Spatial extent of an outbreak in animal epidemics

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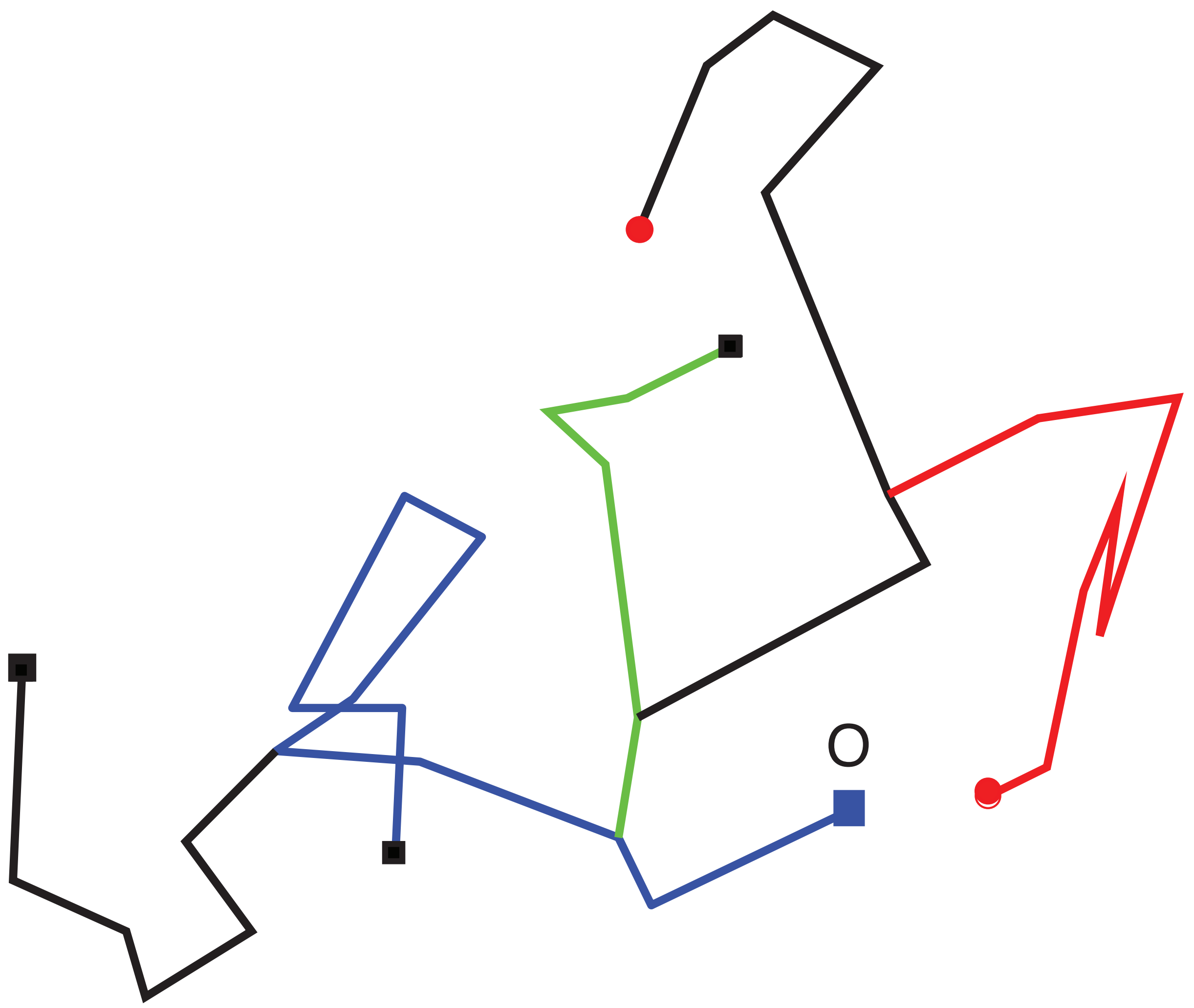

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SIR model for epidemics

Three species : susceptibles (S), infected (I), recovered (R)

$$\frac{dS}{dt} = -\beta\, I\, S$$

$$\frac{dI}{dt} = \beta\, I\, S - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

- mean field fully connected model
- β rate of infection transmission
- γ rate at which an infected recovers

$$I(t) + S(t) + R(t) = N$$

N being the total population

Outbreak of an epidemic

Initial condition : $I(0) = 1,\ S(0) = N - 1 \approx N,\ R(0) = 0$

Outbreak regime

$$\frac{dS}{dt} = -\beta\, I\, S$$
$$\frac{dI}{dt} = \beta\, I\, S - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$

$t \approx 0,\ S \approx N$ ⇒

$$\frac{dI}{dt} \simeq (\beta\, N - \gamma)\, I$$

Reproduction rate: $\boxed{R_0 = \frac{\beta N}{\gamma}}$

Deterministic and stochastic models

SIR is a deterministic model. In the outbreak fluctuations are important

- Stochastic process: Galton-Watson (mean field)
- each infected individual transmits the disease at rate $N\beta$
- each infected individual recovers at rate γ

Reproduction rate: $\boxed{R_0 = \frac{\beta N}{\gamma}}$

- $R_0 < 1$ epidemics extinction
- $R_0 > 1$ epidemics invasion
- $R_0 = 1$ critical case

How far in space can an epidemic spread?

Problem 1: How to model the space?

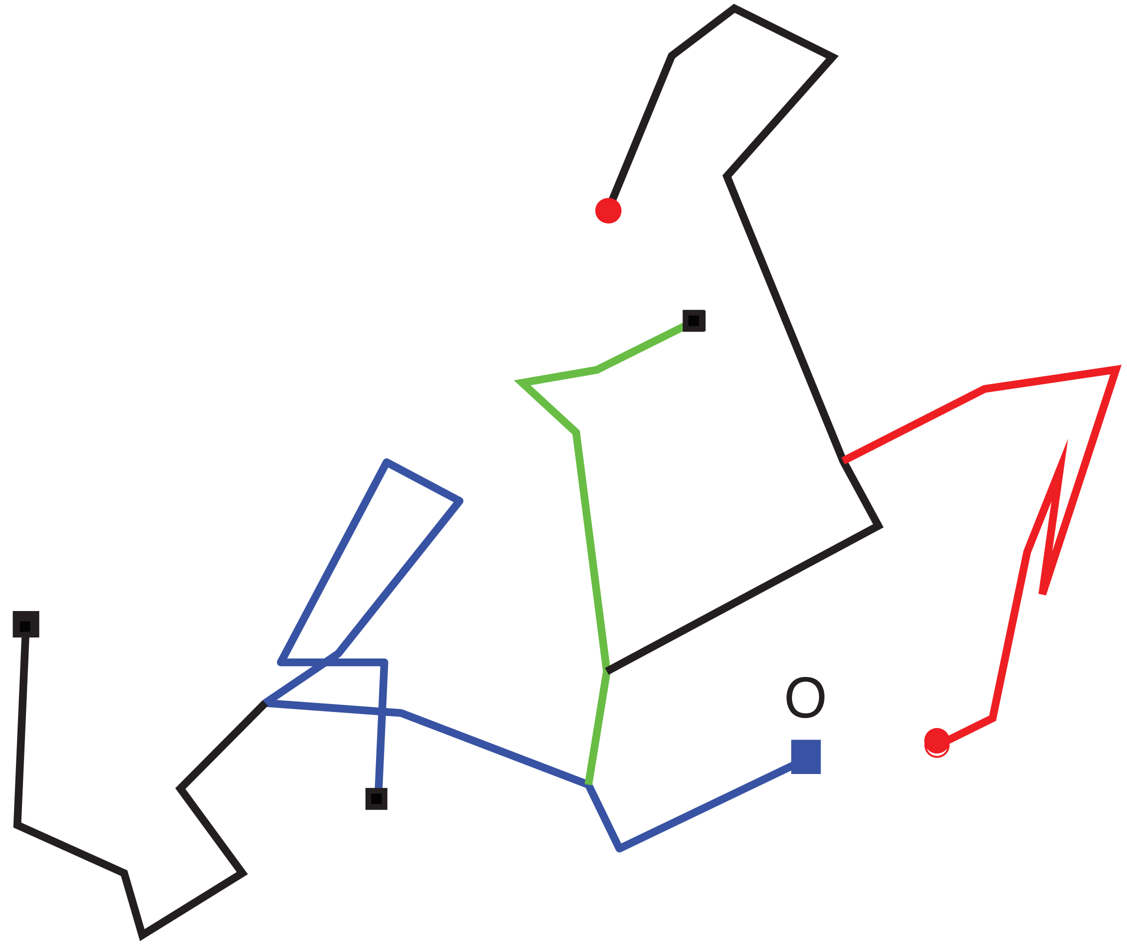
The population is uniformly distributed

At time $t = 0$ an infected individual appears

... and moves in space

Brownian motion is the paradigm of animal migration

while human beings take the plane (even when they are sick)



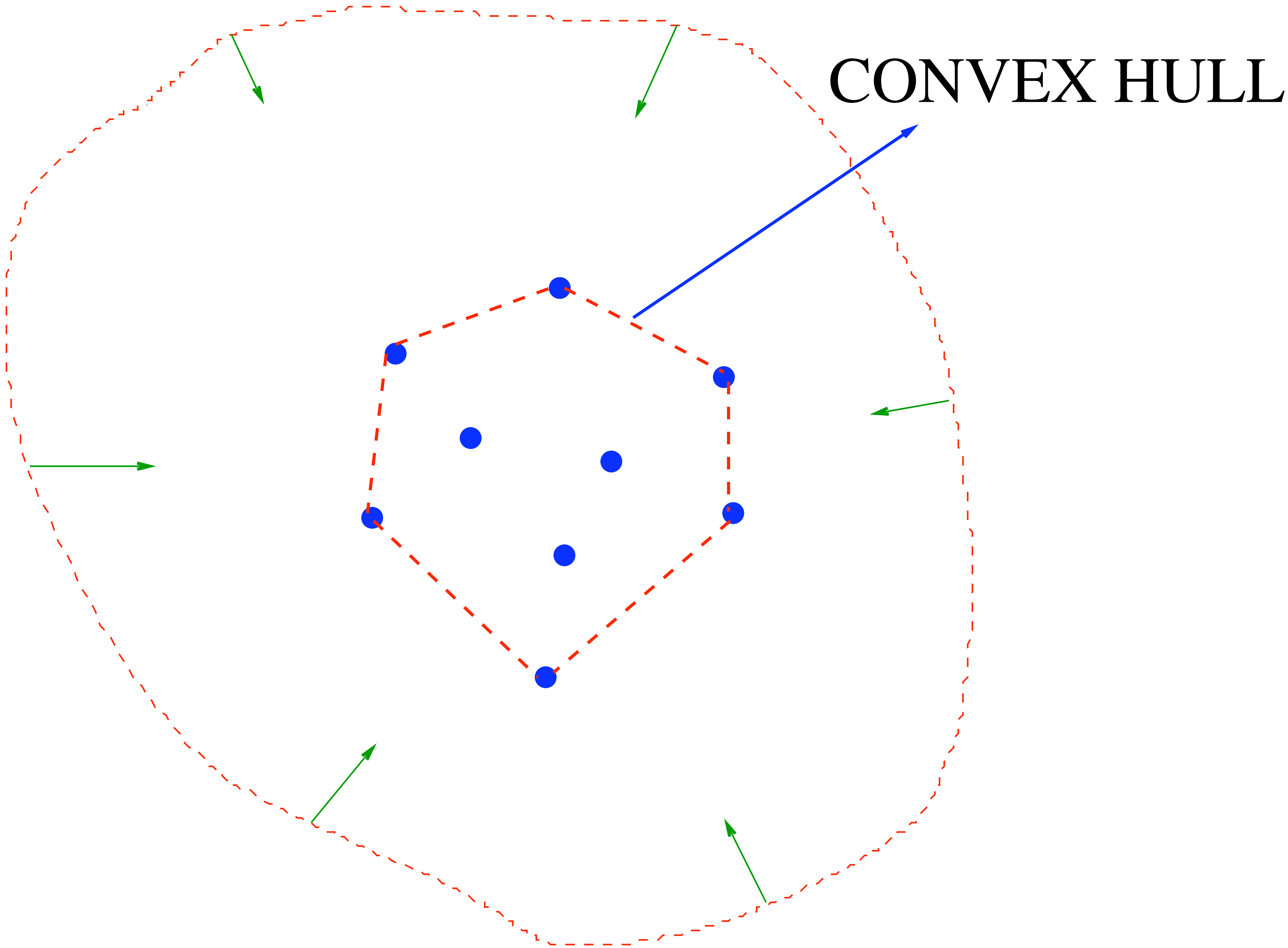
The good candidate: Brownian process with branching and death

In dt, each infected can:

- recovers with probability $\gamma\, dt$
- infects with probability $\beta N\, dt = \gamma R_0 dt$
- otherwiese, it diffuses (D diffusion const.)

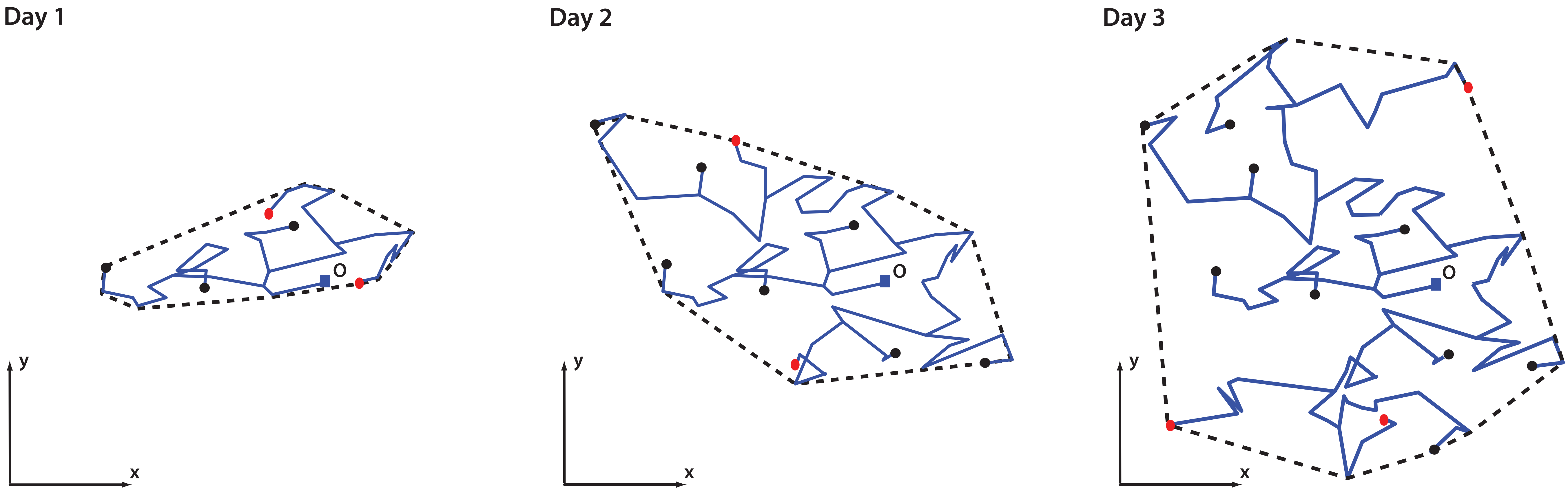
$$\text{Diffusion } \Delta x \sim \sqrt{2Ddt},\ \langle \Delta x \rangle = 0,\ \langle \Delta x^2 \rangle = \sqrt{2Ddt}$$

Problem 2: How to quantify the area that needs to be quarantined?

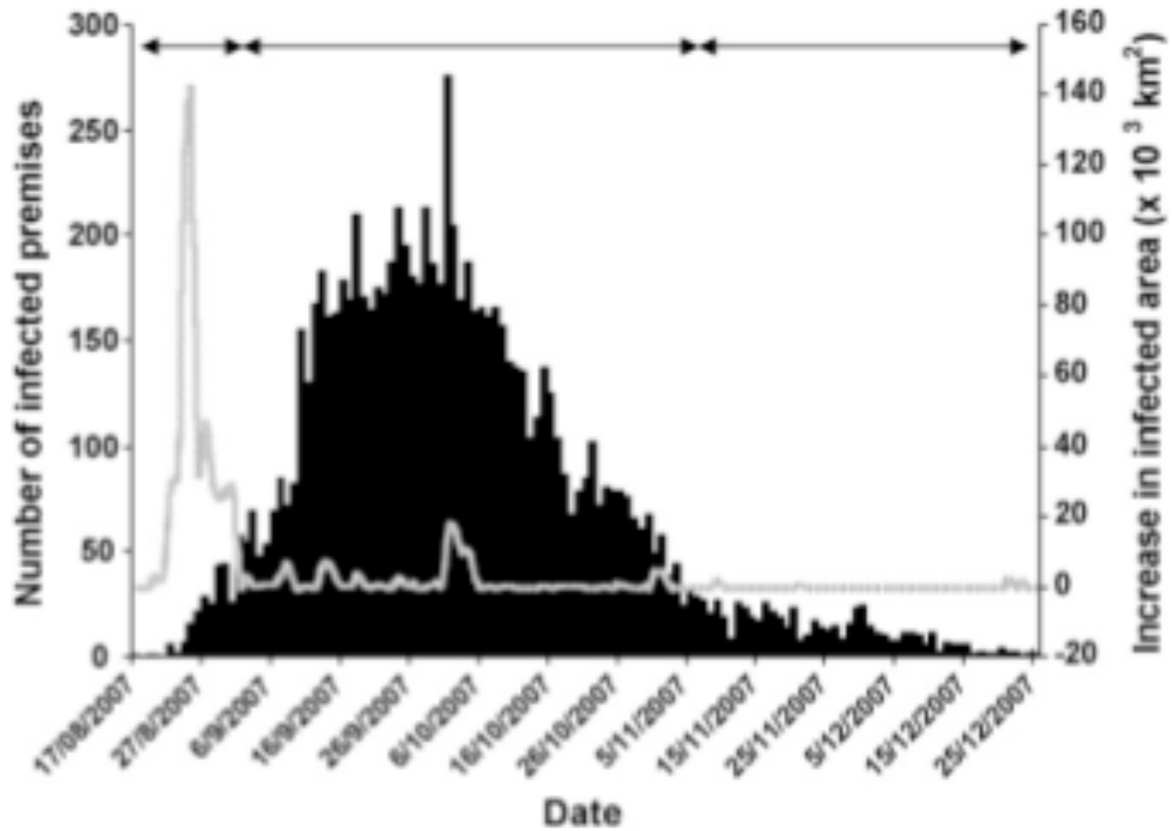


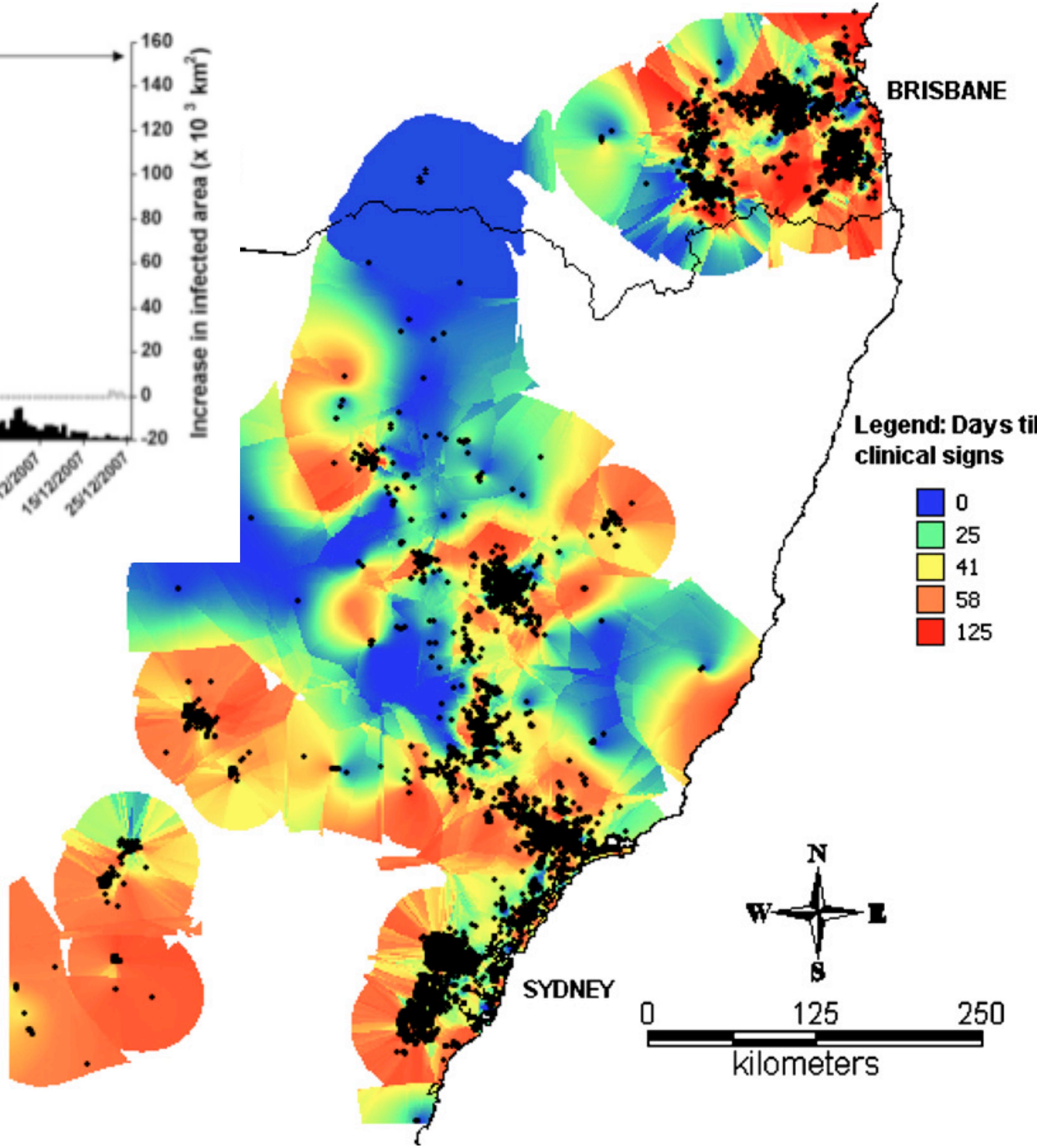
Algorithms: Graham Scan (Nlog(N))

Monitoring the outbreak

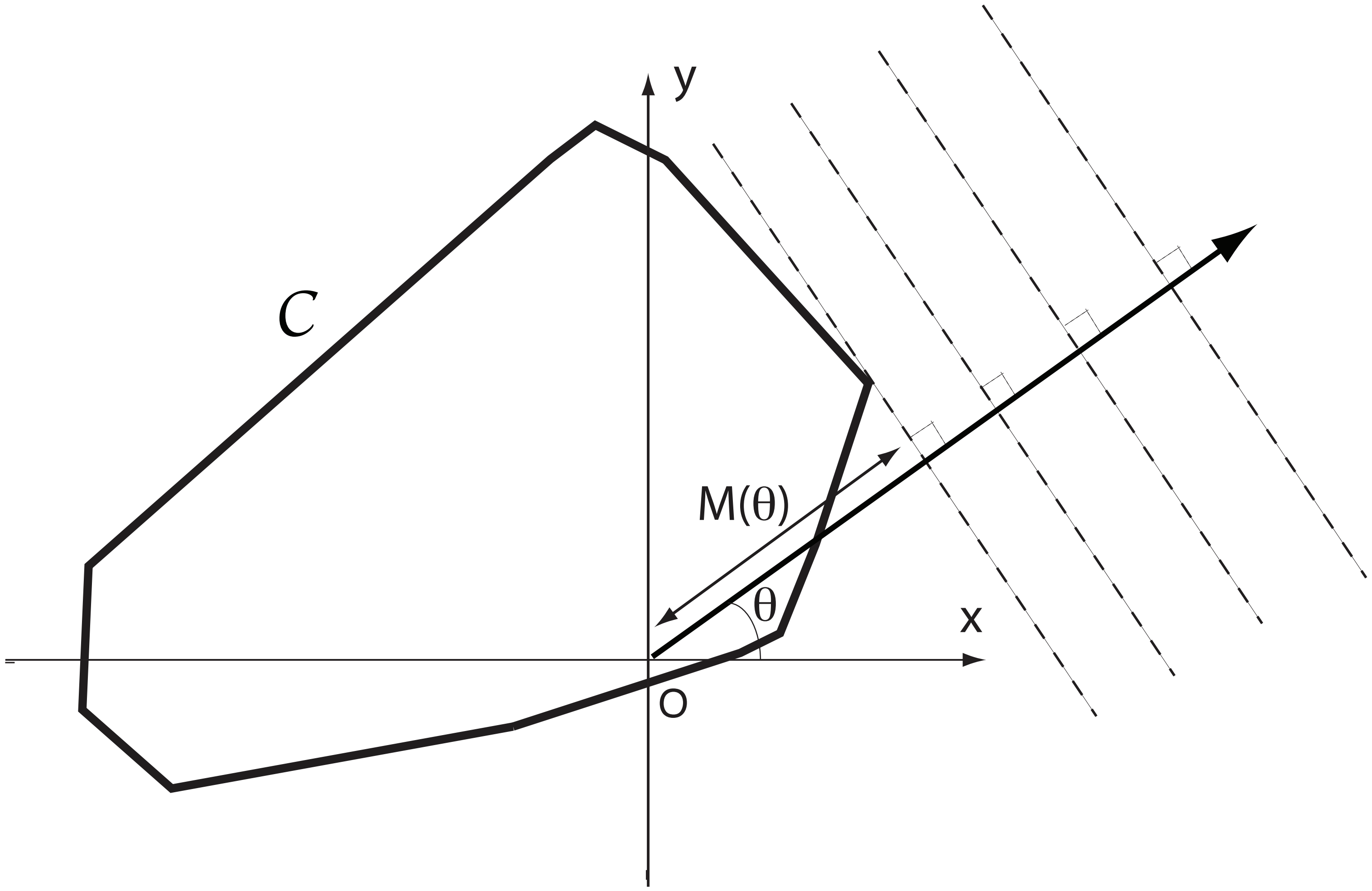


Real applications





How to compute the convex hull of Branching processes?



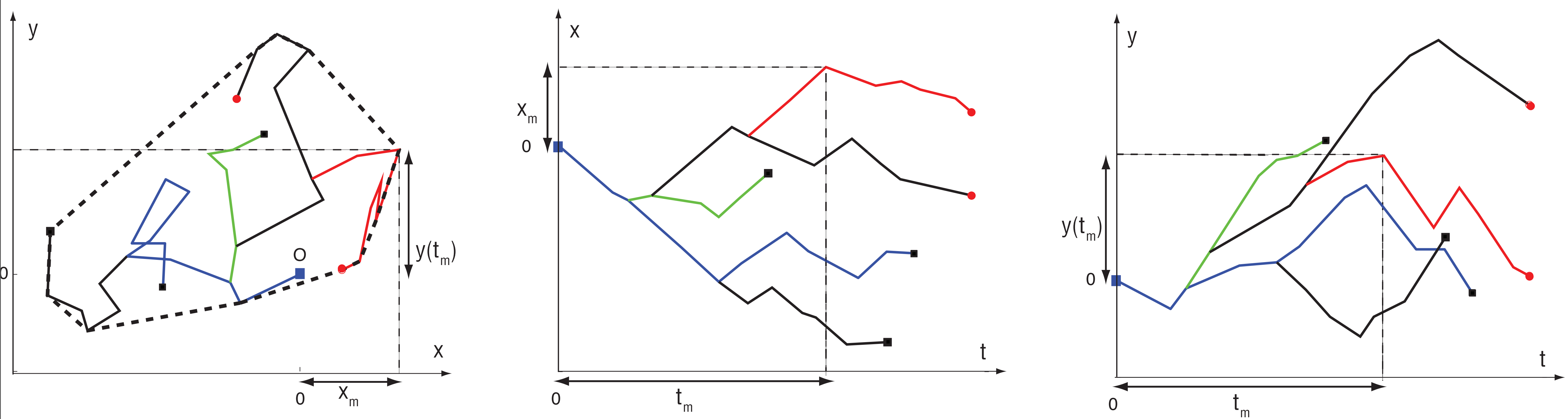
Cauchy formulas

$$L = \int_0^{2\pi} M(\theta)\, d\theta$$

$$A = \frac{1}{2}\int_0^{2\pi} \left[M^2(\theta) - (M'(\theta))^2\right] d\theta$$

Support Function

$$M(\theta) = \max_{0 \leq \tau \leq t} \left[x_\tau \cos\theta + y_\tau \sin\theta \right]$$



$$M(0) = x_{\tau = t_m} = x_m(t)$$

- $x_m(t)$ x-maximum up to time t
- t_m time *location* of the maximum

$$M'(\theta = 0) = -x_{t_m} \sin\theta + y_{t_m} \cos\theta|_{\theta=0} = y_{t_m}$$

Dimensional reduction

$$L = \int_0^{2\pi} M(\theta)\, d\theta$$

$$A = \frac{1}{2}\int_0^{2\pi} \left[M^2(\theta) - (M'(\theta))^2\right] d\theta$$

If the process is rotationally invariant any average is independent of θ

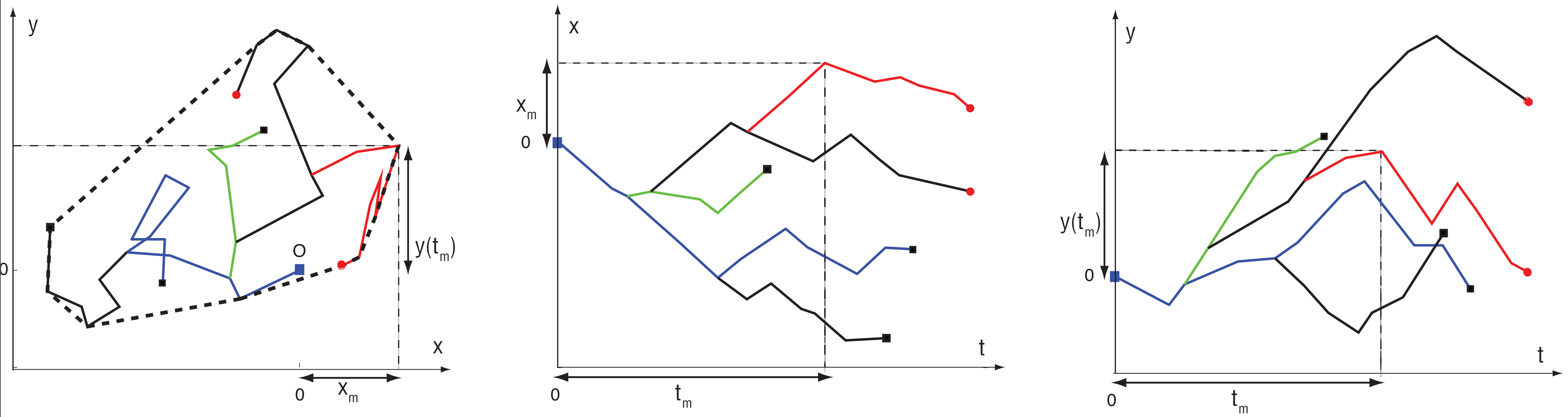
$$\langle L(t)\rangle = 2\pi \langle M(0)\rangle$$

$$\langle A(t)\rangle = \pi \left[\langle M^2(0)\rangle - \langle M'(0)^2\rangle\right]$$

$$\langle L(t)\rangle = 2\pi \langle x_m(t)\rangle$$

$$\langle A(t)\rangle = \pi \left[\langle x_m^2(t)\rangle - \langle y^2(t_m)\rangle\right]$$

This relation is valid ONLY in average



$$\langle L(t) \rangle = 2\pi \langle x_m(t) \rangle$$

$$\langle A(t) \rangle = \pi \left[\langle x_m^2(t) \rangle - \langle y^2(t_m) \rangle \right]$$

consider a 1d branching process evolving in $(0, t)$

- x_m is the global maximum
- t_m is the location of the maximum
- $\langle y^2(t_m) \rangle = \ldots = 2D\, t_m$

Backward Fokker Planck equation

$$Q_t(x_m) = \text{Proba}[\text{global max up to } t < x_m]$$

$$Q_{t+dt}(x_m) = \gamma dt + R_0 \gamma dt Q_t^2(x_m) + [1 - \gamma(R_0+1)] dt \langle Q_t(x_m - \Delta x) \rangle$$

- $\langle Q_t(x_m - \Delta x) \rangle = Q_t(x_m) - \langle \Delta x \rangle Q_t'(x_m) + \langle \frac{\Delta x^2}{2} \rangle Q_t''(x_m) + \ldots$
- $\langle \Delta x \rangle = 0$
- $\langle \Delta x^2 \rangle = 2Ddt$

$$\langle Q_t(x_m - \Delta x) \rangle = Q_t(x_m) + Ddt \partial_x^2 Q_t(x_m) + \ldots$$

$$\frac{\partial}{\partial t}Q = D\frac{\partial^2}{\partial x_m^2}Q - \gamma(R_0+1)Q + \gamma R_0 Q^2 + \gamma$$

- initial condition $Q_{t=0}(x_m) = \theta(x_m)$
- boundary condition $Q_t(x_m < 0) = 0$
- boundary condition $Q_t(x_m \to \infty) = 1$

$$\langle L(t)\rangle = 2\pi \int_0^\infty [1 - Q_t(x_m)]dx_m.$$

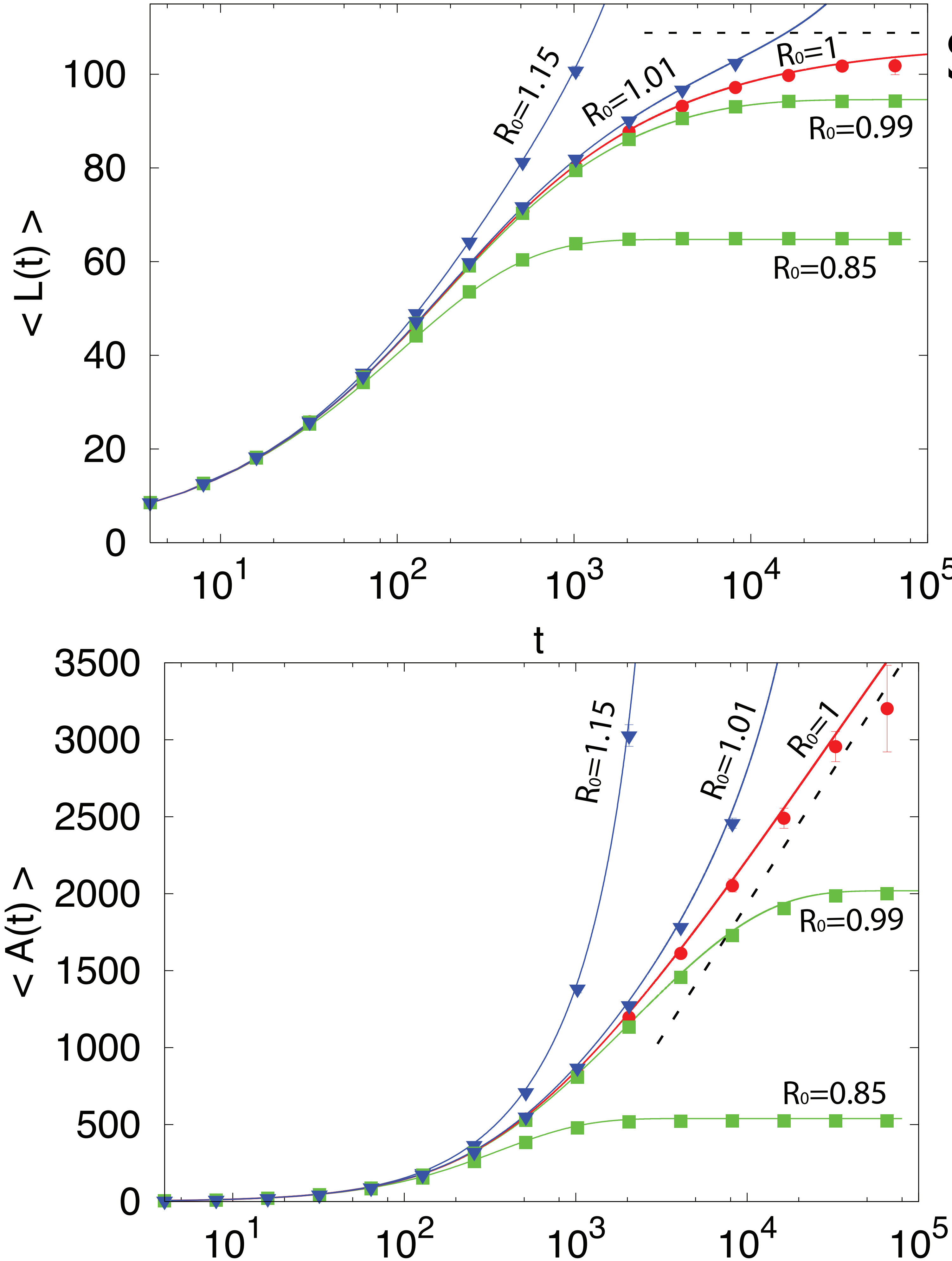
Similar calculations allows to express the mean area as:

$$\langle A(t)\rangle = \pi \int_0^\infty dx_m \left[2x_m(1 - Q_t(x_m)) - T_t(x_m)\right]$$

Where the evolution of $T_t(x_m)$ is governed by:

$$\frac{\partial}{\partial t}T_t + \partial_x Q_t(x_m) = \left[D\frac{\partial^2}{\partial x_m^2} + 2\gamma R_0 Q_t - \gamma\left(R_0 + 1\right)\right]T_t,$$

Both PDE can be integrated numerically and solved in some asymptotic limit



Solid lines: our predictions

Blue lines: super-critical

Red lines: critical

Green lines: sub-critical

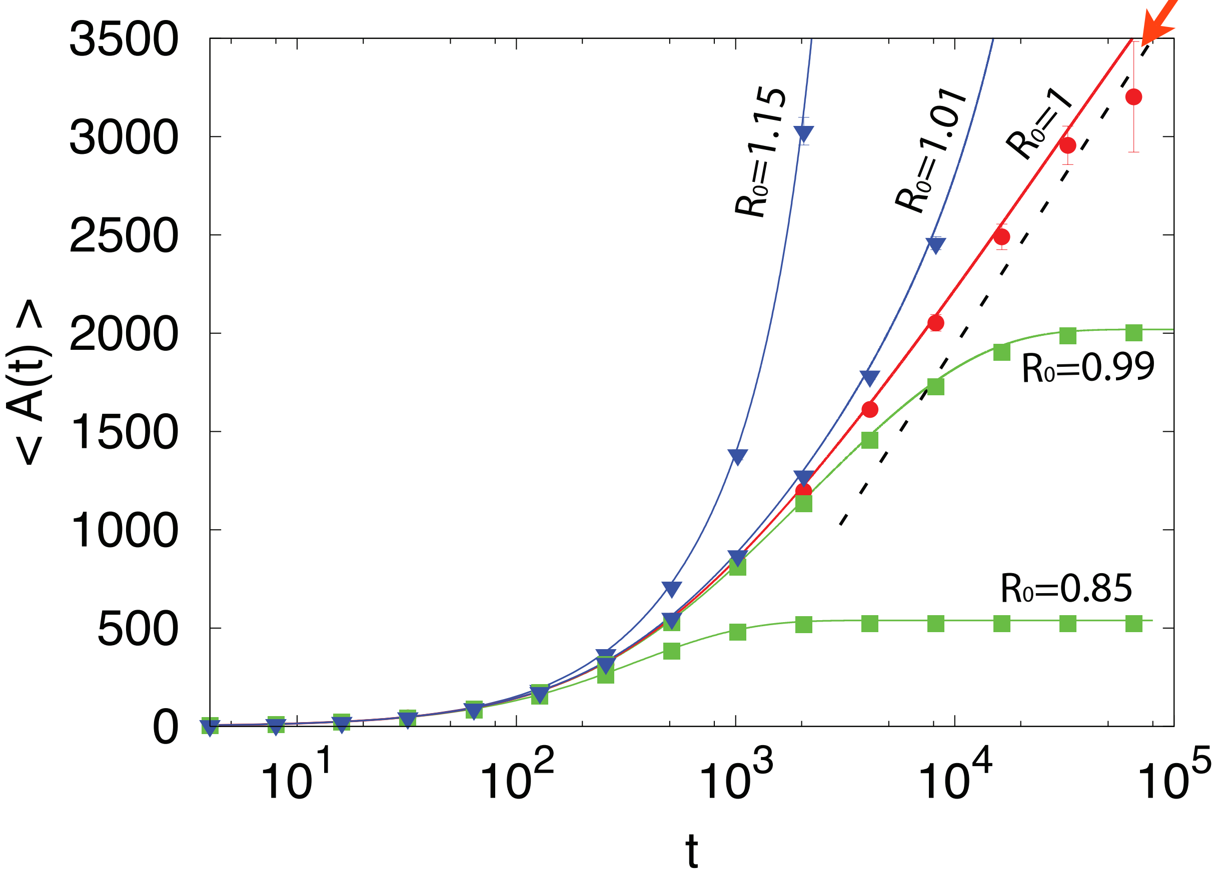
Symbols: Monte Carlo simulations

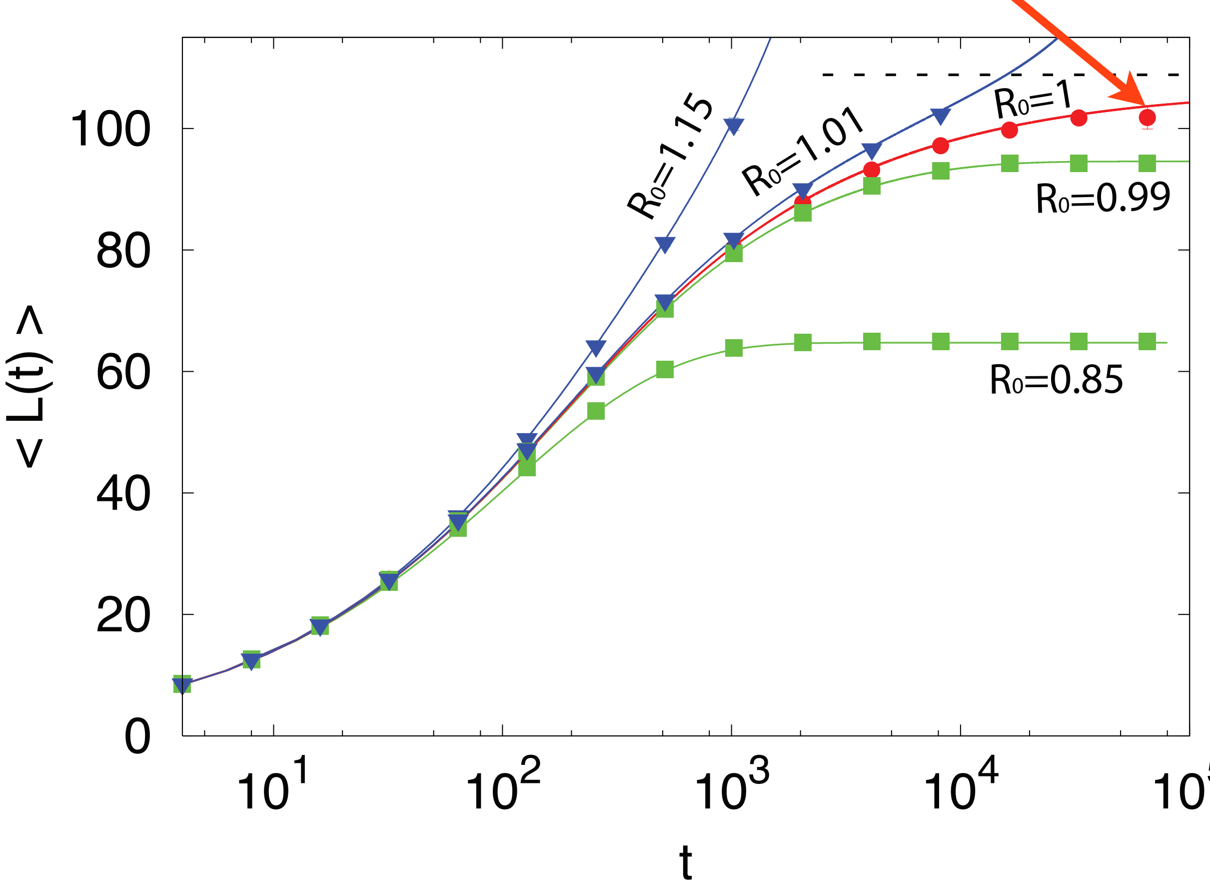
Dashed lines: analytical asymptotic results

The critical case

$$\langle L(t \to \infty) \rangle = 2\pi \sqrt{\frac{6D}{\gamma}} + \mathcal{O}(t^{-1/2})$$

$$\langle A(t \to \infty) \rangle = \frac{24\pi D}{5\gamma} \ln t + \mathcal{O}(1)$$

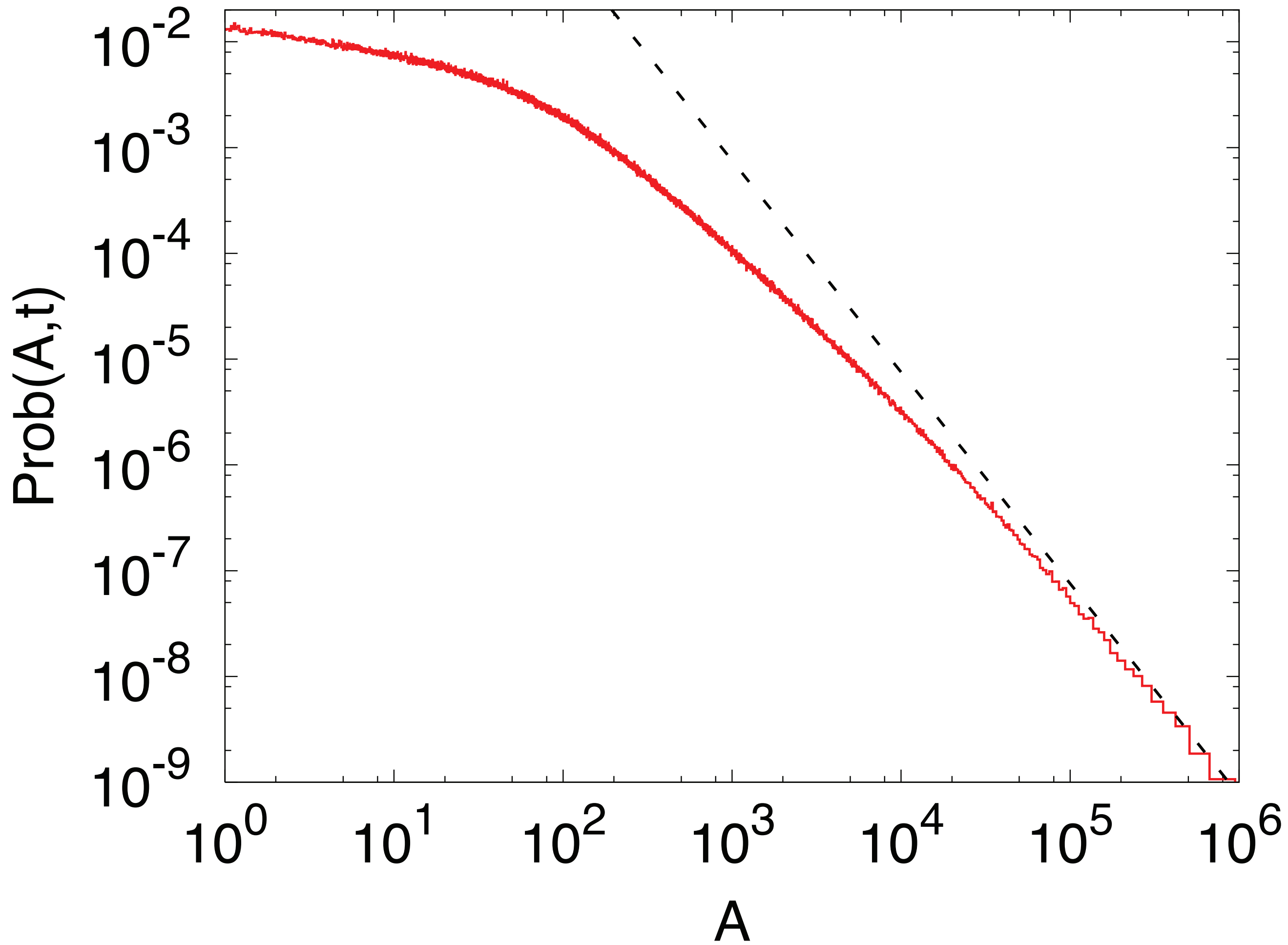


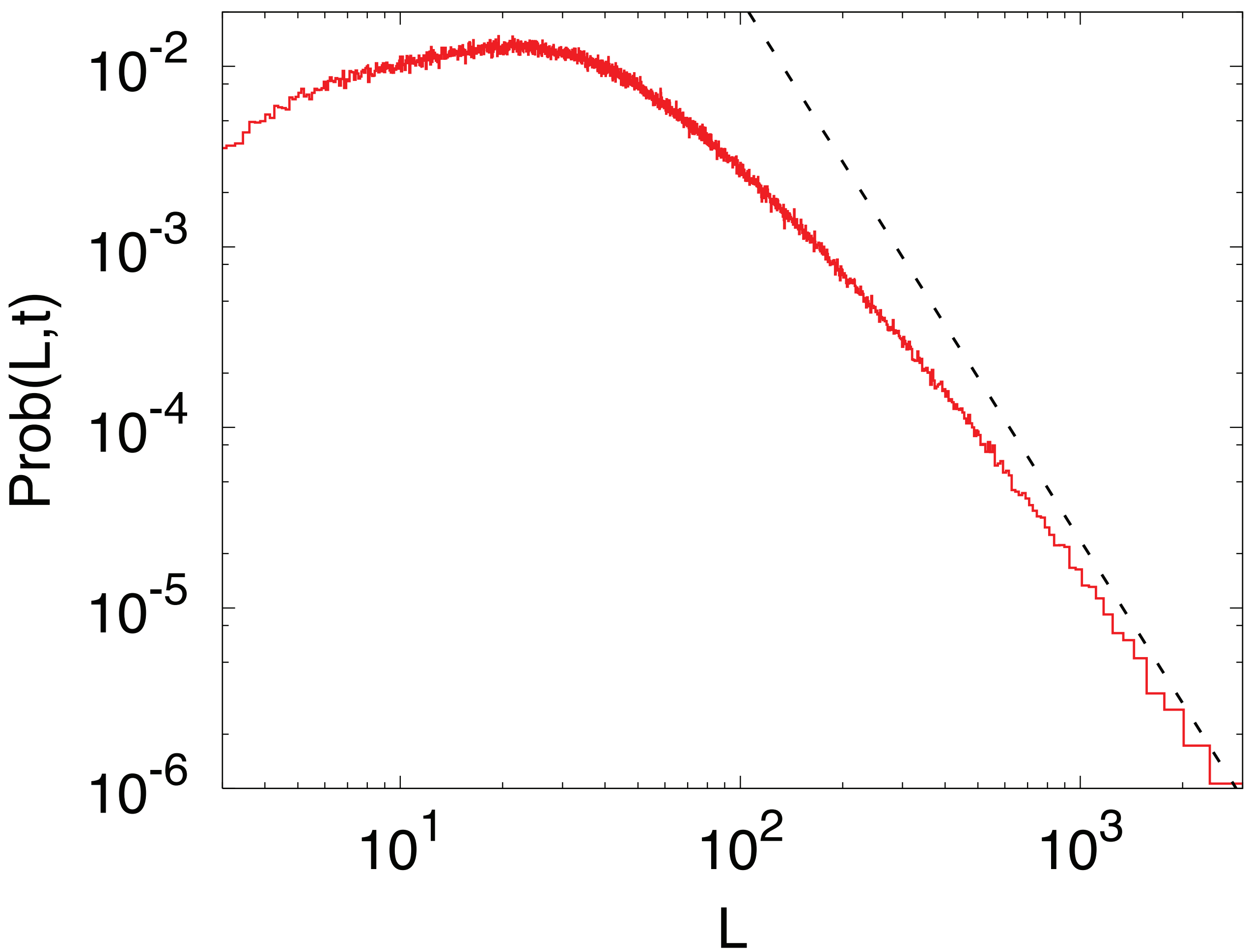


When $t \to \infty$ the perimeter remains finite, but the area diverges!

How it is possible ? ... Fluctuations

$$\mathrm{Prob}(A) \xrightarrow[A\to\infty]{t=\infty} \frac{24\pi D}{5\gamma} A^{-2} \qquad \mathrm{Prob}(L) \xrightarrow[L\to\infty]{t=\infty} L^{-3}$$





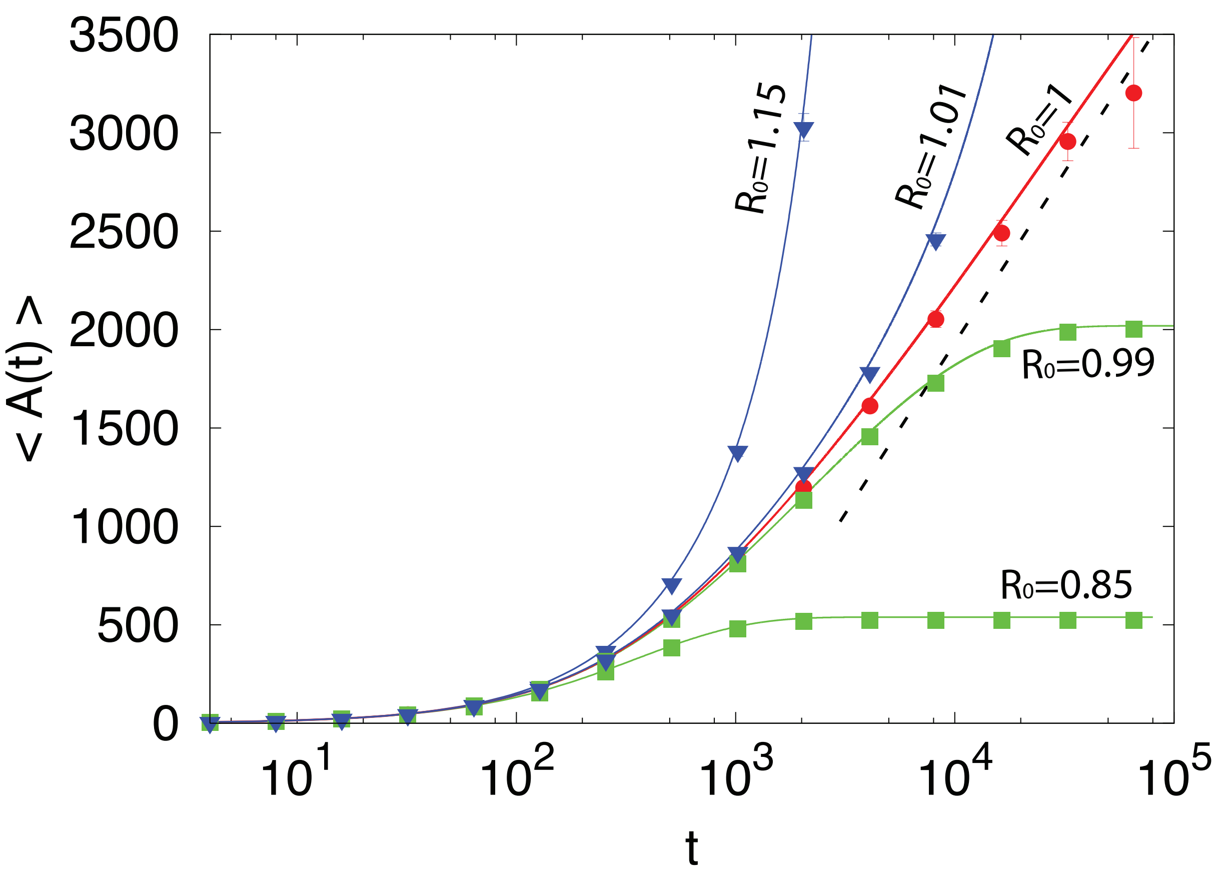
Out of criticality

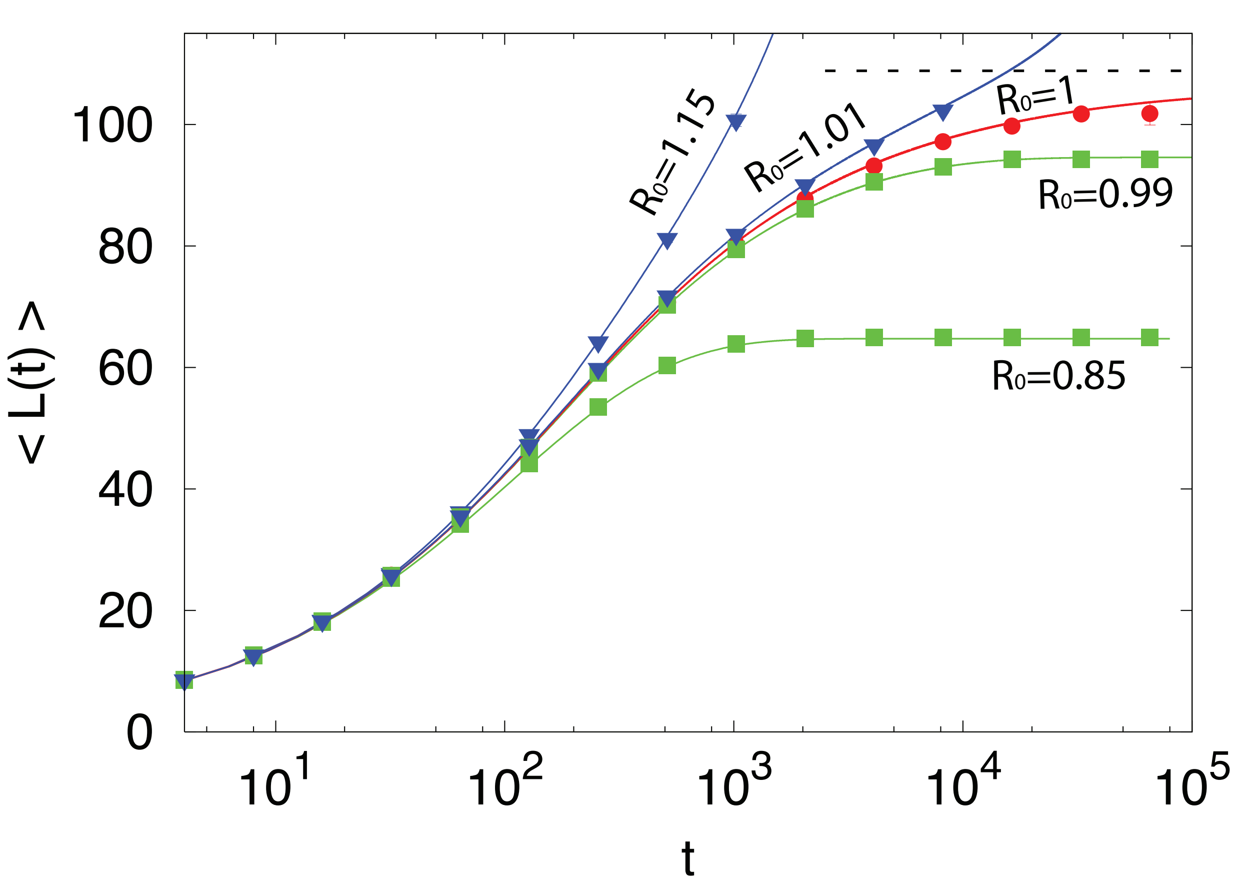
When $R_0 \neq 1$, characteristic time $t^* \sim |R_0 - 1|^{-1}$.

For times $t < t^*$ the epidemic behaves as in the critical regime.

In the *subcritical* regime, for $t > t^*$ the epidemic goes to extinction.

In the *supercritical* regime, with probability $1 - 1/R_0$ epidemic explodes.



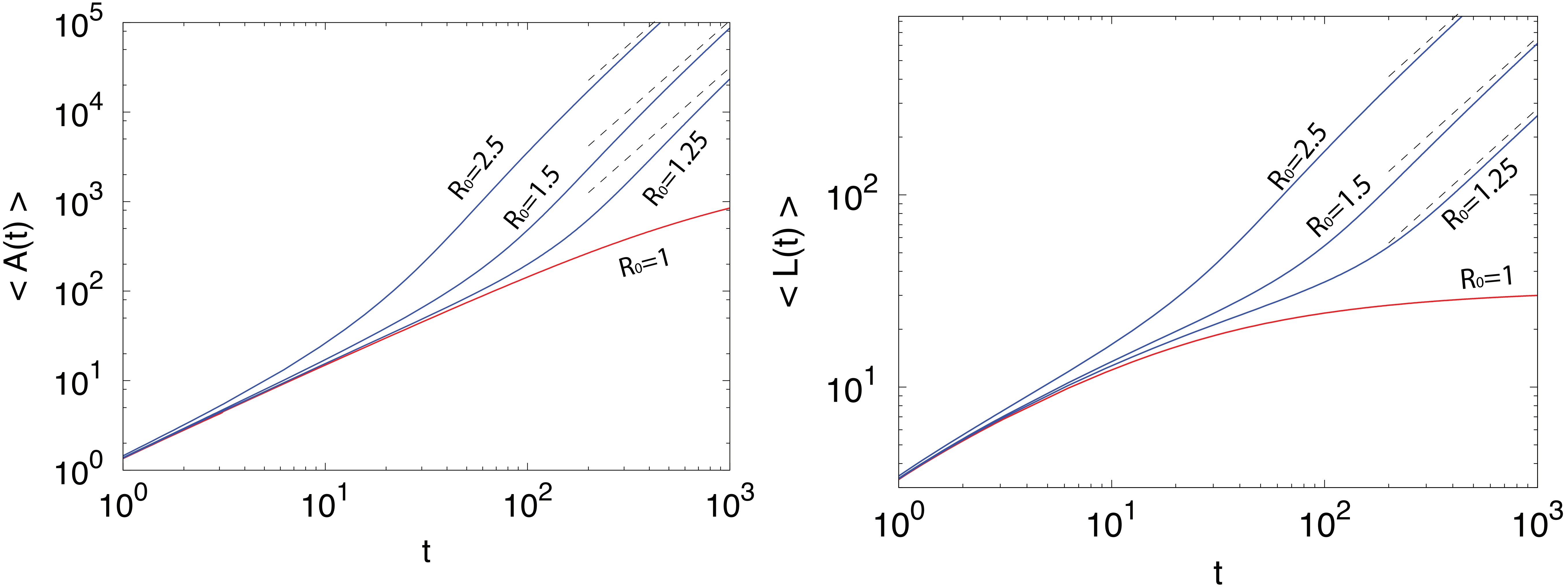


Supercritical

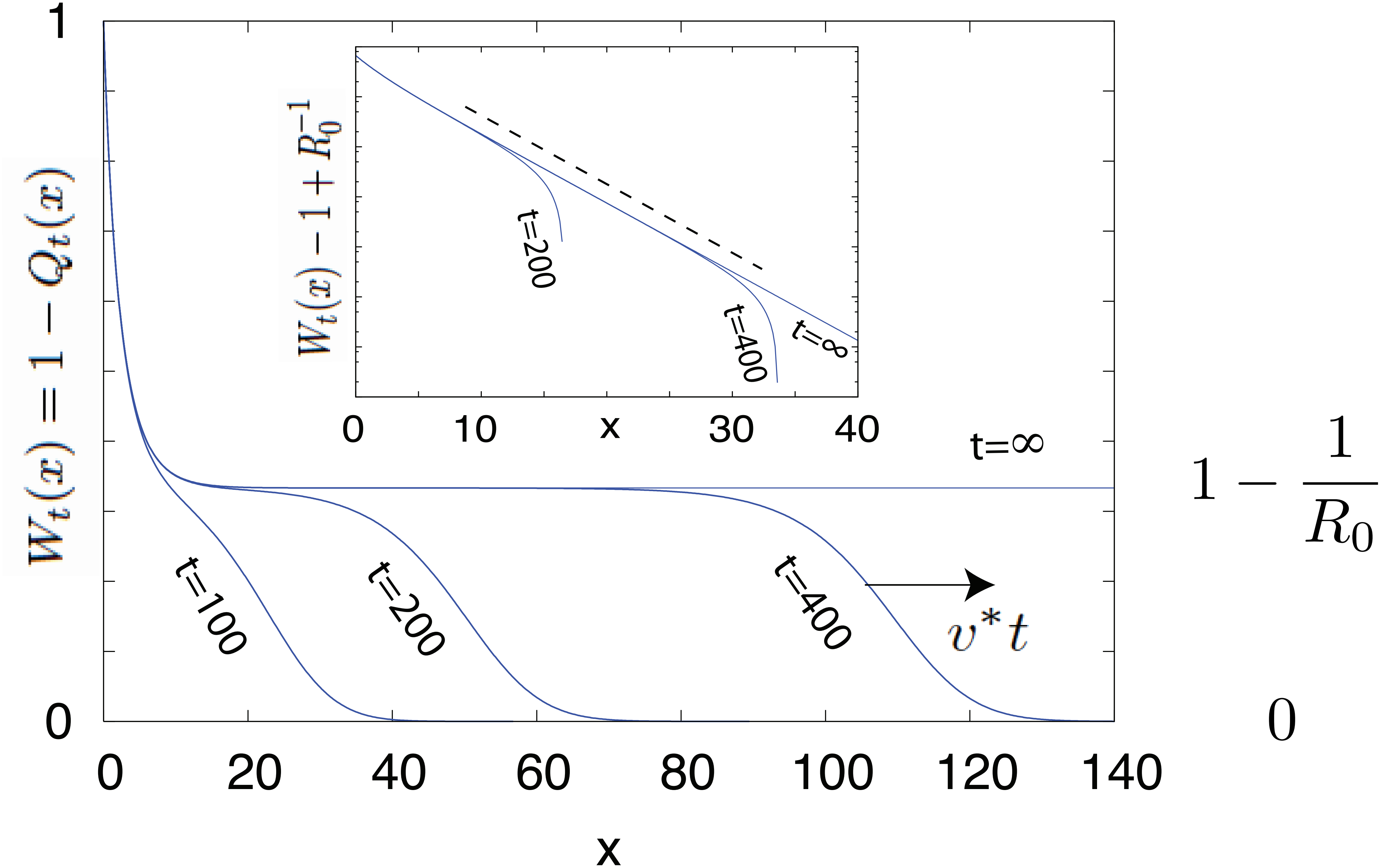
$$\langle L(t \gg t^*) \rangle = 4\pi \left(1 - \frac{1}{R_0}\right) \sqrt{D\,\gamma\,(R_0 - 1)}\,t$$

$$\langle A(t \gg t^*) \rangle = 4\pi \left(1 - \frac{1}{R_0}\right) D\,\gamma\,(R_0 - 1)\,t^2$$

$$t^* \sim |R_0 - 1|^{-1}$$



$$\frac{\partial}{\partial t} W = D \frac{\partial^2}{\partial x_m^2} W + \gamma(R_0 - 1)W - \gamma R_0 W^2$$



Traveling front solution

Conclusions:

- Branching Brownian motion with death as a model for the spatial extent of animal epidemics
- Using Cauchy Formulas we can map the convex hull problem in the extreme statistic of the 1-dimensional process
- Backward F-P equations for the extreme distributions
- Critical case has very large fluctuations
- Super Critical case: traveling front solution