RANDOM WALK IN
DYNAMIC RANDOM ENVIRONMENT

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§ RANDOM WALK IN RANDOM ENVIRONMENT

RWRE has been a highly active area of research since the early 1970's. The object of interest is a random walk in discrete or continuous space-time for which the transition probabilities or transition rates are random themselves.

RWRE is part of the larger area of disordered systems. In $d = 1$ the understanding is fairly complete. In $d \geq 2$ many beautiful results have been obtained, but there are still some very hard open problems.

> What makes RWRE particularly interesting is that new phenomena occur due to slow-down in rare pockets.

§ RANDOM WALK IN DYNAMIC RANDOM ENVIRONMENT

RWDRE is a variant of RWRE where the random transition probabilities or transitions rates evolve with time. The state of the art for RWDRE is rather modest. In fact, RWDRE has started to develop properly only since 2000. Presently there are some 30 papers in the literature.

Three classes of dynamic random environments have been considered so far:

1. Independent in time: globally updated at each unit of time.

2. Independent in space: locally updated according to independent single-site Markov chains.

3. Dependent in space and time.

Very few papers fall into class 3, which is the most challenging. Most of these require additional assmptions, such as:

- fast decay of the space-time correlations in the random environment;

- weak effect of the random environment on the random walk (= perturbative regime).

THIS TALK:

The random environment is taken to be a one-dimensional interacting particle system. The focus will be on a few classical choices:

- spin-flip systems
(stochastic Ising model, contact process, voter model)
- exchange systems
(exclusion process, zero-range process).

MAIN QUESTION:

Does the dynamics destroy the slow-down in rare pockets present in the static situation? Are there interesting new phenomena?

1. DRE. Let

$$\xi = \{\xi(x,t)\colon\ x \in \mathbb{Z},\, t \geq 0\}$$

be a one-dimensional interacting particle system, where $\xi(x,t) = 1$ means that site x is occupied at time t and $\xi(x,t) = 0$ means that it is vacant.

Suppose that ξ has a (not necessarily unique) equilibrium measure μ on $\Omega = \{0,1\}^{\mathbb{Z}}$, which is assumed to be shift-invariant and shift-ergodic. Write $\rho = \mu(\xi(0,0) = 1)$ for the particle density under μ.

For $\eta \in \Omega$, write P_η to denote the law of ξ starting from η, and put

$$\mathbb{P}_\mu(\cdot) = \int_\Omega \mu(\mathrm{d}\eta)\, P_\eta(\cdot).$$

2. RW. Given ξ, let

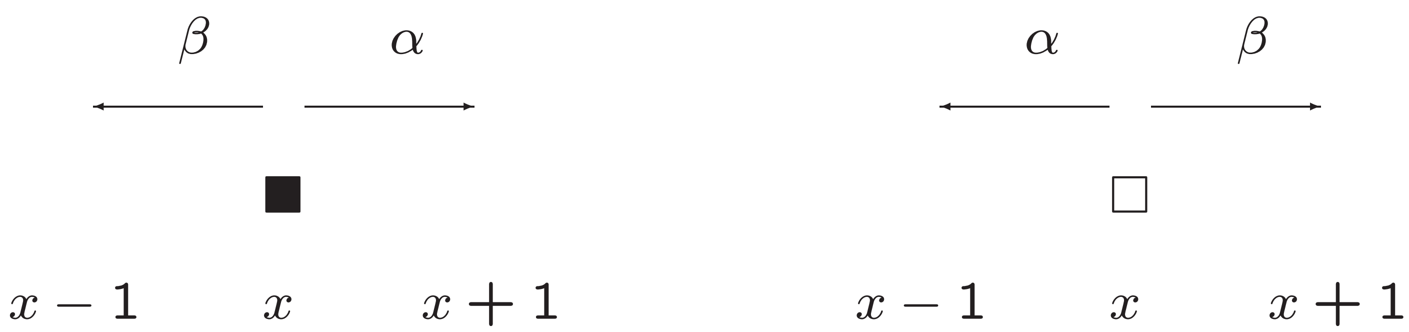
$$X = \{X(t)\colon\ t \geq 0\}$$

be the random walk with transition rates

$$x \to x+1 \quad \text{at rate } \alpha\,\xi(x,t) + \beta\,[1-\xi(x,t)],$$

$$x \to x-1 \quad \text{at rate } \beta\,\xi(x,t) + \alpha\,[1-\xi(x,t)],$$

where $\alpha, \beta > 0$.



Jump rates on top of particles ■ and holes □.

3. RWDRE. Write P_0^ξ to denote the law of X starting from 0 conditional on ξ [= quenched law], and

$$\mathbb{P}_{\mu,0}(\cdot) = \int_{D_\Omega[0,\infty)} \mathbb{P}_\mu(\mathrm{d}\xi)\, P_0^\xi(\cdot)$$

to denote the law of X starting from 0 averaged over ξ [= annealed law], where $D_\Omega[0,\infty)$ is the set of càdlàg paths in Ω.

Without loss of generality may assume that

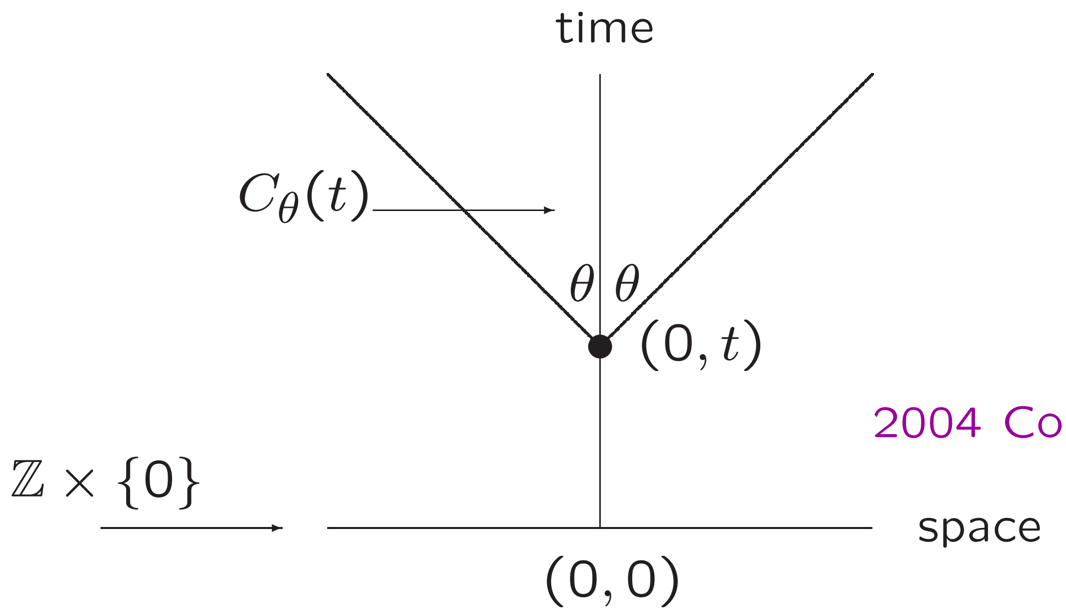
$$\rho \geq \tfrac{1}{2}, \qquad \alpha > \beta > 0.$$

§ LAW OF LARGE NUMBERS

DEFINITION: $\mathbb{P}_\mu$ is said to be cone mixing if, for all $\theta \in (0, \frac{1}{2}\pi)$,

$$\lim_{t\to\infty} \sup_{\substack{A\in\mathcal{F}_{\mathbb{Z}\times\{0\}} \\ B\in\mathcal{F}_{C_\theta(t)}}} \Big| \mathbb{P}_\mu(B|A) - \mathbb{P}_\mu(B) \Big| = 0,$$

where $\mathcal{F}$ stands for sigma-algebra.



2004 Comets, Zeitouni

THEOREM 1 [LLN]

If $\mathbb{P}_\mu$ is cone-mixing, then $\lim_{t\to\infty} X_t/t = v$ *exists and is constant a.s. under the law* $\mathbb{P}_{\mu,0}$.

No information is available on the sign of v [hard problem]!

It is natural to conjecture that

$$v \begin{cases} = 0 & \text{if} \quad \rho = \frac{1}{2}, \\ > 0 & \text{if} \quad \rho > \frac{1}{2}. \end{cases}$$

Positive speed implies transience. Does zero speed correspond to recurrence?

§ SPIN-FLIP SYSTEMS

Suppose that ξ is a spin-flip system with transition rates $c(x,\eta)$, $x \in \mathbb{Z}$, $\eta \in \Omega$, that are shift-invariant. Let

$$M = \sum_{x \in \mathbb{Z} \setminus \{0\}} \sup_{\eta \in \Omega} |c(0,\eta) - c(0,\eta^x)|,$$

$$\epsilon = \inf_{\eta \in \Omega} [c(0,\eta) + c(0,\eta^x)].$$

THEOREM 2 [LLN for spin-flip systems]

If $\mathbb{P}_\mu$ *satisfies* $M < \epsilon$ *and* $0 < \alpha - \beta < \frac{1}{2}(\epsilon - M)$, *then* $v = (\alpha - \beta)(2\tilde{\rho} - 1)$ *with*

$$\tilde{\rho} = \sum_{n=0}^{\infty} (\alpha - \beta)^n c_n(\alpha + \beta; \mathbb{P}^\mu),$$

where $c_0 = \rho$ *and* c_n, $n \in \mathbb{N}$, *are given by a recursive relation.*

The condition $M < \epsilon$ in essence is a large noise or high temperature condition, and guarantees exponentially fast decay of space-time correlations in the random environment.

Additional facts:

1. $c_1 = 0$ when $\mathbb{P}_\mu$ is reversible.
2. $c_1 = 0, c_2 < 0$ for $\rho > \frac{1}{2}$ when $\mathbb{P}_\mu$ is independent spin-flips.

§ EXCHANGE SYSTEMS

Key example:

The exclusion process: independent random walks jumping at rate 1 with hard core repulsion.

The exclusion process is not cone-mixing. Nevertheless, the LLN is expected to hold.

Below we describe simulations.

In the figures below, four speeds are computed via simulation as a function of $p = \alpha/(\alpha + \beta)$ and ρ:

(1) static speed

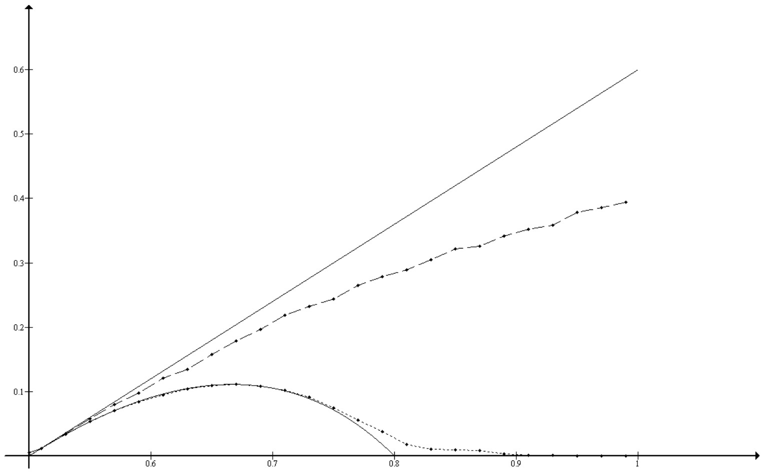
(2) static simulated speed

(3) dynamic simulated speed

(4) average medium speed

Each simulation is based on 10^3 initial configurations of the RE and 10^4 steps of the RE and the RW, both in discrete time.

Quenched and annealed speeds are the same within simulation error.



Speed as a function of p for $\rho = 0.8$.

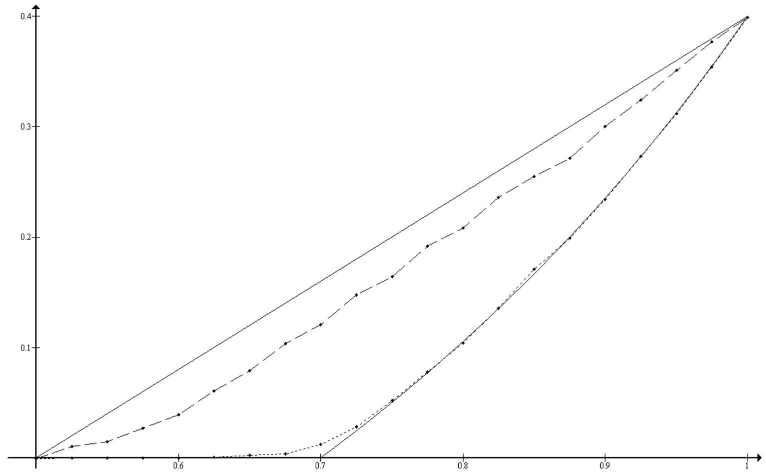
From bottom to top:

(1) static speed

(2) static simulated speed

(3) dynamic simulated speed

(4) average medium speed



Speed as a function of ρ for $p = 0.7$.

From bottom to top:

(1) static speed

(2) static simulated speed

(3) dynamic simulated speed

(4) average medium speed

§ FURTHER WORK ON LLN

- Cone-mixing and random walk with unbounded steps.

  FdH, Renato dos Santos, Vladas Sidoravicius

- Zero-range process at high density.

  FdH, Harry Kesten, Vladas Sidoravicius

  Marcelo Hilario, FdH, Vladas Sidoravicius

- Exclusion process and random walk with uniformly positive drift.

  Luca Avena, Renato dos Santos, Florian Völlering

- Exclusion process and random walk with variable jump rate.

  Luca Avena, Philip Thomann

- Contact process.

  FdH, Renato dos Santos