Non-universality for Crossword Puzzle Percolation

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A percolation model inspired by crossword puzzle games is introduced. A game proceeds by solving words, which are segments of sites in a two-dimensional lattice. As test case, the *iid* variant allows for independently occupying sites with letters, only the percolation criterion depends on the existence of solved words. For the *game* variant, inspired by real crossword puzzles, it becomes more likely to solve crossing words which share sites with the already solved words. In this way avalanches of solved words may occur. Both model variants exhibit a percolation transition as function of the a-priori site or word solving probability, respectively. The *iid* variant is in the universality class of standard two-dimensional percolation. The *game* variant exhibits a non-universal critical exponent ν of the correlation length. The actual value of ν depends on the function which controls how much solved words accelerate the solved of crossing words.

INTRODUCTION

When a statistical physicist looks at a partial solved crossword puzzle, she or he sees immediately a percolation problem: Is there a spanning path consisting of fully solved words? Cross word puzzles have been investigated in the mathematical literature so far with respect to their graph network structure [1, 2].

Percolation problems [3–6] are investigating the conditions for the existence of system-wide connect components, allowing for the transport of matter, information or currents. Percolation is ubiquitous in all fields of sciences like physics, mathematics, computer science, social sciences or biology [5, 7]. Typically, there are one or more external parameters, like the density of relevant objects in the system, which control whether the system is in the percolating phase or in the non-percolating phase. The transition between the two phases occurs often as a second-order phase transition, described by critical exponents, e.g., the critical exponents ν for the correlation length. Interestingly, the percolation transition is highly universal, i.e. for different systems of the same dimensionality, the critical exponents are the same, independent of the physical details, in particular $\nu = 4/3$ in two dimensions.

This is also true not only for different lattice structures, but also when changing the geometric properties of the system's constituents. In particular the percolation transition for independently placed dimers [8, 9], string-like objects as rods and k-mers [10–13] or in general objects with varying aspect-ratio [14] is described by the standard critical exponents.

One variant of the present model, as introduced below, includes a dynamic process which resembles the spread of diseases on networks. For these disease models also percolation-like phase transitions between local and global spread of diseases have been observed. Still, for the ubiquitous susceptible-infected-recovered (SIR) model on two-dimensional lattices, the observed transition exhibits a critical exponent ν which is compatible with the standard value $\nu = 4/3$ [15].

Most of the standard percolation problems studied in statistical mechanics consist of disorder without correlations, although several studies with imposed correlations exist, in particular with power-law correlations [16, 17]. Here indeed a non-universality can arise. For long-range power-law correlated order $C(r) \sim r^{-a}$ and very long-range correlations with $a < 3/2$, the exponent ν is expected to depend on a [16]. But for values $a > 3/2$ the standard 2d value $\nu = 4/3$ for the critical exponent of the percolation length is obtained [17].

The crossword puzzle percolation problem studied here is based on a notion of occupancy, which is based on linear segments of sites, i.e., *words*. This linear property is similar to the percolation problem for rods, which, as mentioned, on its own does not lead to a change of the value of ν. In addition, for the present crossword model correlations arise naturally, because words, where some letters are already known through other solved words, are easier to solve. As it is shown below, the crossword puzzle percolation belongs to a different universality class as compared to standard percolation, and the value critical exponent ν depends on how much one benefits from the partial knowledge of as word.

Note that recently, a model for crossword puzzle was introduced [1], by using quenched normal-distributed *difficulties* for the words. The difficulty is the fraction of letters of a word which have to be known in order to solve the word. The normal distribution allows for negative difficulties, i.e., words which are always known, and for large positive difficulties such that these words cannot be solved. The solution of the words with negative difficulties lead to other words with positive difficulties to be solved as well, i.e., creates a dynamic solution process. The mean and variance of the difficulty distribution determine whether most words can be solved or not. In the study [1] for some parameter values a bimodal distribution for the amount of solved words was observed, which speaks in favor of a first-order phase transition. But no systematic study with respect to averaging the grid structure, or the values of the parameters of the Gaussian distribution or grid sizes was performed. Thus, in particular the question about the universality could not be addressed.

In the present paper a model is studied which is based on similar concepts as Ref. [1], but exhibits higher similarity with respect to the control parameter as standard statistical physics percolation models. Namely, here the a-priori probability of occupying sites or solving words, depending on the model variant, is varied. Also, the present model exhibits a parameter which controls how much one benefits from the partial knowledge of words. Furthermore, the present work contains a comprehensive simulation [18] study including finite-size scaling analyses [19] which allows one to determine critical exponents like ν, showing the non-universality of the model.

The paper is organized as follows: First the model is introduced, for two different variants. Next the algorithm to determine clusters, based on standard depth-first search, is outlined. In the main section, the results for the two variants are shown. Finally, a discussion is given.

MODEL

The model used here, namely the *game* variant, see below, is based on the principle that solving a word makes solving other words wore likely [1]. For a higher flexibility of the present model, a word-solution probability function is used instead of fixed probabilities. Still, since solving a word is a stochastic event, some words with the same probability might be solved while others might not. The relevant word solution probability is set at a given initial probability, which is the main control parameter. Hence it states the a-priori global fraction of solved words. Thus, the model is more in the spirit of a standard percolation probability and correspondingly leads to a second-order phase transition.

More formally, a realization of the problem is given by variables $s(\mathbf{x}) \in \{-1, 0, 1\}$ for sites $\mathbf{x}$ on a d-dimensional lattice of site L^d. The three different values correspond to *black*, *empty* and *occupied* sites, respectively. With respect to a real crossword puzzle, an occupied site contains a letter. Black sites cannot contain letters. Sites which are not black are also called *white*. Here, the simple quadratic, i.e., 2d, case with periodic boundary conditions in all directions is considered, a generalization to other dimensions d is straightforward.

Realizations are generated as follows: First, each site is set to *black* with probability p_{b}. All other sites are white, i.e., empty or occupied with a letter. Two examples are shown in Fig. 1. The black sites partition the system into *words*, i.e. segments of horizontal or vertical white sites bounded by black sites. Note that the black sites are assigned independently of each other, such that short words like of length one might occur, in contrast to typical real-world crossword puzzles. Also very long words might occur: For the special case that a row or column does not contain any black site, the full row or column is a single word of length L, but this occurs only for very small values of p_{b} and L. Technically, for all simulations, the words of a given realization are determined after the black sites have been assigned.

In order to assign the white sites, to be occupied or empty, two variants are considered, the *iid* and the *game* variant. For the *iid* case, each white site is assigned the state occupied independently with the identical probability p. With probability $1 - p$ the site is empty. The occupation rule of this variant is similar to standard percolation and is used for comparison.

The *game* variant mimics the way a human would try to solve a crossword puzzle. The a-priori probability p_{w} describes the probability to solve a word, i.e., occupy all sites of a word, if no letters are present so far. Furthermore, if some words are solved, corresponding to partial knowledge of "neighbouring" words which share a letter with the solved word, this will increase the probability that a neighboring word can be solved as well. This is here modeled by the word-solution probability

$$p_{\mathrm{cw}}(x) = p_{\mathrm{w}} + (1 - p_{\mathrm{w}})x^{\omega}\,, \qquad (1)$$

where $x \in [0, 1]$ is the fraction of already known letters, i.e., occupied sites, of a word. Note that $p_{\mathrm{cw}}(1) = 1$ which is consistent. The *benefit exponent* ω, describes how much knowing some letters helps in solving a word. For $\omega \to \infty$, one does not benefit much, since $p_{\mathrm{cw}}(x) \approx p_{\mathrm{w}}$ for almost all $x \in [0, 1]$. For values $\omega < 1$ the benefit grows in particular quickly.

To generate a realization of disorder for the *game* variant, one draws for each word w a fixed random number $r(w)$. Then, one iterates over all words and solves a word, i.e. occupies all its sites $\mathbf{x}$, if $r(w) < p_{\mathrm{cw}}(x(w))$ where $x(w)$ is the current occupation fraction of word w, which is initially zero for all words. Occupying the letters for a word leads to an increase of $x(w')$ for words w' which share sites with w. This makes it more likely that w' can be solved as well. The process is iterated again over all words, until no more additional words are solved.

Hence, in the *game* variant crossword model, a word has two states, *solved*, i.e., all sites are occupied, and *unsolved*, i.e., not all sites are occupied. The state solved of a word increases the probability of neighbors to be solved, i.e., avalanches of solved words occur dynamically. This resembles the spread of diseases, where infected sites with some likelihood infect their neighbors. A two state model for infectious diseases is the susceptible-infected (SI) model [20]. However, for the standard version of the SI model, in the limit of infinite time, all nodes of a given system become infected. In contrast, for the present model not all words will be solved,

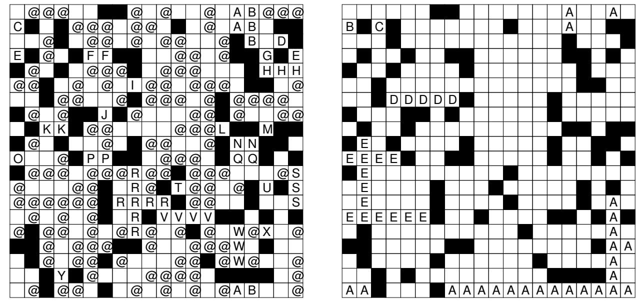

FIG. 1. Sample realizations for $L = 20$, $p_{\mathrm{b}} = 0.2$. Black sites are black, empty sites white, occupied sites contain symbols and letters. Any segment of sites bordered by black sites is a word. The letters A,B,... denote the clusters formed by solved words, while the @ symbol denotes occupied sites which do not belong to solved words. In the left an *iid* sample is shown for $p = 0.5$. In the right an *game* sample is shown for $p_{\mathrm{w}} = 0.053$, $\omega = 1$.

typically. Thus, the three-state SIR model [21] shares a higher similarity with the present model, since, depending on the infection parameters, only a certain fraction of nodes will experience an infection followed by a recovery. Anyway, in detail the present model is very different from the SIR model and exhibits also very different results.

ALGORITHM

To analyze a given realization, let it be the *iid* or the *game* variant, the following procedure is applied: First, all solved words are determined, i.e., those words where all its sites are occupied. Now, two words are called *connected* if they are both solved and if they share one site. In particular, they run perpendicular to each other on the grid, crossing at the shared site. Thus, two word which run parallel next to each other are not connected. Now, *clusters* of words are determined by a standard depth-first search [22] as the transitive closure of connected words. The size s of a cluster is the number of occupied sites. A realization is considered *spanning* if there is a cluster which exhibits at least one occupied site in each row or in each column, i.e. it connects the any row (column) with any other row (column).

RESULTS

Simulations were performed for several system sizes between $L = 20$ and $L = 1000$, for the *iid* and the *game* variant. For each realization, determined also by the parameters p_b, p or ω, p_w, an average over a number of realizations was taken, between 10^4 for system sizes $L \leq 200$ and 2×10^3 for $L = 1000$.

First, results for the *iid* case are presented. The probability P_{span} of a spanning cluster is shown in Fig. 2 as function of the site-occupation probability p, for the case of a fraction $p_b = 0.2$ of black sites. A clear increase of P_{span} near $p = 0.73$ is visible, where the curves of different sizes cross. This is a clear indication for a percolation phase transition.

A simple quantitative finite-size analysis of the phase transition is possible by considering the variance of the spanning probability which is simply given by $\text{Var}(P_{\text{span}}) = P_{\text{span}}(1 - P_{\text{span}})$, which is shown in Fig. 3 as an example for size $L = 100$. An estimate of the finite-size transition point is given by the maximum $p_{\max}$ of the variance, where the sample-to-sample fluctuations are largest. Note that this also corresponds to $P_{\text{span}} = 1/2$. The values of $p_{\max}$ were estimated by fitting Gaussians near the peak.

To extrapolate to the infinite system size, the peak positions are fitted to the standard finite-size scaling form, well known for standard percolation [5, 23] and other phase transitions [19]

$$p_{\max}(L) = p_{\text{c}} + aL^{-1/\nu}\,, \tag{2}$$

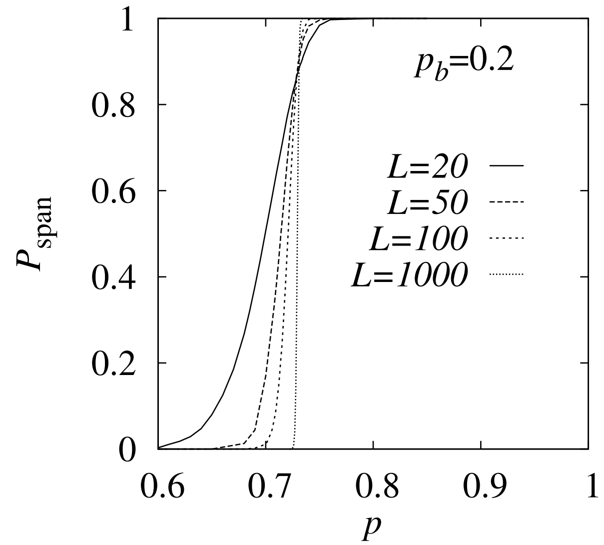

FIG. 2. Spanning probability P_{span} as function of the *iid* site occupation probability p for a fraction $p_{\text{b}} = 0.2$ of black sites.

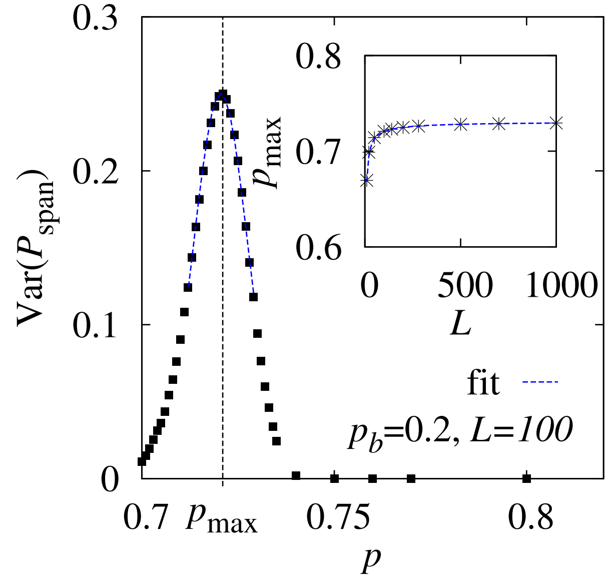

FIG. 3. Variance of the spanning probability P_{span} as function of the *iid* site occupation probability p for $p_{\text{b}} = 0.2, L = 100$. Near the peak variance, a Gaussian is fitted to obtain the peak position $p_{\max}$. The dependence of $p_{\max}$ as function of system size L is shown in the inset and allows for an extrapolation to $L \to \infty$ by using Eq. (2).

where p_c is the critical point and ν the critical exponent which describes the divergence of the correlation length. For $p_b = 0.2$ the fit for $L > 140$ results in a value of $\nu = 1.31(4)$ which is compatible with the exponent $\nu = 4/3$ for standard 2d percolation. The resulting critical value is $p_c = 0.7313(1)$.

The behavior of the critical point p_c as a function of the fraction of black sites is shown in Fig. 4. For the border case of no black sites $p_{\text{b}=0}$, all words are occupying a full row or column respectively. Here the system behaves essentially one-dimensional. Therefore, all sites of a row or column have to

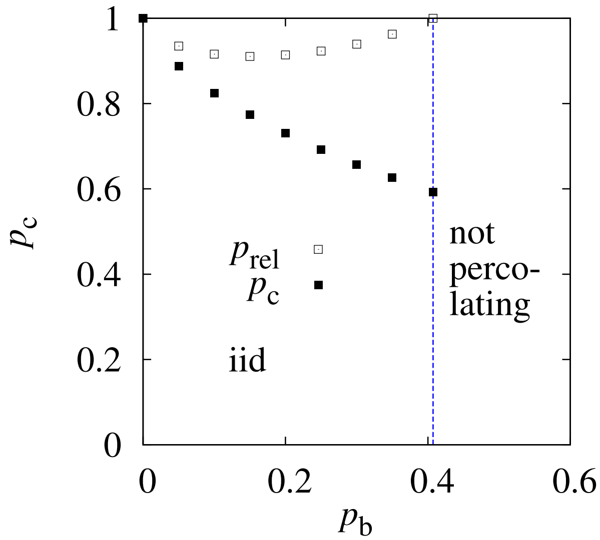

FIG. 4. Phase diagram if the *iid* case, i.e., critical threshold p_{c}, together with $p_{\mathrm{rel}} = p_{\mathrm{c}}/(1-p_{\mathrm{b}})$, as function of the fraction p_{b} of black sites.

be occupied for a solved word, i.e., $p = 1$. This is shown in the upper left corner of Fig. 4.

On the other hand, when the fraction $1 - p_{\mathrm{b}}$ of white sites reaches the percolation threshold $p_{\mathrm{c}}^{(d=2)} \approx 0.59274621$ [23] of standard percolation, all white sites have to be occupied for a spanning path, i.e., $p = p_{\mathrm{c}}$. For $p_{\mathrm{b}} \geq 1 - p_{\mathrm{c}}^{(d=2)}$ the system cannot percolate for any value of p. Note that p is measured as fraction of all lattice sites. One can also measure the fraction of occupied sites among the white sites, i.e., the relative fraction of occupied sites. The corresponding critical fraction is $p_{\mathrm{rel}} = p_{\mathrm{c}}/(1-p_{\mathrm{b}})$ which is also shown in Fig. 4. Interestingly, since trivially $p_{\mathrm{rel}}(p_{\mathrm{b}} = 0) = 1$ and $p_{\mathrm{rel}}(p_{\mathrm{b}} = 1 - p_{\mathrm{c}}^{(d=2)}) = 1$, the behavior of $p_{\mathrm{rel}}(p_{\mathrm{b}})$ is non-monotonous and exhibits a minimum. According to the figure this is located near $p_{\mathrm{b}} \approx 0.15$.

Next, the results for *game* variant are shown, restricted to the case $p_{\mathrm{b}} = 0.2$. First, the correlations of the disorder correlations are quantified via analyzing the density-density correlation function

$$C(r) = \frac{[s(\mathbf{x})s(\mathbf{x}+\mathbf{r})]_{0/1} - [s(\mathbf{x})]_{0/1}^2}{[s(\mathbf{x})^2]_{0/1} - [s(\mathbf{x})]_{0/1}^2} \tag{3}$$

where $r = |\mathrm{r}|$ amd r is for simplicity along the x direction. The average $[\ldots]_{0/1}$ is over the disorder realizations and over white sites only because the location of black sites is not correlated anyway. Note that $[s(\mathbf{x})^2]_{0/1} = [s(\mathbf{x})]_{0/1}$ is just the resulting density of occupied sites among the white sites.

In Fig. 5 the correlations are shown near the critical points as determined below, together with fits to sums of two exponentials $e^{-r/\lambda_1} + e^{-r/\lambda_2}$ with $\lambda_1 > \lambda_2$. For all values of ω, clear exponential decreases are visible. The dominating length scales λ_1 as obtain from the fit are for $\omega \in [0.8, 2]$ small, i.e. $\lambda_1 \leq 7$. For $\omega = 0.6$ a much longer scale $\lambda_1 = 137(4)$ is obtained. Note that for the latter case below $p_c = 0$ is estimated, so the simulations were performed at the values $p_w = 0.00004$ close to p_c. For larger distances r, depending on ω, the correlations fluctuate around zero due to the finite statistics. Thus, the correlations are exponentially decreasing, i.e., short-ranged, and one could expect that, in particular for $\omega \geq 0.8$, they have no influence on larger length scales such that the critical behavior of standard percolation is obtained.

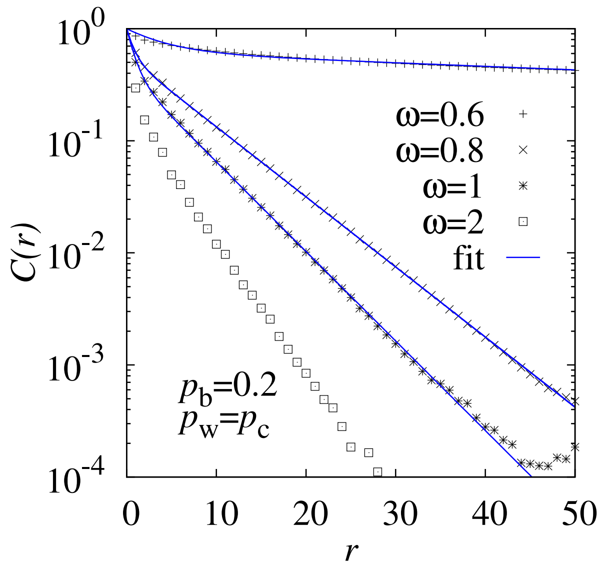

FIG. 5. Density-density correlations of the *game* realizations for the considered values of ω for the critical value $p_{\mathrm{w}} = p_{\mathrm{c}}$, respectively The lines show fits to a sums of two exponentials.

But this is actually not the case. The spanning probability P_{span} as function of the word probability p_{w} looks qualitatively similar to the *iid* case, thus it is not shown here. Although for each realization avalanches of solved words occur, the function $P_{\mathrm{span}}(p_{\mathrm{w}})$ looks very smooth, no indication of a first-order step-like behavior is visible. One can again fit Gaussians near the peaks of the variance Var(P_{span}) to obtain finite-size critical points $p_{\max}(L)$. Fitting to Eq. (2) yields extrapolated critical points p_{c} and critical exponents ν of the correlation length. In Fig. 6 the results for $p_{\max}(L) - p_{\mathrm{c}}$ are shown as function of L. For $L \geq 100$ clear power laws are visible, but with different exponents $-1/\nu$ as compared to the standard percolation value $-3/4$. For $\omega \geq 0.8$ here always larger critical exponents ν are found, see Tab. I. This shows that the crossword percolation is in a different universality class than standard percolation. Note that with increasing value of ω, the critical exponent ν moves towards the standard value. This makes sense, because the benefit becomes smaller and smaller when ω increases, such that the percolation problem of independent rods should be obtained, which is known to be in the standard universality class [10–13] also for polydisperse systems [14].

The table shows that the critical point p_{c} decreases, as the benefit exponent ω decreases, since the growing benefit means

ω	0.6	0.8	1.0	1.5	2.0
p_c	0	0.037(5)	0.0948(2)	0.204(1)	0.264(1)
ν	0.56(3)	3.0(10)	1.96(2)	1.78(9)	1.65(11)
β	–	0.05(2)	0.09(2)	0.20(2)	0.19(2)
τ	–	2.06(5)	2.05(5)	2.07(5)	2.05(5)

TABLE I. Values of the critical point p_c and the critical exponents ν for the *game* crossword percolation, for various values of the benefit exponent ω.

that less words have to be solved without the help of some letters. Interestingly, for a small values of $\omega = 0.6$, a critical value which is compatible with $p_c = 0$ is obtained. Thus, the benefit from solving words partially is so large that an infinite small a-prior solution probability is sufficient to solve a full puzzle. Also, an even more non-standard value of the critical exponent ν is found. In particular the $L \to \infty$ approach to the critical point is now, not from below but from above (not shown). This is natural, because no probabilities $p < p_c = 0$ exist, in contrast to larger values of ω or the *iid* case, where the $L \to \infty$ approach is from below as visible in Figs. 3 and 6.

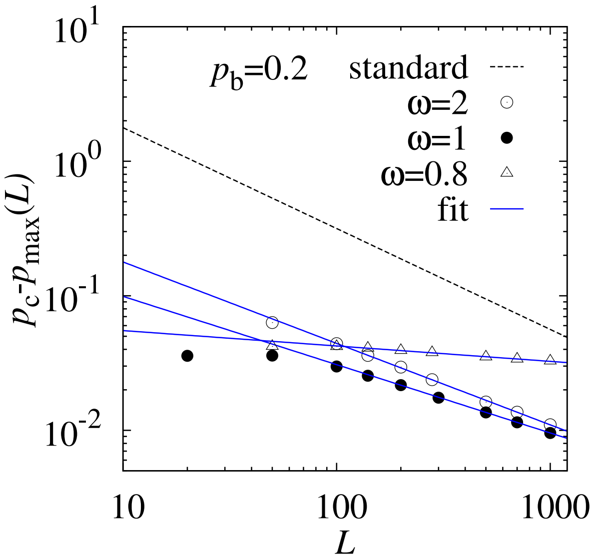

FIG. 6. Approach of the finite-size critical points $p_{\max}$ for the *game* variant to the one extrapolated by Eq. (2), for various values of ω. The slope in the log-log plot corresponds to the critical exponent $-1/\nu$ and shows the non-universality of the model.

To gain further insight, the order parameter, i.e., the average size $s_{\max}$ of the larges component as function of p_w was studied. For all values of ω, a smooth behavior was found, see Fig. 7 for the case $\omega = 1$. At the corresponding critical points, power laws $\sim L^{-\beta/\nu}$ were observed, except for $\omega = 0.6$ where it is not possible to generate data at the critical point $p_c = 0$. In all cases, values of β as reported in Tab. I are rather small, such that the differences to the standard value $\beta = 5/36 \approx 0.139$ are not too large. Still, for values $\omega < 1$ the value of β appears to be significantly smaller than the standard value. Since the scaling of the absolute cluster sizes also defines via $L^d s_{\max} \sim L^{d_f}$ the fractal dimension d_f, this leads to the hyper-scaling relation $\beta/\nu = (d - d_f)$, i.e., the fractal dimension $d_f \leq d$ is close to 2, in particular for small values of ω, which means that the clusters are almost space filling.

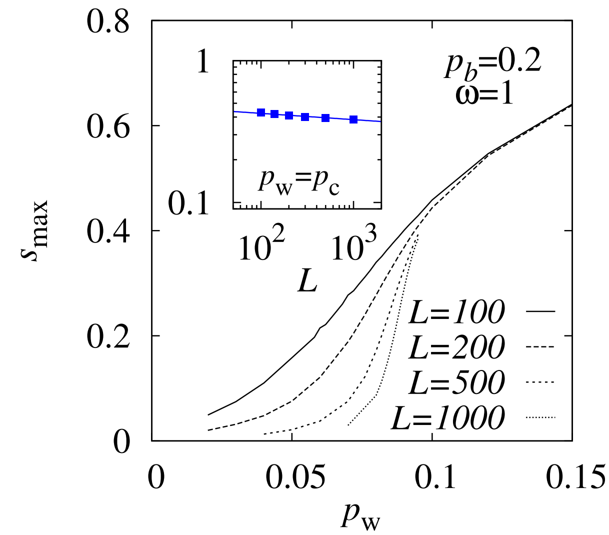

FIG. 7. Relative size $s_{\max}$ of the largest component as function of the word probability p_w for the $\omega = 1$ *game* case, for several system sizes L. The inset shows the behavior at the critical point $p_w = p_c$ as function of system size L together with a fit to a power law $\sim L^{-\beta/\nu}$.

Furthermore the distributions of sizes for the non-percolating clusters at the corresponding critical points were studied, i.e., the probabilities $P(s)$ that a cluster has size s. For $\omega = 0.6$, the critical estimated point is $p_c = 0$, so it is not possible to obtain $P(s)$ at the critical point. For $\omega \geq 0.8$, mainly a power-law behaviors $P(s) \sim s^{-\tau}$ were observed, each with an exponential cut off due to the finite system sizes. Two sample results, for the *iid* case and for $\omega = 1$ of the *game* case are shown in Fig.8. Fits to

$$P(s) = Z_s s^{-\tau} e^{-s/l_s} \tag{4}$$

with suitable length scale l_s and normalization Z_s were performed. Since the data does not exhibit perfect power laws over larger ranges of the size, the result for τ depends a bit on the fitting range. Thus, no very precise results for the exponent τ could be obtained, but they are all scattered near the standard value $\tau = 187/91 \approx 2.05$ as quoted in Tab. I. This holds also for the *iid* case. From the standard hyper scaling relation $\tau = d/d_f + 1$ this is compatible with the observed high similarity of d_f to the standard value.

DISCUSSION

In the present work, crossword-puzzle percolation is introduced, where letters or words are occupied with independent or neighbor-dependent probabilities. In the model, letters correspond to sites and words to segments of sites, bordered by

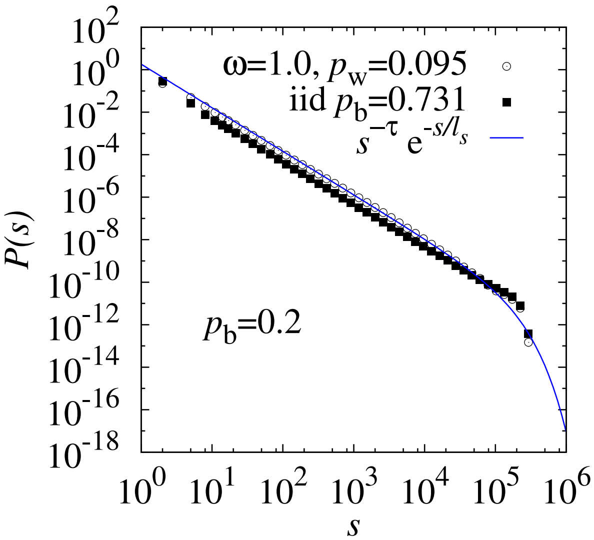

FIG. 8. Distribution of the sizes of the non-percolating clusters at the critical points for the *iid* case and for the $w = 1$ *game* case along with a fit to Eq. (4) for the $\omega = 1$ case.

black sites. The model comprises properties of several other non-standard percolation models: Like in rod percolation, the percolation objects are linear segments. Like in bootstrap percolation or in models of infectious diseases, the occupation of the sites, here words, influences neighboring sites. Like for long-range correlated disorder, the resulting critical exponents differ from the exponents of standard percolation. Note that other models like bootstrap, rod-like or disease percolation on two-dimensional lattices behave like standard percolation. Still, the density-density correlations of the model are only short range. Thus, it appears that crossword-puzzle percolation comprises a new type of universal behavior. One possible explanation for the non-universality could be that the word-solving probabilities, although not long-range correlated, change during the solving process, i.e., solving some word leads to an acceleration or improvement of solving other words. These dynamically created correlations bear some similarity with directed percolation [24], which is a model for non-equilibrium dynamic processes. An indeed, directed percolation is characterized by different values of the critical exponents as compared to standard percolation.

Also, it is remarkable that for very small values of the benefit exponents, the numerical results indicate that the percolation transition appears at infinitely small strength of the disorder. This is also a feature which is not present in standard percolation.

For further studies it would be certainly of interest to look in more detail into the model, e.g. by studying fractal properties of clusters, the backbone and other characteristic quantities. Certainly, one should also consider higher dimensions in order to see whether the non-universality occurs there as well. Also, one could determine the upper critical dimension, which could be the same as for standard percolation, or not.

Also the process of generating a *word* realization could be studied: Since solving a word leads to an increase of the probability of solving neighboring words, this leads to further iterations, i.e., avalanches of solving words. Some test runs revealed that these avalanches seem to be largest right at the critical point, leading to longer running times there. This phenomenon could be similar to critical slowing down as observed for Monte Carlo simulations of various models like Ising models [25–27], or related to avalanches in systems exhibiting self-organized criticality [28, 29].

Furthermore, so far it is not clear why the model does not belong to the universality class of standard percolation although the density-density correlations are short range. Therefore, it would be very interesting to consider an analytical calculation for this model.

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