\frac{f_1(x)}{2} - x^{g+1}$ only has degree g.

Proposition 4.9 with $\epsilon_i = 1$ for $1 \le i \le g+1$ recovers the "generic" case of [DdJZ06, Example 7.9]. Moreover, [DdJZ06, Example 10.5] is a special case of Proposition 4.9 when $g = 2$.

As Example 4.7 suggests, the curves we obtain using Propositions 4.6 and 4.9 may have rather unwieldy coefficients, which complicates, for instance, numerically verifying Conjecture 2.1. Instead, we can prescribe $m \le g$ vertical tangents at rational points using Lemma 4.3. Soling the resulting system of linear equations, the parametrized set of solutions may yield curves whose coefficients are relatively small and for which we know at least m representatives of elements of $K_2^T(C)/\text{torsion}$. In light of Conjecture 2.1, this is especially interesting when $m = g$. We do not go into detail, but note that taking $g = m = 2$, $d = 5$ recovers [DdJZ06, Example 7.3] and hence [DdJZ06, Example 10.1]. Taking $g = 3$, $d = 6, m = 2, a_1 = 1$ and $a_2 = 1/2$ recovers [DdJZ06, Example 10.6]. Furthermore, taking g arbitrary, $d = 2g+2$, $m = g$ and $\epsilon_i = -1$ for $1 \le i \le g$ recovers [DdJZ06, Example 10.8]. This example is especially important, because de Jeu [dJ05] showed that for this family one gets g *independent* elements of $K_2^T(C)/\text{torsion}$ using a limit formula for the regulator of these elements, see also [LdJ15, §6].

Remark 4.11. Note that in all examples for $d = 2g+2$ considered in [DdJZ06], the prescribed roots of the 2-torsion polynomial $t(x)$ were either all roots of the first factor (corresponding to $\epsilon_i = -1$) or all roots of the second factor (corresponding to $\epsilon_i = -1$) in the factorisation (8). Therefore we can easily construct examples which do not appear in [DdJZ06], by considering a "mixed" situation, where not all ϵ_i are the same. This observation is due to Rob de Jeu.

Example 4.12. We apply Proposition 4.9 with $g = 2$, $\epsilon_1 = 1$ and $\epsilon_2 = \epsilon_3 = -1$. Setting

$$\gamma = a_1^2 - a_1a_2 - a_1a_3 + a_2a_3,$$

this leads to the following family, where $a_1, a_2, a_3 \in \mathbb{Q}^\times$ are pairwise distinct:

$$C : y^2 + \left(-\frac{4a_1^3a_2a_3}{\gamma} + \frac{4a_1^3(a_2+a_3)}{\gamma}x - \frac{4a_1^3}{\gamma}x^2 + 2x^3\right)y + x^6 = 0$$

Here we have 3 elements $\{\infty, O, P_i\} \in K_2^T(C)/\text{torsion}$, where $P_1 = (a_1, a_1^3)$, $P_2 = (a_2, -a_2^3)$, $P_3 = (a_3, -a_3^3) \in C(\mathbb{Q})$.

Remark 4.13. So far we have only considered hyperelliptic curves with a unique point at infinity. The main reason is that in this situation Remark 3.7 applies, so we can use Construction 3.1. In the situation where O is a point of maximal contact, but there are two points at infinity lying above the singular point at infinity on C', the construction discussed in the present section breaks down. In Section 7 we will show how to construct examples of hyperelliptic curves with two points at infinity with a given element of $K_2^T(C)/\text{torsion}$ using a different method, see Example 7.5.

4.1. **Integrality.** Lemma 4.3 provides a method for constructing hyperelliptic curves $C/\mathbb{Q}$ with given elements of $K_2^T(C)/\text{torsion}$ as in Propositions 4.6 and 4.9. Provided a simple condition is satisfied, it is easy to deduce from the results of [DdJZ06] that these elements are actually integral. Recall that we call an element of $K_2^T(C)/\text{torsion}$ integral if it lies in $K_2(C;\mathbb{Z})$, the subgroup of $K_2^T(C)/\text{torsion}$ that appears in the formulation of Beilinson's Conjecture 2.1. Dokchitser, de Jeu and Zagier prove the following result:

Theorem 4.14. *Let $g \geq 1$, let $d \in \{2g+1, 2g+2\}$ and let f_1 be as in (4) such that f_1 has integral coefficients. Consider a rational root α of $t(x) = -x^d + f_1(x)^2/4$ and let $P = (\alpha, -f_1(\alpha)/2) \in C(\mathbb{Q})$, where C is the hyperelliptic curve given by the affine equation $y^2 + f_1(x)y = x^d$. Then, if $1/\alpha \in \mathbb{Z}$, the element $2\{\infty, O, P\}$ is integral.*

Proof. This is a special case of [DdJZ06, Theorem 8.3]. □

Corollary 4.15. *In the situation of Proposition 4.6 or Proposition 4.9, let $1/a_i \in \mathbb{Z}$. Then the element $2\{\infty, O, P_i\}$ is integral.*

Proof. Recall that the points $P_i = (x_i, y_i)$ are precisely the rational affine Weierstrass points P which appear in Proposition 4.1. So if $f_1 \in \mathbb{Z}[x]$ and $1/x_i \in \mathbb{Z}$, then Theorem 4.14 implies that $2\{\infty, O, P_i\}$ is integral.

If $f_1 \notin \mathbb{Z}[x]$, then we can find a polynomial $\widetilde{f}_1 \in \mathbb{Z}[x]$ and a rational number $a \in \mathbb{Q}^\times$ such that the given model of C can be transformed into

$$\widetilde{C} : y^2 + \widetilde{f}_1(x)y + x^d = 0$$

via the transformation $\psi : C \to \widetilde{C}$ taking $(x, y) \in C$ to $(x, ay) \in \widetilde{C}$. Hence the reciprocal of the x-coordinate of $\psi(P_i)$ is equal to $1/x_i$. So if $1/x_i \in \mathbb{Z}$, then

$$2\psi(\{\infty, O, P_i\}) = 2\{\psi(\infty), \psi(O), \psi(P_i)\}$$

is integral by Theorem 4.14, and therefore $2\{\infty, O, P_i\}$ is integral as well. □

In particular, for each $m \leq g+1$, we can find families of hyperelliptic curves in m integral parameters having m explicitly given elements of $K_2(C;\mathbb{Z})$. Unfortunately, it is in general not at all easy to check how many of these elements are actually independent, but see [LdJ15].

5. Elements on quartics coming from line configurations

We saw in the previous section that by results of [DdJZ06] and [dJ05] one can construct hyperelliptic curves C of arbitrary genus g with at least g independent elements of $K_2^T(C)/\text{torsion}$ (and even $K_2(C;\mathbb{Z})$) using vertical tangent lines and lines having maximal contact multiplicity with C. We would like to adapt this strategy to non-hyperelliptic curves. Because all genus 2 curves are hyperelliptic, it is natural to consider smooth plane quartics, since every non-hyperelliptic curve of genus 3 can be canonically embedded into $\mathbb{P}^2$ as a smooth plane quartic.

Similar to Section 4, we will use smooth projective plane curves C of degree 4 having two rational points ∞ and O of maximal contact with lines L_∞ and L_O, respectively. Such points are usually called *hyperflexes*. The difference is that in the quartic situation, such a curve can be embedded as a smooth curve into $\mathbb{P}^2$ (without weights), so that we can work directly with the smooth curve C and the lines are simply the respective tangent lines to C. Using a transformation, if necessary, we might as well assume that $\infty = (0:1:0)$ is the unique point at infinity on C and that $O = (0:0:1)$. Vermeulen [Ver] showed that in this case C can be described as the projective closure in $\mathbb{P}^2$ of the affine curve given by an equation

$$F(x,y) := y^3 + f_2(x)y^2 + f_1(x)y + x^4 = 0, \tag{9}$$

where $\deg(f_1) \leq 2$ and $\deg(f_2) \leq 1$. For the remainder of the present section, we assume that C is of this form. Then the divisor $(O) - (\infty)$ is a torsion divisor induced by the tangent line $L_O : y = 0$, since

$$\mathrm{div}(y) = 4(O) - 4(\infty).$$

For our purposes, the smooth plane quartic analogue of a rational point on a hyperelliptic curve with a vertical tangent is an inflection point having a vertical tangent $L_\alpha = x - \alpha$, where $\alpha \in \mathbb{Q}$. Such a point $P \in C$ has the property that L_α intersects C in P with multiplicity 3 and in ∞ with multiplicity 1. If such a point P exist, then we are in the situation of Construction 3.1 and hence we get an element

$$\{\infty, O, P\} \in K_2^T(C)/\text{torsion}.$$

We first translate this geometric condition on the point P (or rather the line L_α) into a condition on the polynomials f_1 and f_2

Lemma 5.1. *The vertical line L_α intersects C in the point $P = (\alpha, \beta) \in C(\mathbb{Q})$ with multiplicity 3 if and only if*

$$\beta^3 = -\alpha^4, \quad f_1(\alpha) = 3\beta^2 \ \textit{ and } \ f_2(\alpha) = -3\beta. \tag{10}$$

Proof. In analogy with the proof of Lemma 4.3, the conditions (10) are equivalent to

$$F(\alpha, y) = (y - \beta)^3. \tag{11}$$

The assertion follows, because the roots of $F(\alpha, y)$ are the y-coordinates of the affine intersection points between C and L_α. □

For a positive integer m, we can try to use Lemma 5.1 to construct smooth plane quartics $C : F = 0$, where F is as in (9), which have m inflection points P_i with vertical tangent lines, giving rise to m elements $\{\infty, O, P_i\} \in K_2^T(C)/\text{torsion}$ by virtue of Construction 3.1, similar to Proposition 4.6 and 4.9. This amounts to solving a system of linear equations, given by (10). But we can only expect polynomials f_1 and f_2 as in the Lemma to exist if $m \leq 2$, because $\deg(f_2) \leq 1$. The case $m = 2$ would lead to a smooth plane quartic C for which we have elements

$$\{\infty, O, P_1\}, \{\infty, O, P_2\}, \{\infty, P_1, P_2\} \in K_2^T(C)/\text{torsion}.$$

However, by a calculation analogous to [DdJZ06, Example 5.2], the element $\{\infty, P_1, P_2\}$ is trivial in $K_2^T(C)/\text{torsion}$, so that for $m \leq 2$ we only get at most 2 independent elements of $K_2^T(C)/\text{torsion}$ because of parts (iii), (iv) and (v) of Proposition 3.2.

Remark 5.2. Suppose that F is of the form (9) and that $C : F = 0$ is the corresponding smooth plane quartic and such that there are 3 distinct inflection points $P_i = (x_i, y_i) \in C(\mathbb{Q})$ having vertical tangents. Then it turns out that such a curve corresponds to a nonsingular $\mathbb{Q}$-rational point (a_1, a_2, a_3) on the projective curve defined in $\mathbb{P}^2$ by

$$a_1^2a_2^2 + a_1^2a_2a_3 + a_1^2a_3^2 + a_1a_2^2a_3 + a_1a_2a_3^2 + a_2^2a_3^2 = 0,$$

where $x_i = a_i^3$ and $y_i = -a_i^4$. Using (for instance) the computer algebra system `Magma` [BCP97], one sees easily that this curve has 3 singular points $(0:0:1), (0:1:0)$ and $(1:0:0)$ and its normalization is isomorphic over $\mathbb{Q}$ to the conic defined by

$$X^2 + XY + Y^2 + XZ + YZ + Z^2 = 0\,,$$

which has no rational points. Hence, no such curve C can exist.

Therefore inflection points with vertical tangent lines are not sufficient to construct 3 independent elements of $K_2^T(C)/\text{torsion}$ for a smooth plane quartic C with two hyperflex points in ∞ and O. Instead, we will only use one such inflection point. In addition, we will construct an inflection point whose tangent line also intersects the quartic in O (instead of ∞). To this end, we work on a different model D, where the parts of O and ∞ are interchanged and use an inflection point on D with a vertical tangent. These conditions lead to a system of linear equations whose solution gives us the family presented in Theorem 5.3 below. See Figure 2 on page 17 for an example of this geometric configuration.

Recall that the *discriminant* of a projective plane quartic curve is defined to be the discriminant of the ternary quartic form defining the curve, cf. [Sal60]. It vanishes if and only if the curve is singular.

Theorem 5.3. *Let $a, b \in \mathbb{Q}^\times$ such that $a \neq b$, let $c \in \mathbb{Q}$ and let*

$$f_1(x) = a^6b^6 + a^3b^3cx + (3a^2 - b^6 - b^3c)x^2 \quad \text{and} \quad f_2(x) = 3a^4 - a^3c + cx. \tag{12}$$

Suppose that $\operatorname{disc}(C) \neq 0$. *Then the smooth plane quartic C given by the affine equation*

$$F(x,y) = y^3 + f_2(x)y^2 + f_1(x)y + x^4 = 0$$

has the following properties:

(i) We have $P = (a^3, -a^4) \in C(\mathbb{Q})$ and the tangent line $L_P : x = a^3$ through P has contact multiplicity 3 with C; the other point of intersection is ∞.

(ii) We have $Q = (-a^2b^3, -a^2b^6) \in C(\mathbb{Q})$ and the tangent line $L_Q : y = b^3x$ through Q has contact multiplicity 3 with C; the other point of intersection is O.

(iii) We get the following elements of $K_2^T(C)/\text{torsion}$:

$$\{\infty, O, P\},\ \{\infty, O, Q\},\ \{\infty, P, Q\}$$

Proof. We have

$$(a^3)^4 = -(-a^4)^3, \quad f_1(a^3) = 3a^8 \quad \text{and} \quad f_2(a^3) = 3a^4,$$

so that (i) follows immediately from Lemma 5.1.

To prove (ii), we again want to use Lemma 5.1. To this end, we apply the projective transformation

$$\varphi : C \to D, \quad (X:Y:Z) \mapsto (X : a^2b^2Z : Y),$$

where D is the smooth plane quartic given by the affine equation

$$D : G(w,z) = z^3 + g_2(w)z^2 + g_1(w)z + w^4 = 0.$$

This transformation maps O to the unique point at infinity $(0:1:0) \in D$, it maps ∞ to $(0,0) \in D$ and Q to $\varphi(Q) = (\frac{1}{b^3}, -\frac{1}{b^4})$. Because we also have

$$g_1\left(\frac{1}{b^3}\right) = \frac{3}{b^8} \quad \text{and} \quad g_2\left(\frac{1}{b^3}\right) = -\frac{3}{b^4},$$

Lemma 5.1 is applicable exactly as in (i) and assertion (ii) follows.

It remains to prove (iii). For this it suffices to note that by (i) we have

$$\operatorname{div}(x-a^3) = 3(P) - 3(\infty)$$

and by (ii) we have

$$\operatorname{div}\left(\frac{y-b^3x}{y}\right) = 3(Q) - 3(O),$$

where we consider the functions on the left hand sides as functions on C. Hence the pairwise differences $(R) - (S)$ are torsion divisors for all R, S in the set $\{\infty, O, P, Q\}$ and the result follows from Proposition 3.2. $\square$

Remark 5.4. If $c = 0$ in Theorem 5.3, then we have the additional rational points $P' = (-a^3, -a^4)$ and $Q' = (a^2b^3, -a^2b^6)$ on C. The respective tangent lines $L_{P'} : x = -a^3$ and $L_{Q'} : y = -b^3x$ through P' and Q' also have contact multiplicity 3 with C at these points and intersect C in ∞ and O, respectively. This yields the following additional elements of $K_2^T(C)/\text{torsion}$:

$$\{\infty, O, P'\}, \{\infty, O, Q'\}, \{\infty, P', Q'\}, \{\infty, P, Q'\}, \{\infty, P', Q\}, \{\mathcal{O}, P', Q\}, \{\infty, P, Q'\}$$

Remark 5.5. Invariants of non-hyperelliptic curves of genus 3 can be computed, for instance, using David Kohel's `Magma` package `Echidna` [Koh]. With its aid, we find that the discriminant of the curve C from Theorem 5.3 is equal to

$$\operatorname{disc}(C) = a^{42}b^{36}(a+b^3)^3(2a-2b^3-c)^6(3a-2b^3-c)(3a-b^3-c)(6a+2b^3+c)^2q(a,b,c),$$

where

$$\begin{aligned} q(a,b,c) = {} & 9a^5 + 30a^4b^3 - 24a^4c + 4a^3b^6 + 23a^3b^3c + 16a^3c^2 + 12a^2b^9 + 3a^2b^6c + 12a^2b^3c^2 \\ & - 3ab^{12} - 3ab^9c - 2b^{15} - 5b^{12}c - 4b^9c^2 - b^6c^3, \end{aligned}$$

making it easy to check whether C is smooth or not.

Remark 5.6. It is tempting to generalize the approach described above to non-hyperelliptic curves of genus $g > 3$ which have two points of maximal contact with lines and have tangent lines which pass through one of these two points and precisely one other point, However, we have found that this does not allow us to construct g elements of $K_2^T(C)/\text{torsion}$, since the dimensions of the solution spaces of the resulting systems of linear equations are too small.

5.1. **Integrality.** Now we investigate when the elements of $K_2^T(C)/\text{torsion}$ constructed in Theorem 5.3 are integral, It turns out that we can easily write down a family in three integral parameters such that two of the elements are integral, but if we want all three of them to be integral, then we have to restrict to a one parameter subfamily.

Proposition 5.7. *Let $a, b \in \mathbb{Q}^\times$ such that $a \neq b$, let $c \in \mathbb{Q}$ and let C be as in Theorem 5.3 such that* $\operatorname{disc}(C) \neq 0$.

(i) If $1/a, b, c \in \mathbb{Z}$, then $\{\infty, O, P\}$ and $\{\infty, O, Q\}$ are integral.
(ii) If $a = \pm\frac{1}{2}, b = \mp 1, c \in \mathbb{Z}$, then $2\{\infty, P, Q\}$ is integral.

Proof. To prove part (i), suppose that $d := 1/a, b, c \in \mathbb{Z}$. Then the smooth plane quartic given by the affine model

$$\widetilde{C} : d^{18}y^3 + d^{12}f_2(x)y^2 + d^6f_1(x)y + x^4$$

has coefficients in $\mathbb{Z}$ and is isomorphic to C via the transformation

$$\psi : C \to \widetilde{C}; \qquad (x, y) \mapsto (x, a^6y).$$

By functoriality, the element $\{\infty, O, P\}$ induces an element of $K_2(C;\mathbb{Z})$ if and only if its image under ψ induces an element of $K_2(\widetilde{C};\mathbb{Z})$.

Applying Construction 3.1 to the points ∞, O and P, we find that

$$\{\infty, O, P\} = \left\{-\frac{y}{a^4}, 1 - \frac{x}{a^3}\right\}.$$

Hence we get

$$S_1 := \psi(\{\infty, O, P\}) = \{\psi(\infty), \psi(O), \psi(P)\} = \{h_1, h_2\} \in K_2^T(\widetilde{C})/\text{torsion},$$

where $h_1 = -d^{10}y$ and $h_2 = 1 - d^3x$.

We will show integrality of this element using a strong desingularization $\widetilde{\mathcal{C}}$ (cf. [Liu02, §8.3.4]) of the Zariski closure $\widetilde{C}^Z$ of $\widetilde{C}$ in $\mathbb{P}^2_{\mathbb{Z}}$. In other words, $\widetilde{\mathcal{C}}$ is a proper regular model of $\widetilde{C}$ such that there exists a proper birational morphism

$$\pi : \widetilde{\mathcal{C}} \to \widetilde{C}^Z$$

which is an isomorphism outside the singular locus of $\widetilde{C}^Z$. Following the proof of [DdJZ06, Theorem 8.3], we will prove the integrality of the class of S_1 using the behaviour of the functions h_1 and h_2 on $\widetilde{C}^Z$.

If p is a prime such that $\mathrm{ord}_p(d) > 0$, then $h_2 = 1$ on $\widetilde{\mathcal{C}}_p$ and hence $T_\Gamma(S_1) = 1$ for every irreducible component Γ of $\widetilde{\mathcal{C}}_p$.

Hence we may assume that $\mathrm{ord}_p(d) = 0$. If Γ is an irreducible component of $\widetilde{\mathcal{C}}_p$ such that $\pi(\Gamma) = \Delta$ is an irreducible component of $\widetilde{C}^Z_p$, then it is easy to see from the given equation of $\widetilde{C}$ that

$$\mathrm{ord}_\Gamma(h_1) = \mathrm{ord}_\Gamma(h_2) = 0,$$

which implies that $T_\Gamma(S_1) = 1$.

In order to finish the proof that S_1 induces an element of $K_2(\widetilde{C};\mathbb{Z})$, we have to consider the case of a prime number p such that $\mathrm{ord}_p(d) = 0$ and such that $\pi(\Gamma) = P_0$ is a singular point of $\widetilde{C}^Z$. One checks directly that P_0 must be an affine point of $\widetilde{C}^Z_p$, say $P_0 = (x_0, y_0)$. Now we apply a case distinction: If $x_0 = 0$, then $h_2(P_0) = 1$ follows, implying that $T_\Gamma(S_1) = 1$. If, on the other hand, $x_0 \neq 0$, then we can only have $\mathrm{ord}_\Gamma(h_2) \neq 0$ if $x_0 = a^3$, implying $y_0 = -a^{10}$ and hence $h_1(P_0) = 1$ and $T_\Gamma(S_1) = 1$. This proves that the class of S_1 in $K_2^T(C)/\text{torsion}$ is integral.

Next we show that the element $\{\infty, O, Q\}$ is integral. We work on the curve $\widetilde{D}$ which is defined as the image of $\widetilde{C}$ under the transformation

$$\widetilde{\varphi}(x, y) = (w, z) = \left(\frac{x}{y}, \frac{1}{y}\right).$$

There is a commutative diagram

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\psi"] & D \arrow[d] \\ \widetilde{C} \arrow[r, "\widetilde{\varphi}"] & \widetilde{D}, \end{tikzcd}$$

where D is as in the proof of Theorem 5.3 and the vertical morphism on the right maps (w, z) to (w, a^2z). Therefore it follows that

$$\widetilde{\varphi}(\psi(\{\infty, O, Q\})) = \{-b^6z, 1 - b^3w\}$$

is integral using an argument analogous to the one employed above for S_1. This finishes the proof of (i).

Now we move on to a proof of (ii). Suppose that $d = 1/a, b, c \in \mathbb{Z}$. As in the proof of (i), we work on the curve $\widetilde{C}$ and we compute

$$S_2 := \psi(\{\infty, P, Q\}) = \{h_3, h_4\},$$

where

$$h_3 = -\frac{a^2(b^3+a)}{x-a^3}, \quad h_4 = -\frac{(y-a^6b^3x)^4}{a^{26}(b^3+a)^4y}.$$

If p is a prime such that $\mathrm{ord}_p(a) < 0$, then a simple calculation shows that $h_3 = 1$ on $\widetilde{\mathcal{C}}_p$ and hence $T_\Gamma(S_2) = 1$ for every irreducible component Γ of $\widetilde{\mathcal{C}}_p$. Hence S_2 is integral for $p = 2$ when $a = \pm\frac{1}{2}$ and $b = \mp 1$.

Suppose that p is a prime such that $\mathrm{ord}_p(a) = 0$ and $\mathrm{ord}_p(b^3+a) = 0$. Note that the second condition is satisfied for every prime $p \neq 2$ when $a = \pm\frac{1}{2}$ and $b = \mp 1$. If Γ is an irreducible component of $\widetilde{\mathcal{C}}_p$ such that $\pi(\Gamma) = \Delta$ is an irreducible component of $\widetilde{C}^Z_p$, then the conditions on p imply $\mathrm{ord}_\Gamma(h_3) = \mathrm{ord}_\Gamma(h_4) = 0$ and thus $T_\Gamma(S_2) = 1$.

It remains to consider the situation where Γ is an irreducible component of $\widetilde{\mathcal{C}}_p$ such that $\pi(\Gamma) = P_0$ is a singular point of $\widetilde{C}^Z_p$. Such a point P_0 must be an affine point because $\mathrm{ord}_p(a) = 0$, say $P_0 = (x_0, y_0)$. We distinguish cases as follows:

If $x_0 = 0$, then we have

$$h_3(P_0) = \frac{b^3+a}{a} \quad h_4(P_0) = -\frac{y_0^3}{a^{26}(b^3+a)^4}.$$

Hence we deduce $\mathrm{ord}_\Gamma(h_3) = 0$ and, if $y_0 \neq 0$, also $\mathrm{ord}_\Gamma(h_4) = 0$. If, on the other hand, $y_0 = 0$, then we find $\mathrm{ord}_\Gamma(h_4) > 0$ which could potentially cause problems. However, if $a = \pm\frac{1}{2}$ and $b = \mp 1$, then we actually have $\frac{b^3+a}{a} = -1$, so that $T_\Gamma(2S_2) = 1$.

If $x_0 \neq 0$, then we must also have $y_0 \neq 0$, due to the defining equation of $\widetilde{C}$. This final case is easy because of our knowledge of the zeros and poles of h_4 and h_3 on $\widetilde{C}$: Namely, if $\mathrm{ord}_\Gamma(h_4) \neq 0$, then P_0 is the reduction of $\psi(Q)$ and therefore, by construction, $h_3(P_0) = 1$. Similarly, P_0 must be the reduction of $\psi(P)$ whenever $\mathrm{ord}_\Gamma(h_3) \neq 0$; hence $h_4(P_0) = 1$.

We conclude that if $a = \pm\frac{1}{2}$ and $b = \mp 1$, then $T_\Gamma(2S_2)$ is trivial for all irreducible components. The integrality of $2\{\infty, P, Q\}$ follows. □

Remark 5.8. If $c = 0$ in Theorem 5.3, then by Remark 5.4, we have the additional points $P' = (-a^3, -a^4)$, $Q' = (a^2b^3, -a^2b^6) \in C(\mathbb{Q})$ and the following additional elements of $K_2^T(C)/\mathrm{torsion}$:

$$\{\infty, O, P'\}, \{\infty, O, Q'\}, \{\infty, P', Q'\}, \{\infty, P, Q'\}, \{\infty, P', Q\}$$

If $1/a, b \in \mathbb{Z}$, then at least the first two of these are integral.

Using Theorem 5.3 and Proposition 5.7 we can write down a one-parameter family of plane quartics C_t such that any smooth C_t with $t \in \mathbb{Z}$ has at least 3 representatives of elements of $K_2(C; \mathbb{Z})$ which we can describe explicitly.

Corollary 5.9. *Consider the family of plane quartics defined by*

$$C_t : y^3 + txy^2 + (1/8t + 3/16)y^2 - (t + 1/4)x^2y - 1/8txy + 1/64y + x^4 = 0$$

where $t \in \mathbb{Q}$. Then we have the following properties:

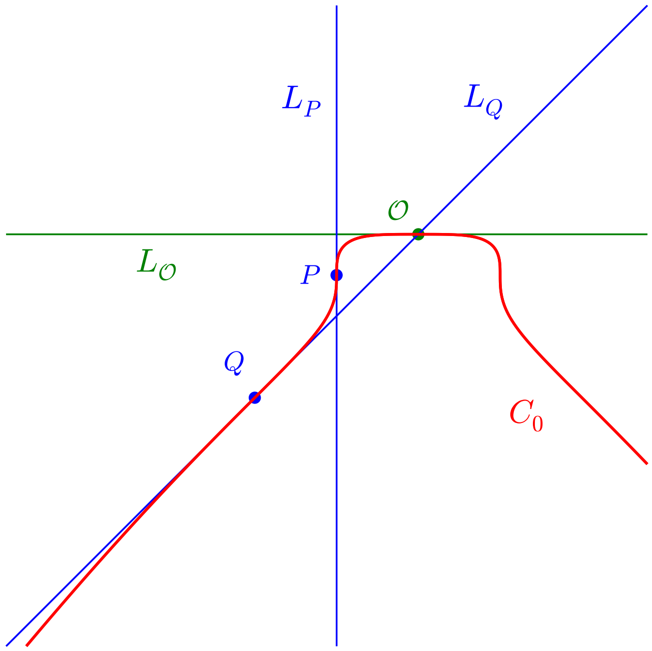

FIGURE 2. The curve C_0 from Corollary 5.9

(i) The curve C_t is smooth unless $t \in \{1, -3, -5/2, -7/2\}$.
(ii) If $t_1, t_2 \in \mathbb{Z}$ are distinct, then C_{t_1} and C_{t_2} are not isomorphic.
(iii) If $t \in \mathbb{Z} \setminus \{1, -3\}$, then the following elements of $K_2^T(C_t)/\mathrm{torsion}$ are integral:

$$\{\infty, O, P\}, \{\infty, O, Q\}, 2\{\infty, P, Q\},$$

where $P = (-\frac{1}{8}, -\frac{1}{16})$, $Q = (-\frac{1}{4}, -\frac{1}{4}) \in C_t(\mathbb{Q})$.
If $t \in \mathbb{Z} \setminus \{1, -3\}$, then by means of the tangent lines $L_P : x = -1/8$ and $L_Q : y = x$ at the points $P = (-\frac{1}{8}, -\frac{1}{16})$, $Q = (-\frac{1}{4}, -\frac{1}{4}) \in C_t(\mathbb{Q})$ we get the following integral elements of $K_2^T(C_t)/\mathrm{torsion}$:

$$\{\infty, O, P\}, \{\infty, O, Q\}, 2\{\infty, P, Q\},$$

Proof. The discriminant of C_t factors as

$$2^{-52}(t-1)^2(t+3)^6(2t+5)(2t+7)(32t^3+96t^2-12t+5),$$

proving (i).

To prove (ii) it suffices to compute the first Dixmier-Ohno invariant (see [Dix87]) I_3 of C_t which is equal to $I_3 = 2^{-7}(16t^3+84t^2+98t-15)$ and vanishes if and only if C_t is singular. Elementary arguments show that no two distinct integers can lead to the same I_3.

The equation for C_t is derived from the equation of the curve C in the Theorem 5.3 by specialising to $a = -\frac{1}{2}$ and $b = 1$ and replacing c by t. Hence (iii) follows immediately from Proposition 5.7. □

See Figure 2 on page 17 for the case $t = 0$.

Remark 5.10. Instead of choosing $a = -\frac{1}{2}$ and $b = 1$ in (12), we could have chosen the family of curves C'_t obtained by setting $a = \frac{1}{2}$ and $b = -1$. But if $t \in \mathbb{Q}$, then we have $C_t \cong C'_{-t}$ via $(x, y) \mapsto (-x, y)$.

6. Elements on quartics coming from conics and lines

If we want to apply Construction 3.1, then it is not necessary to restrict to the situation where the elements in K_2^T are constructed using lines. Instead it is also possible to use curves of higher degree and we give an example of such a construction: We find a family of smooth plane quartics over $\mathbb{Q}$ that have elements in K_2^T which are constructed using lines and conics. See Figure 3 on page 20.

As in Section 5, we work on smooth plane quartics with two hyperflex points $\infty = (0 : 1 : 0)$ and $O = (0, 0)$ with respective tangent lines L_∞ (the line at infinity) and $L_O : y = 0$, so we may assume that they are given as in (9), namely by an equation

$$F(x, y) = y^3 + f_2(x)y^2 + f_1(x)y + x^4 = 0, \tag{13}$$

where $f_1,\ f_2 \in \mathbb{Q}[x]$, $\deg(f_1) \le 2$ and $\deg(f_2) \le 1$.

Suppose that $D/\mathbb{Q}$ is a projective conic, defined by the affine equation

$$(y + d_1x + d_2)^2 + d_3x + d_4x^2 = 0, \tag{14}$$

where $d_1, \dots, d_4 \in \mathbb{Q}$. Then we can construct an equation (13) of a plane quartic C having maximal contact with D.

Lemma 6.1. *Let $d_1, \dots, d_4 \in \mathbb{Q}$ and let*

$$F(x, y) = ((y + d_1x + d_2)^2 + d_3x + d_4x^2)y + x^4. \tag{15}$$

Then the plane quartic C defined by $F(x, y) = 0$ has contact multiplicity 8 with the conic D at the point $R = (0, -d_2)$. Moreover, if C is smooth, then we have an element

$$\{\infty, O, R\} \in K_2^T(C)/\text{torsion}.$$

Proof. Since we can add a multiple of the defining equation of D to the defining equation of C without changing the intersection multiplicity $I_R(C, D)$ between C and D at R, the latter is equal to $4I_R(L, D)$, where L is the y-axis. Hence $I_R(C, D) = 8$. This means that the function

$$h = (y + d_1x + d_2)^2 + d_3x + d_4x^2 \in K(C)^\times$$

satisfies

$$\operatorname{div}(h) = 8(R) - 8(\infty).$$

Because we have

$$\operatorname{div}(y) = 4(O) - 4(\infty)$$

as before, Construction 3.1 is applicable and the claim follows. □

So if C is as in Lemma 6.1, and C is smooth, then we know one representative of an element of $K_2^T(C)/\text{torsion}$. Smoothness can be checked by computing the discriminant of C using `Echidna` [Koh]; it turns out that, in particular, C is singular when $d_2 = 0$ or $d_3 = 0$. In order to construct further elements, we can combine Lemma 6.1 with the approaches discussed in Section 5. Namely, we can choose the coefficients d_i so that, in addition to intersecting D in R with multiplicity 8, C has contact multiplicity 3 with a vertical tangent line L_P in a $\mathbb{Q}$-rational point P. This forces certain conditions on the d_i to be satisfied; these are spelt out in Lemma 5.1.

Example 6.2. Let $a, d_1, d_4 \in \mathbb{Q}$ and let

$$d_2 = \frac{3}{2}a^4 - a^3 d_1 \ \text{ and } \ d_3 = \frac{3}{4}a^5 - d_4 a^3.$$

Then, if the discriminant of the plane quartic $C : F = 0$ is nonzero, where F is as in (13), the conditions of Lemma 5.1 are satisfied for the point $P = (a^3, -a^4) \in C(\mathbb{Q})$. Namely, C is smooth and the vertical line $L_P : x = a^3$ intersects C in P with multiplicity 3 and in ∞ with multiplicity 1. By Lemma 5.1 and Lemma 6.1, the following are elements of $K_2^T(C)$/torsion:

$$\{\infty, O, P\}, \{\infty, O, R\}, \{\infty, P, R\}$$

Here the discriminant of C factors as

$$a^{42}(3a^2 - 4d_4)^2(3a - 2d_1)^8(13a^2 - 4ad_1 - 4d_4)^3 q(a, d_1, d_4),$$

where q is homogeneous of degree 12 if we endow a and d_1 with weight 1 and d_4 with weight 2.

In fact we can do better: We can find a subfamily of the family from Example 6.2 in two parameters such that there are two vertical tangent lines L_{P_1} and L_{P_2} intersecting C in $\mathbb{Q}$-rational points P_1 and P_2, respectively, with contact multiplicity 3. To this end, we simply have to solve a system of linear equations in $d_1, \ldots, d_4$, given by the conditions of Lemma 5.1.

Example 6.3. Let $a_1,\ a_2 \in \mathbb{Q}$ and set

$$d = a_1^2 + a_1a_2 + a_2^2,\ d_1 = \frac{3}{2d}(a_1^3 + a_1^2a_2 + a_1a_2^2 + a_2^3),\ d_2 = -\frac{3}{2d}a_1^3a_2^3,$$

$$d_3 = -\frac{3}{4d}a_2^3a_1^3(a_1 + a_2),\ d_4 = \frac{3}{4d}(a_1^4 + a_1^3a_2 + a_1^2a_2^2 + a_1a_2^3 + a_2^4).$$

Let F be as in (13) and let C be the projective closure of the affine curve given by $F = 0$. Then, if $\operatorname{disc}(C) \neq 0$, the vertical line $L_{P_i} : x = a_i^3$ is tangent to C in the point $P_i = (a_i^3, -a_i^4) \in C(\mathbb{Q})$, where $1 \le i \le 2$. In this case, the following are elements of $K_2^T(C)$/torsion:

$$\{\infty, O, P_1\}, \{\infty, O, R\}, \{\infty, R, P_1\}, \{\infty, O, P_2\}, \{\infty, R, P_2\}$$

The discriminant of C factors as

$$\begin{aligned}\operatorname{disc}(C) = -&\frac{43046721a_1^{42}a_2^{42}}{67108864(a_1^2 + a_1a_2 + a_2^2)^{22}}(4a_1^3 + 8a_1^2a_2 + 12a_1a_2^2 + 3a_2^3)^3(a_1 - a_2)^4(a_2 + a_1)^2\\ &(a_1^8 + 22a_1^7a_2 + 67a_1^6a_2^2 + 140a_1^5a_2^3 + 161a_1^4a_2^4 + 140a_1^3a_2^5 + 67a_1^2a_2^6 + 22a_1a_2^7 + a_2^8)\\ &(3a_1^3 + 12a_1^2a_2 + 8a_1a_2^2 + 4a_2^3)^3.\end{aligned}$$

Finally, we can also combine Lemma 6.1 with Theorem 5.3 and choose the d_i in such a way that C has contact multiplicity 3 with the tangent line to C in a $\mathbb{Q}$-rational point P (resp. Q) and such that this tangent line also intersects C in ∞ (resp. O). This leads to the following family in two parameters.

Example 6.4. Let $a, b \in \mathbb{Q}$ such that

$$d_1 = b^3 + \frac{3}{2}a,\ d_2 = -a^3b^3,\ d_3 = 2a^3(2b^3 + 3a)b^3,\ d_4 = -4b^6 - 6ab^3 + \frac{3}{4}a^2;$$

Let F be as in (13) and let C be the projective closure of the affine curve given by $F = 0$. If

$$\begin{aligned}\operatorname{disc}(C) =& -4a^{42}b^{42}(-8b^{18} - 36ab^{15} - 18a^2b^{12} + 189a^3b^9 + 351a^4b^6 + 162a^5b^3 + 27a^6)\\ &(2b^3+3a)^2(144b^{12} + 576ab^9 + 504a^2b^6 + 112a^3b^3 + 9a^4)^2\\ &\neq 0,\end{aligned}$$

then C is smooth and the following are elements of $K_2^T(C)/\text{torsion}$

$$\{\infty, O, P\}, \{\infty, O, R\}, \{\infty, R, P\}, \{\infty, O, Q\}, \{\infty, R, Q\}, \{\infty, P, Q\},$$

where $P = (a^3, -a^4) \in C(\mathbb{Q})$ and $Q = (-a^2b^3, -a^2b^6) \in C(\mathbb{Q})$. See Figure 3 on page 20 for the curve $C : x^4 + 1/64((2x-8y-1)^2 - 52x^2 + 8x)y = 0$, corresponding to the choice $a = 1/2, b = -1$.

Remark 6.5. It would be interesting try to find 3 integral elements in the families discussed above. We have not checked under which conditions the elements constructed in the present section are integral, as our focus is on the geometric picture: One can force a curve C to have specific intersection properties with other curves of varying degrees to produce elements of $K_2^T(C)/\text{torsion}$.

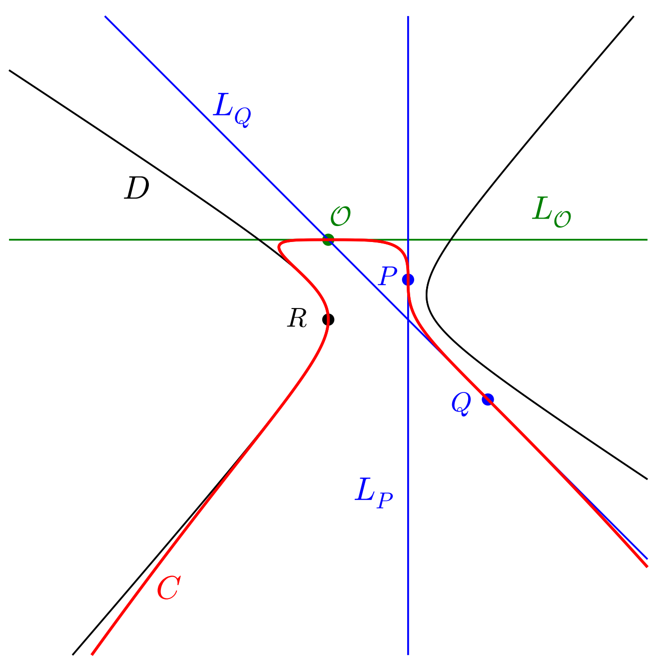

FIGURE 3. The curve $C : x^4 + 1/64((2x-8y-1)^2 - 52x^2 + 8x)y = 0$

7. Nekovář-type constructions

In [RS98] a family of elliptic curves E_a equipped with an element in $K_2^T(E_a)\otimes\mathbb{Q}$ is studied. The remarkable fact about this family is that this element corresponds to a non-torsion divisor. The authors of [RS98] attribute this family to J. Nekovář. We will recover this and similar construction in our setup. The theoretical basis of such constructions can be phrased geometrically, as in the following lemma. In its formulation we denote, by abuse of notation as before, the defining polynomial of a plane curve by the same letter as the curve itself.

Lemma 7.1. *Let C be a smooth projective plane curve defined over $\mathbb{Q}$. We assume that there are 3 pairwise distinct plane curves E, G and H, defined over $\mathbb{Q}$, of the same degree and having pairwise distinct intersection with C, which define functions g and h on C via the quotients of their defining polynomials as $g := G/E$ and $h := H/E$, satisfying the following properties:*

- *There is a rational point ∞ such that C and E have maximal contact in ∞.*
- *For all P in the intersection of G and C the divisor $(P) - (\infty)$ is a torsion divisor, i.e. if*
$$C \cap G = \sum a_i P_i$$
then there exist functions g_i and integers m_i with
$$\operatorname{div}(g_i) = m_i(P_i) - m_i(\infty).$$
- *There is a constant $\kappa \in \mathbb{Q}^\times$ such that for all Q in the intersection of H and C the value of g at Q equals κ.*

Then replacing g by $\tilde{g} := g/\kappa$ and setting $\kappa_i = T_{P_i}(\{\tilde{g}, h\})$, we get the element

$$m\{\tilde{g}, h\} - \sum_i \frac{m}{m_i}\{\kappa_i, g_i\} \in K_2^T(C), \tag{16}$$

where $m = \operatorname{lcm}(\{m_i\})$ is the least common multiple of the multiplicities m_i.

Proof. It suffices to prove that the tame symbol equals 1 at all critical points except ∞; because of the product formula (2) the claim then follows. At each Q in the support of $H \cap C$ we have $T_Q(\{\tilde{g}, h\}) = 1$ by construction and $T_Q(\{\kappa_i, g_i\}) = 1$ by assumption. For each P_i in the support of $G \cap C$ we get $T_{P_i}(\{\kappa_j, g_j\}) = 1$ if $i \neq j$ and else

$$T_{P_i}(\{\kappa_i, g_i\}) = (-1)^{\operatorname{ord}_{P_i}(\kappa_i)\operatorname{ord}_{P_i}(g_i)} \frac{\kappa_i^{\operatorname{ord}_{P_i}(g_i)}}{g_i^{\operatorname{ord}_{P_i}(\kappa_i)}}(P_i) = \frac{\kappa_i^{m_i}}{g_i^0}(P_i) = \kappa_i^{m_i} = T_{P_i}(m_i\{\tilde{g}, h\})$$

□

Observe that by Galois descent the Nekovář element in (16) is defined over $\mathbb{Q}$, although the P_i or the Q_j might not be rational points. If we divide the Nekovář element in (16) by m we get an element in $K_2(C) \otimes \mathbb{Q}$ instead. Thus in the case of elliptic curves we recover the construction 5.1 of [RS98]. We do not study the integrality of such elements in our examples, instead we refer the interested reader to [RS98, §5.2] for the case of elliptic curves.

Remark 7.2. The assumptions of the Lemma can be weakenend as follows: Firstly, we do not have to assume that the curves E, G and H have the same degree, because we can replace the equation of a curve by some power, if needed. Secondly, if the values of the function g at the Q_j differ by some root of unity, then we just need to replace g by some power. Thirdly, the point ∞ can be a singular point (but there are no other singular points). In this case the condition on F means that on the normalization of C, the zero divisor of g_i is a multiple of (P_i) and the pole divisor is supported in the points mapping to ∞. If there is only one such point, then the requirements of Remark 3.7 are satisfied, and the proof of Lemma 7.1 goes through, yielding an element in $K_2^T \otimes \mathbb{Q}$ of the normalization of C. If there are several points

$\infty_1, \dots, \infty_m$ above ∞ in the normalization of C, Remark 3.7 does not apply, but if we can show that for the elements in question, the tame symbol at all ∞_i is 1, then the conclusion of the Lemma still holds.

Example 7.3. We want to apply the Lemma 7.1 to a family of elliptic curves. We let E be the tangent line at ∞ and for G we take the line $y = 0$; the points P_i are therefore the affine 2-torsion points $(x_i, 0)$ and the functions g_i are given by the vertical tangent lines $x - x_i$ in the points P_i.

Consider the family

$$C_r : y^2 = x^3 + \left(-\frac{1}{3} + \frac{2}{3}r - \frac{4}{3}r^2\right)x + \frac{2}{27} - \frac{2}{9}r - \frac{5}{9}r^2 + \frac{16}{27}r^3,$$

where we exclude the finite set of those $r \in \mathbb{Q}$ for which the curve becomes singular. Then for each $r \in \mathbb{Q}$ the line

$$H_r : y = -x + \frac{1}{3} - \frac{1}{3}r$$

meets C_r in the points $Q_1 = (-\frac{4}{3}r + \frac{1}{3}, r)$ and $Q_2 = (\frac{2}{3}r + \frac{1}{3}, -r)$ to order 1 and 2, respectively. Finally, since $|y(Q_1)| = |y(Q_2)| = r$ the assumptions of Lemma 7.1 are satisfied. See Figure 4 for the case $r = 3/4$. Note that the geometric configuration is the same as in the family of elliptic curves due to Nekovář and discussed in [RS98]. In particular, that family can also be constructed using our approach. Nevertheless, one can show that it is not isomorphic over $\mathbb{Q}$ to the family $\{C_r\}$.

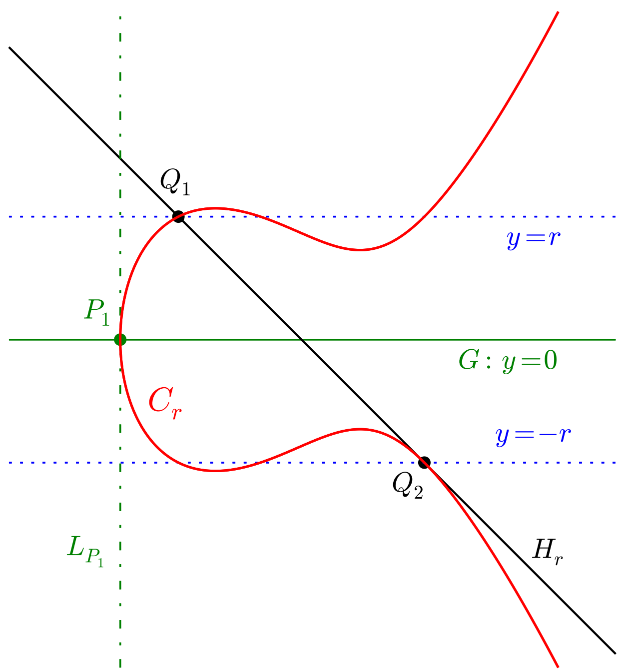

FIGURE 4. The curve C_r from Example 7.3 for $r = 3/4$

Example 7.4. Lemma 7.1 also applies to the family of elliptic curves given by

$$C_r : y^2 + f_1(x)y + x^3 = 0,$$

where $f_1(x) = r - (4r+1)x$ and where $r \in \mathbb{Q} \setminus \{0, -1, 1/8\}$. There is only one point ∞ at infinity and its tangent to C has maximal contact for all r. The line G given by $y = 0$ meets C only in the origin $(0,0)$, which is therefore a 3-torsion point and we can take $g_1 = y$. Finally the line $H : y = x - r$ intersects C in exactly two points $Q_1 = (0, -r)$ and $Q_2 = (2r, r)$.

Example 7.5. We can extend the approach of Example 7.4 to curves genus $g = 2$ (we have not attempted to treat hyperelliptic curves of higher genus). This means that we want our curve to have a point O of maximal contact multiplicity, so we are essentially in the situation discussed in detail in Section 4.

In order to construct our family, we work with affine models of the form

$$y^2 + f_1(x)y + x^5 = 0, \tag{17}$$

where f_1 has degree at most 3. Note that in Section 4, we required $\deg(f_1) \le 2$ in the case of curves of genus 2, which guarantees that there is a unique point at infinity on a smooth model. Geometrically, we again work on the singular model C' defined as the projective closure of the affine curve given by (17).

In the language of Lemma 7.1, we let $E = L_\infty$ be the line intersecting C' in the point ∞ at infinity with maximal contact multiplicity. The point $O = (0,0)$ is a point of maximal contact between C' and the tangent line $G = L_O :\ y = 0$ to C' in O, so we take $g_1 = y$.

In order to apply Lemma 7.1, we want a line H, which intersects C' in exactly two points Q_1 and Q_2, whose y-coordinates have absolute value equal to the same rational number r. For instance, we can require that Q_1 is a 3-contact point and that Q_2 is a 2-contact point. Rewriting this condition as a system of equations, we find that there is no such example if $\deg(f_1) \le 2$. Hence we have to allow $\deg(f_1) = 3$. We find the family of genus 2 curves C_r given by (17) with

$$f_1 = r - x + 4rx^3;$$

these curves are smooth unless $r \in \{0, 1/3\}$. Here, in analogy with Example 7.4, the line $H_r : y = x - r$ meets C in the points $Q_1 = (0, -r)$ and $Q_2 = (2r, r)$ with multiplicities 3 and 2 respectively. See Figure 5 for the case $r = 1/2$.

It remains to check that we actually get an element of $K_2^T(C_r)$. The only potential problem is that we need to check that the tame symbols at the two points at infinity on a nonsingular model of C_r are equal to 1, but in our specific situation this is easily seen to hold by a simple computation once we have scaled h_r by r^{-1}.

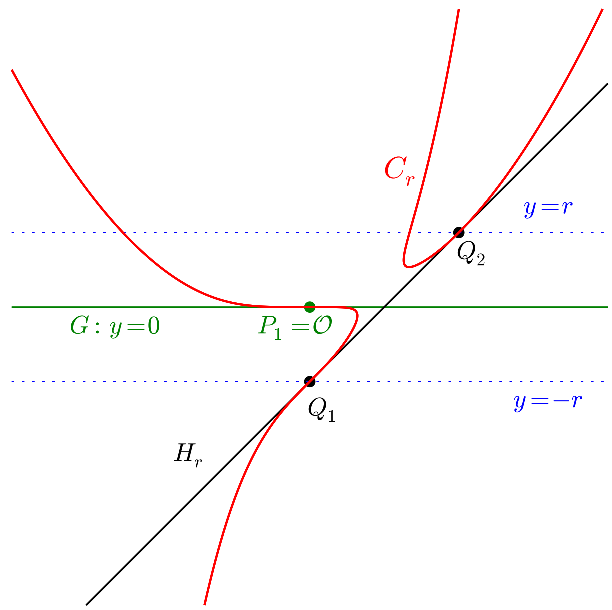

FIGURE 5. The curve C_r from Example 7.5 for $r = 1/2$

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