Exact percolation thresholds on various lattice

- Archimedean lattices and their duals – some have been extensively studied

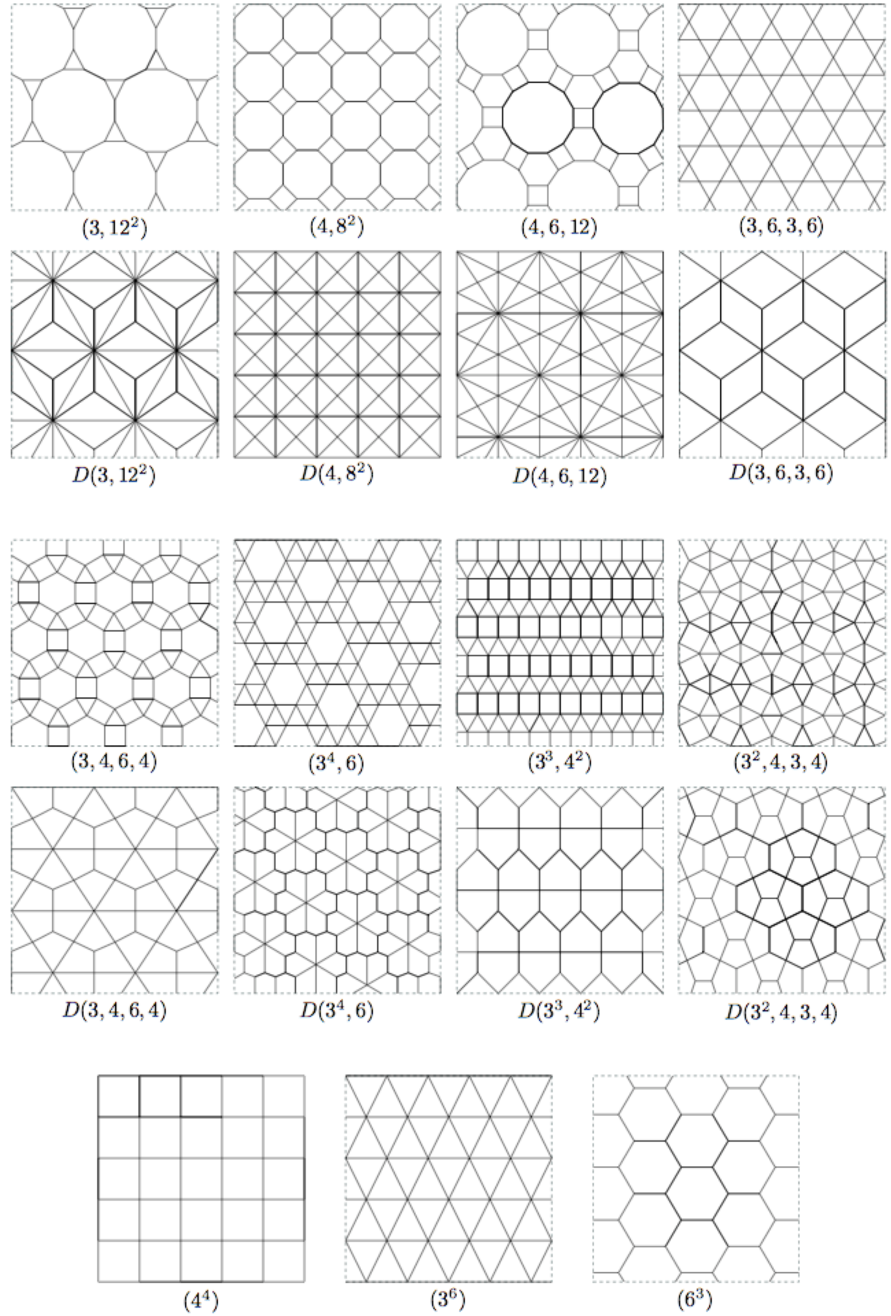


Figure 2.4: The 11 Archimedean lattices and their dual Laves lattices.

- Thresholds (from wikipedia page, “Percolation Thresholds”)

Lattice	z	Site Percolation Threshold	Bond Percolation Threshold
$(3, 12^2)$	3	0.807900764... = $(1 - 2\sin(\pi/18))^{1/2}$ [3]	0.74042195(80)[4], 0.74042081(10) [5]
(4, 6, 12)	3	0.747806(4) [3]	0.69373383(72)[4]
$(4, 8^2)$	3	0.729724(3) [3]	0.67680232(63)[4]
honeycomb (6^3)	3	0.697043(3)[3] 0.6970413(10) [5]	0.652703645... = $1-2\sin(\pi/18)$, $3p^2 - p^3 = 1$ [6]
kagomé (3, 6, 3, 6)	4	0.652703645... = $1 - 2\sin(\pi/18)$ [6]	0.5244053(3) [7], 0.52440516(10) [5], 0.52440499(2)[8]
(3, 4, 6, 4)	4	0.621819(3) [3]	0.52483258(53)[4]
square (4^4)	4	0.59274621(13) [9], 0.59274621(33) [10], 0.59274598(4) [11][12], 0.59274605(3)[8]	1/2
$(3^4, 6)$	5	0.579498(3) [3]	0.43430621(50)[4]
$(3^2, 4, 3, 4)$	5	0.550806(3) [3]	0.41413743(46)[4]
$(3^3, 4^2)$	5	0.550213(3) [3]	0.41964191(43) [4]
triangular (3^6)	6	1/2	0.347296355... = $2\sin(\pi/18)$, $3p - p^3 = 1$ [6]

Exact

p_c for site percolation on a square lattice -- what is the seventh digit?

• Elliott, Heap, Morgan, Rushbrooke	1960	series	0.48
• Domb & Sykes	1961	series	0.55
• Frisch, Sonnenblick, Vyssotsky, Hammersley	1961	MC	0.581(15)
• Dean	1963	MC	0.580(18)
• Sykes & Essam	1964	series	0.59(1)
• Sykes, Gaunt & Glen	1976	series	0.593(2)
• Rossenq, Clerc, Giraud, Guyon & Ottavi	1976	MC	0.595
• Stauffer	1976	series	0.591(1)
• Hoshen, Kopelman & Monberg	1978	MC	0.5927(3)
• Reynolds, Stanley & Klein	1980	MC	0.5931(6)
• Derrida & de Seze	1982	TM	0.5927(2)
• Djordjevic, Stanley, Margolina	1982	series	0.5923(7)
• Gebele	1984	MC	0.59277(5)
• Rapaport	1985	MC	0.5927(1)
• Rosso, Gouyet & Sapoval	1985	MC	0.59280(1)
• Derrida & Stauffer	1985	TM	0.59274(10)
• Ziff	1986	MC	0.59275(3)
• Kertész	1986	TM	0.59273(6)
• Ziff & Sapoval	1986	MC	0.592745(2)
• Ziff & Stell	1989	MC	0.5927460(5)
• Ziff	1992	MC	0.5927460(5)
• Newman & Ziff	2000	MC	0.5927462(1)
• de Oliveira, Nóbrega, Stauffer	2003	MC	0.59274621(33)
• Deng & Blöte	2005	MC	0.5927465(4)
• Mike Lee	2007	MC	0.59274603(9)
• Mike Lee	2008	MC	0.59274598(4)
• Feng, Deng & Blöte	2008	MC	0.59274605(3)

- Looks like the seventh digit is zero!! 0.5927460
- (Pushing the limits of computation, finite-size scaling, random number generators…)

The triangle-triangle transformation

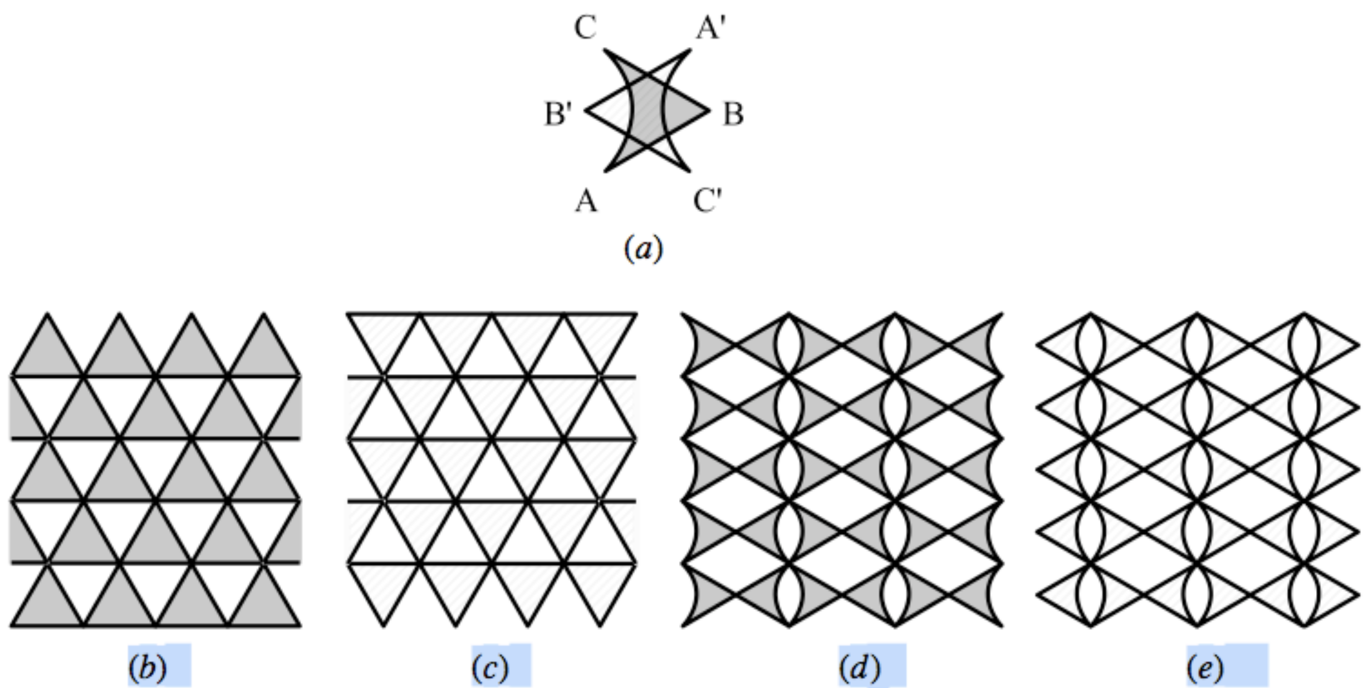
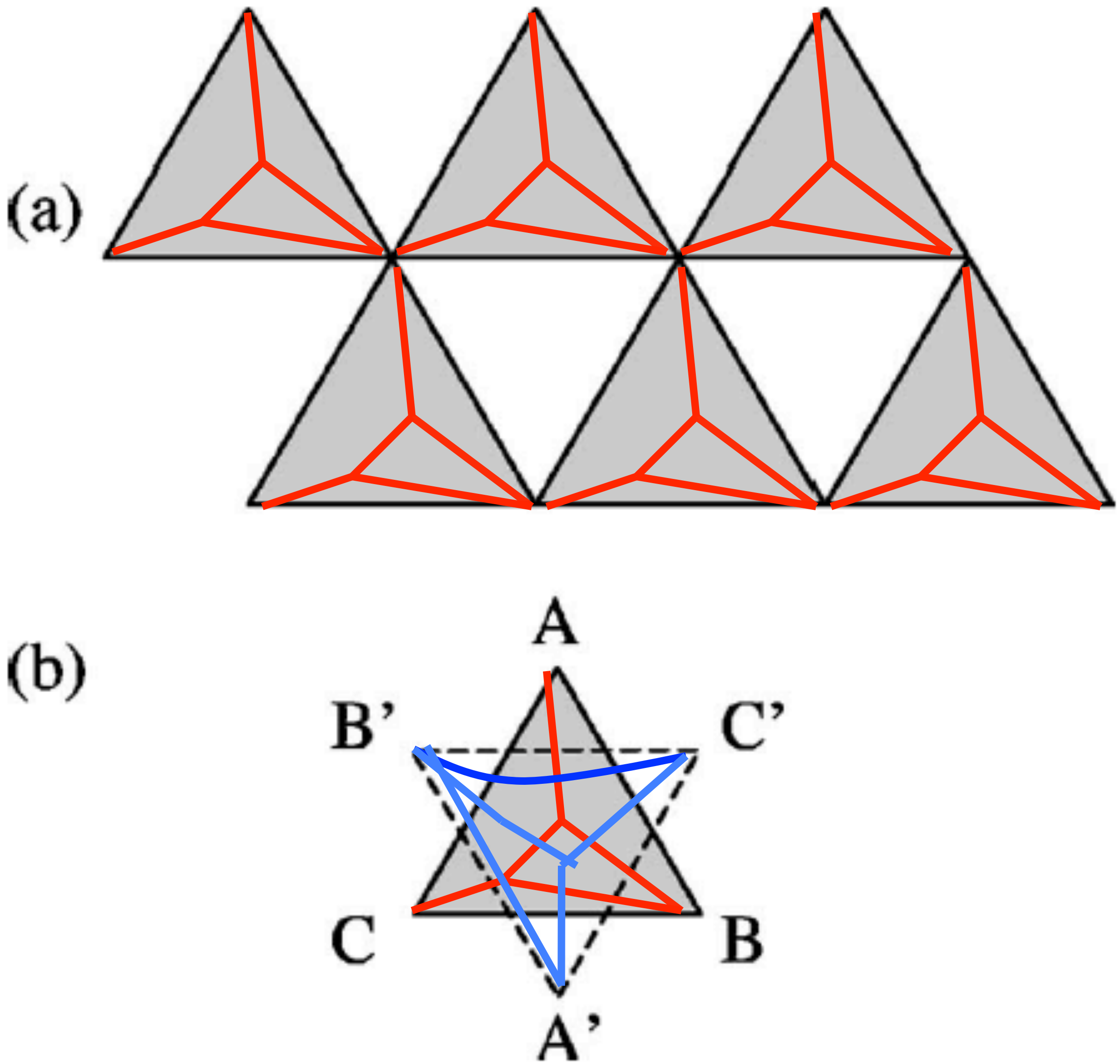


Figure 1. The triangle–triangle transformation is shown in (*a*). The lattices in (*b*) and (*d*) are invariant under this transformation, as shown in (*c*) and (*e*).


(a)
(b)
A
B’
C’
C
B
A’

’

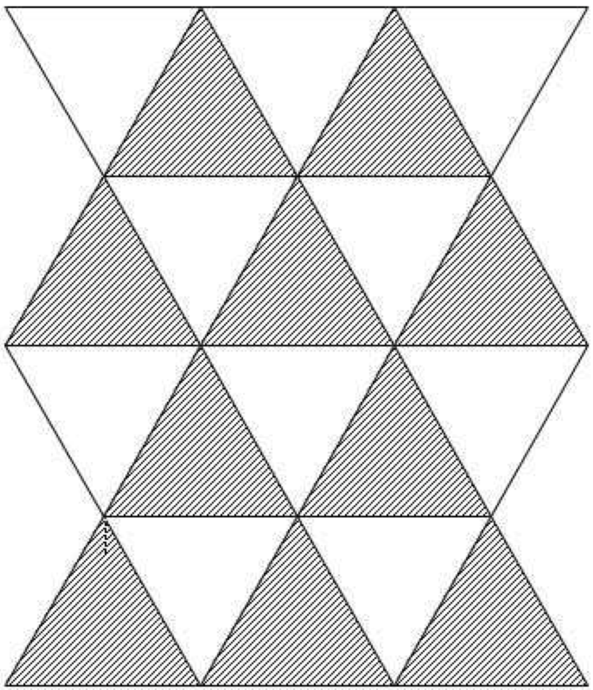
Threshold condition

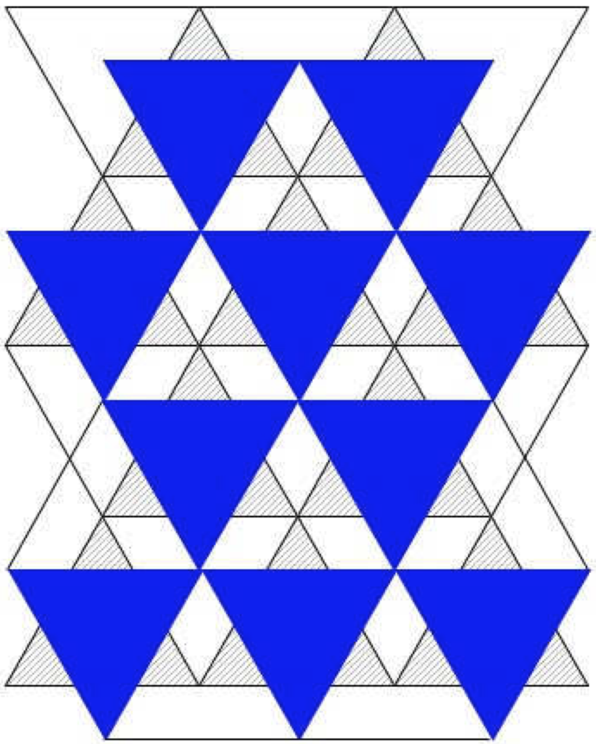
- If the triangle-triangle transformation leads to the same lattice, then the threshold is given by the solution to

$$P(ABC) = P(\overline{ABC})$$

or Prob(all) = Prob(none), for whatever connection of bonds (including correlated) within the triangle.

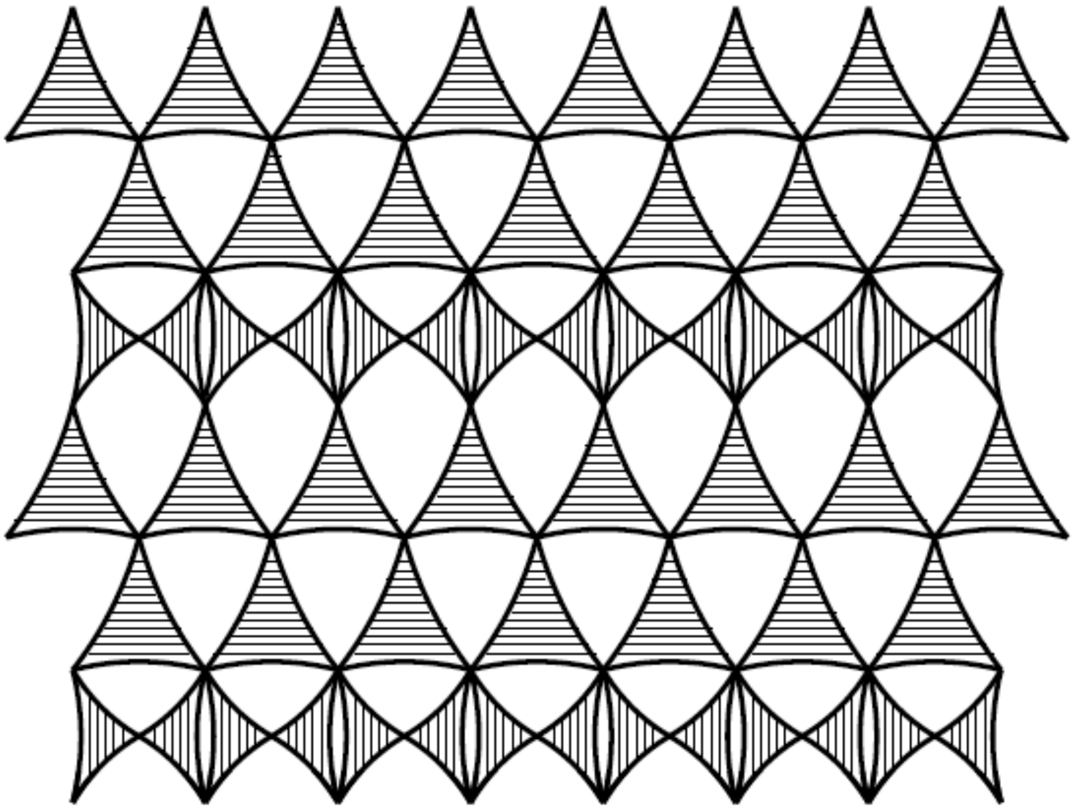
These are special cases of a more general triangle-triangle transformation:



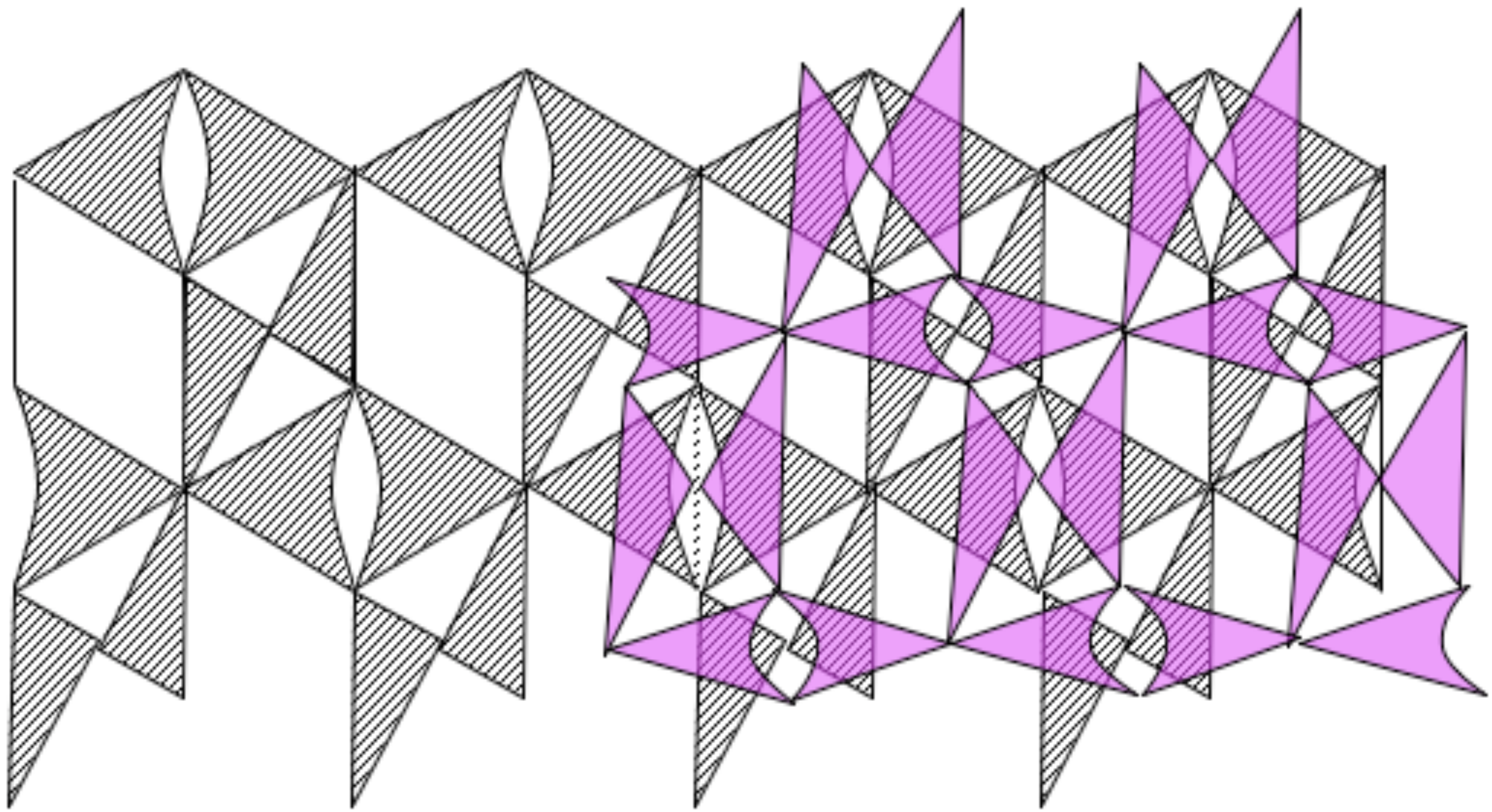


The triangle can contain *any* collection of bonds, including correlated ones

Self dual:



Self-dual:



"Martini" lattices (C. Scullard)

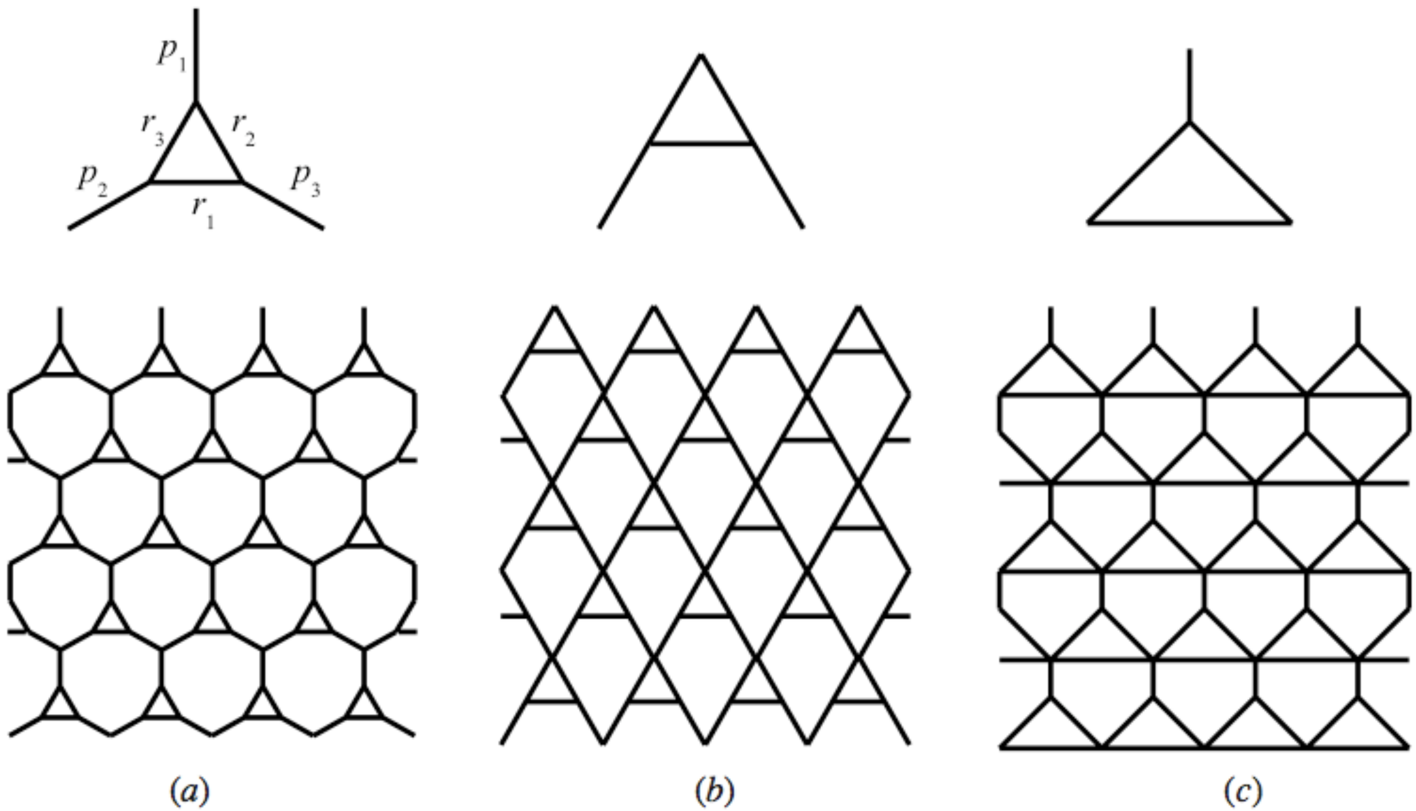


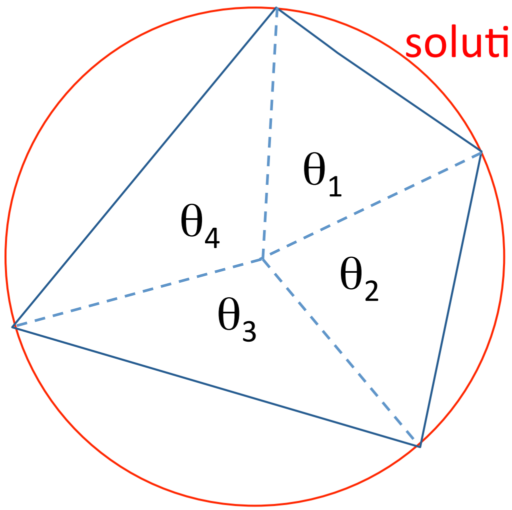
Figure 3. Some examples of lattices created by using different generators in the arrangement of figure 1(*b*). (*a*) The martini lattice, (*b*) The martini-A lattice, (*c*) the martini-B lattice.

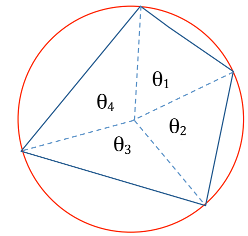
Site and bond thresholds

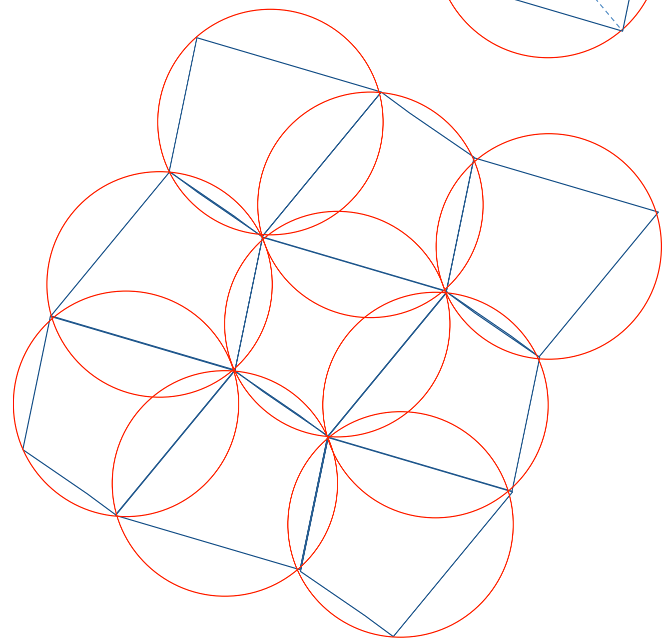
lattice	p_c^{site}	equation
martini	$0.764826\ldots$	$p^4 - 3p^3 + 1 = 0$
martini-A	$1/\sqrt{2} \approx 0.707107\ldots$	$2p^2 - 1 = 0$
martini-B	$1/\phi \approx 0.618034\ldots$	$p^2 + p - 1 = 0$

lattice	p_c^{bond}	equation
martini	$1/\sqrt{2} \approx 0.707107\ldots$	$(2p^2 - 1)(p^4 - 3p^3 + 2p^2 + 1) = 0$
martini-A	$0.625457\ldots$	$p^5 - 4p^4 + 3p^3 + 2p^2 - 1 = 0$
martini-B	$1/2$	$(2p - 1)(p^2 - p - 1) = 0$

Grimmett and Manolescu, 2012
A new class of systems with exact solutions







“Isoradial” graphs

BOND PERCOLATION ON ISORADIAL GRAPHS

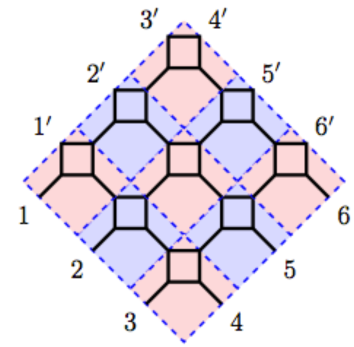
GEOFFREY R. GRIMMETT AND IOAN MANOLESCU

ABSTRACT. In an investigation of percolation on isoradial graphs, we prove the criticality of canonical bond percolation on isoradial embeddings of planar graphs, thus extending celebrated earlier results for homogeneous and inhomogeneous square, triangular, and other lattices. This is achieved via the star–triangle transformation, by transporting the box-crossing property across the family of isoradial graphs. As a consequence, we obtain the universality of these models *at* the critical point, in the sense that the one-arm and $2j$-alternating-arm critical exponents (and therefore also the connectivity and volume exponents) are constant across the family of such percolation processes. The isoradial graphs in question are those that satisfy certain weak conditions on their embedding and on their track system. This class of graphs includes, for example, isoradial embeddings of periodic graphs, and graphs derived from rhombic Penrose tilings.

$$\frac{p_i}{1-p_i} = \frac{\sin([\pi - \theta_i]/3)}{\sin(\theta_i/3)}$$

New transfer matrix method achieves 7-digit accuracy

Transfer matrix computation of generalised critical polynomials in percolation



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Abstract.

Percolation thresholds have recently been studied by means of a graph polynomial $P_B(p)$, henceforth referred to as the critical polynomial, that may be defined on any periodic lattice. The polynomial depends on a finite subgraph B, called the basis, and the way in which the basis is tiled to form the lattice. The unique root of $P_B(p)$ in $[0,1]$ either gives the exact percolation threshold for the lattice, or provides an approximation that becomes more accurate with appropriately increasing size of B. Initially $P_B(p)$ was defined by a contraction-deletion identity, similar to that satisfied by the Tutte polynomial. Here, we give an alternative probabilistic definition of $P_B(p)$, which allows for much more efficient computations, by using the transfer matrix, than was previously possible with contraction-deletion.

We present bond percolation polynomials for the $(4,8^2)$, kagome, and $(3,12^2)$ lattices for bases of up to respectively 96, 162, and 243 edges, much larger than the previous limit of 36 edges using contraction-deletion. We discuss in detail the role of the symmetries and the embedding of B. For the largest bases, we obtain the thresholds $p_c(4,8^2) = 0.676\,803\,329\cdots$, $p_c(\text{kagome}) = 0.524\,404\,998\cdots$, $p_c(3,12^2) = 0.740\,420\,798\cdots$, comparable to the best simulation results. We also show that the alternative definition of $P_B(p)$ can be applied to study site percolation problems.