A bottom-up approach to the Kardar-Parisi-Zhang equation for interface growth : recent developments

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Introduction

The Kardar-Parisi-Zhang (KPZ) equation describes an interface growth with a height field $h(x,t)$

$$\partial_t h(x,t) = \partial_{xx}^2 h(x,t) + (\partial_x h(x,t))^2 + \sqrt{2}\xi(x,t)$$

ξ is a unit white noise $\mathbb{E}\left[\xi(x,t)\xi(y,t')\right] = \delta(x-y)\delta(t-t')$

This describes an out-of-equilibrium physics problem.

My work

My interest lies in the connexions between asymptotic random matrix ensemble distributions, non-interacting fermions at finite temperature in quantum mechanics and solutions of the KPZ equation.

I want to extract information on the distribution of the solution of the KPZ equation and especially the large deviations away from its typical behavior.

This research is made within the group of Pierre Le Doussal (ENS), Satya N. Majumdar and Gregory Schehr (Orsay).

Plan of the talk

1. Physics motivation and some experimental observations
2. Mapping to the directed polymer, the replica method, the quantum delta Bose gas
3. Exact solutions of the KPZ equation at all times and Fredholm determinants
4. Exact solutions at short time and the matching with high precision simulations

Physics motivation

Since its birth in 1986, the KPZ equation was applied to describe

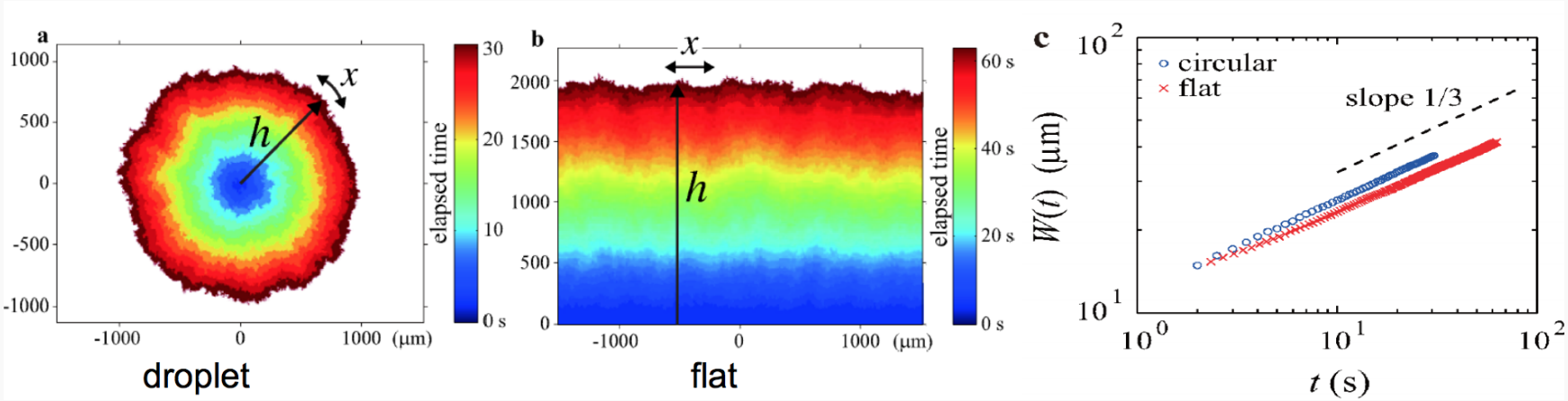
- Growth of interfaces
- Burgers turbulence
- Directed polymers in random media
- Chemical reaction fronts
- Slow combustion
- Coffee stains
- Conductance fluctuations in Anderson localization
- Bose Einstein superfluids
- Quantum entanglement growth

Physics motivation - experiment on liquid-crystal turbulence

K. Takeuchi, M. Sano, *Evidence for Geometry-Dependent Universal Fluctuations of the Kardar-Parisi-Zhang Interfaces in Liquid-Crystal Turbulence* J. Stat. Phys. (2012)

(videos on youtube)

Physics motivation - experiment on liquid-crystal turbulence



Define the temporal fluctuations

$$W(t) = \sqrt{\langle [h(x,t) - \langle h \rangle]^2 \rangle} \propto t^{1/3}$$

Similarly, take the roughness

$$C(\ell, t) = \sqrt{\langle [h(x+\ell,t) - h(x,t)]^2 \rangle} \propto \ell^{1/2}$$

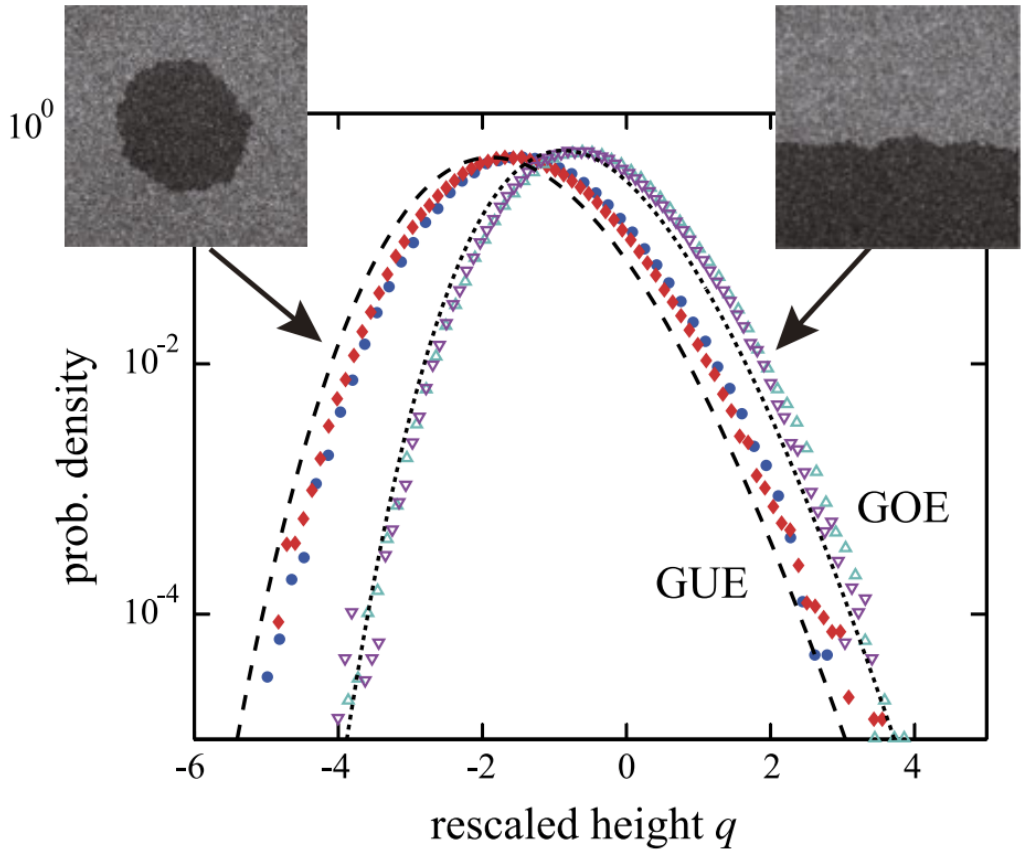
The scaling exponents for the 1D KPZ equation are

$$\delta h \propto t^{1/3} \propto x^{1/2}$$

The large time limit is $h(x,t) =_{t \gg 1} v_\infty t + \chi t^{1/3}$, where χ is a random variable

Physics motivation - experiment on liquid-crystal turbulence

Fig. 8 Histogram of the rescaled local height $q \equiv (h - v_\infty t)/(\Gamma t)^{1/3}$ for the circular (*solid symbols*) and flat (*open symbols*) interfaces. The *blue circles* and *red diamonds* display the histograms for the circular interfaces at $t = 10$ s and 30 s, respectively, while the *turquoise up-triangles* and *purple down-triangles* are for the flat interfaces at $t = 20$ s and 60 s, respectively. The *dashed* and *dotted curves* show the GUE and GOE TW distributions, respectively, defined by the random variables χ_{GUE} and χ_{GOE}. (Color figure online)



χ is called a Tracy-Widom random variable, it describes the distribution of the largest eigenvalue of a class of random matrices.

Initial conditions

Full-space:
$x \in \mathbb{R}$

- Flat, $h(x, t = 0) = 0$
- Droplet (wedge), $h(x, t = 0) = -w|x| + \log(\frac{w}{2})$, $w \gg 1$
- Brownian, $h(x, t = 0) = B(x)$

Half-space:
$x \in \mathbb{R}^+$ with the b.c. $\partial_x h(x, t) \,|_{x=0} = A$, $\forall t > 0$.
It corresponds to the presence of a wall at the origin.

Cole-Hopf mapping

Defining $Z(x,t)$ the partition function as

$$Z(x,t) = \exp\left(h(x,t)\right)$$

It verifies in the Ito sense the stochastic heat equation

$$\partial_t Z(x,t) = \partial_{xx}^2 Z(x,t) + \sqrt{2} Z(x,t)\xi(x,t)$$

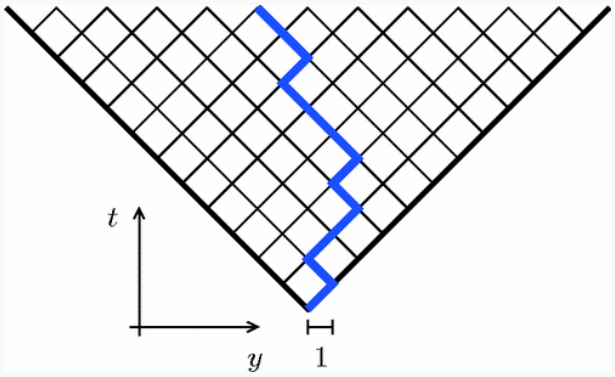
Directed polymer mapping I

Take a polymer on a rotated square lattice with coordinates (y, t).

The allowed moves are

- $(y, t) \to (y \pm 1, t + 1)$

Define random site variables $V_{x,t}$, a temperature T and the associated Boltzmann weight $\exp(-\frac{V_{x,t}}{T})$.



Take one path $\gamma : (0, 0) \to (x_f, L)$, its weight is defined by

$$w_\gamma = \prod_{(x,t)\in\gamma} e^{-\frac{V_{x,t}}{T}}$$

Define the partition sum for all such paths γ, $Z_{x_f,L} = \sum_\gamma w_\gamma$

Discretized version of the SHE

$$Z_{x,t+1} = \left(Z_{x-1,t} + Z_{x+1,t}\right) e^{-\beta V_{x,t+1}}$$

Directed polymer mapping II

The algorithmic method to solve this problem is called the *transfer matrix method*.

If we want to compute the partition sum up to time t, the complexity is of order $\mathcal{O}(t^2)$.

At zero temperature, $T \to 0$, defining the free energy $F_{x,t} = -T \log Z_{x,t}$ we have

$$F_{x,t+1} = \min(F_{x-1,t}, F_{x+1,t}) + V_{x,t+1}$$

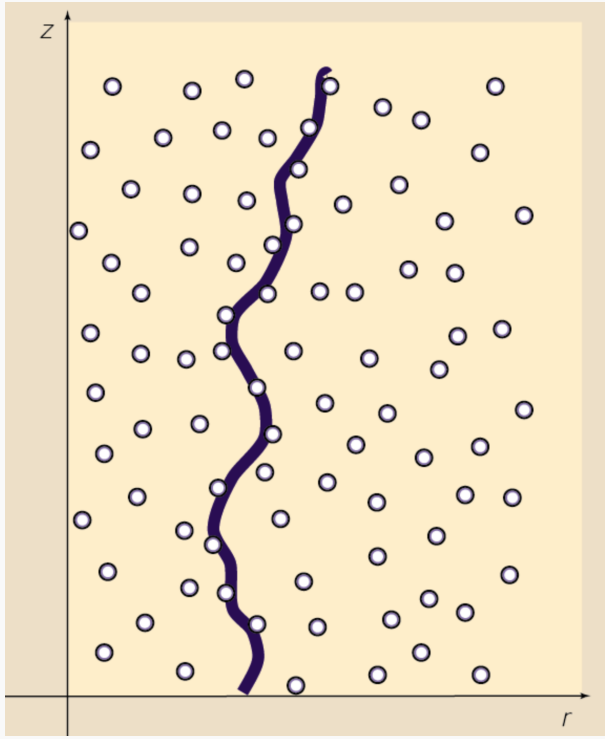
which in the math literature is sometimes referred to as the *Bellman equation* or *Dijkstra's algorithm*.

High temperature limit of the polymer

In the high temperature regime, $T \to \infty$, we obtain the continuum polymer using the mapping

$$\tilde{x} = \frac{4x_f}{T^2} \quad \tilde{t} = \frac{2L}{T^4}$$

According how you choose your end points, you obtain the solution of the stochastic heat equation for different initial conditions.



For a fixed end point x_f, $Z_{x_f,L} \to Z(\tilde{x}, \tilde{t})$, solution of the SHE for droplet initial condition (point to point polymer).

Stat. mech. problem at equilibrium at temperature T in the canonical ensemble $\Rightarrow$ out-of-equilibrium problem

The replica method

We define the n-th replica at equal time as

$$Z_n(x_1, \dots, x_n, t) = \mathbb{E}\left[\prod_{i=1}^{n} Z(x_i, t)\right]$$

where the average is taken over the KPZ white noise and initial condition (if random).

What do we call solving the KPZ equation ?

Taking the replica at equal position, we obtain the n-th moment of the partition function

$$Z_n(x, \dots, x, t) = \mathbb{E}\left[Z^n(x,t)\right]$$

Formally, we can define the generating function of Z as

$$\begin{aligned} g_t(z) &= 1 + \sum_{n=1}^{\infty} \frac{(-z)^n}{n!} \mathbb{E}\left[Z^n(x,t)\right] \\ &= \mathbb{E}\left[\exp\left(-zZ(x,t)\right)\right] \\ &= \mathbb{E}\left[\exp\left(-ze^{h(x,t)}\right)\right] \end{aligned}$$

From $g_t(z)$ one can in principle obtain the distribution of $h(x,t)$ by doing an inverse Laplace transform.

Lieb-Liniger equation I

Using Ito formalism, the n-th replica verifies an imaginary time Schrödinger equation

$$\partial_t Z_n = -\mathcal{H}_n Z_n$$

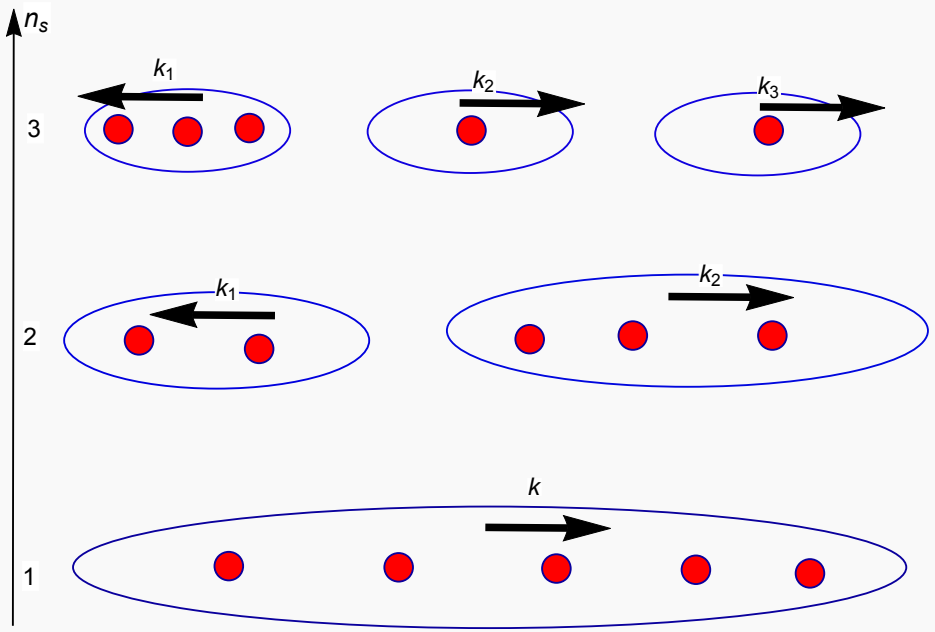
$$\mathcal{H}_n = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - 2 \sum_{1 \leq i < j \leq n} \delta(x_i - x_j)$$

This is called the attractive delta Bose gas model or attractive Lieb-Liniger model.

Because the interaction is attractive, particles want to form bound states, *clusters*. In momentum space, the structures formed are called *strings*.

Lieb-Liniger equation II

Each string has a certain number of particles and a momentum k_j.
Ground state ($n_s = 1$) is $\Psi(x_1, \ldots, x_n) \propto \exp(-\frac{1}{2}\sum_{i<j}|x_i - x_j|)$



Fixed number of particles $\rightarrow$ fixed number of strings

$$\mathbb{E}[Z^n] \rightarrow \tilde{Z}_{n_s}$$

Lieb-Liniger equation III

For some initial conditions

$$\tilde{Z}_{n_s} = \int \cdots \int \mathrm{d}v_1 \ldots \mathrm{d}v_{n_s} \det \left[\bar{K}_{t,z}(v_i, v_j) \right]_{i,j=1}^{n_s}$$

Then the moment generating function becomes

$$\begin{aligned} g_t(z) &= 1 + \sum_{n=1}^{\infty} \frac{(-z)^n}{n!} \mathbb{E}\left[Z^n(x,t) \right] \\ &\to 1 + \sum_{n_s=1}^{\infty} \frac{(-1)^{n_s}}{n_s!} \tilde{Z}_{n_s} \\ &= 1 + \sum_{n_s=1}^{\infty} \frac{(-1)^{n_s}}{n_s!} \int \mathrm{d}v_1 \ldots \int \mathrm{d}v_{n_s} \det \left[\bar{K}_{t,z}(v_i, v_j) \right]_{i,j=1}^{n_s} \\ &= \mathrm{Det}\left[I - \bar{K}_{t,z} \right] \Rightarrow \text{FREDHOLM DETERMINANT} \end{aligned}$$

Applications and evaluation of Fredholm determinants

Fredholm determinants appear in

- Random matrix theory
- One dimensional path integrals in field theory
- Partition functions in string theory
- Fermionic systems at finite temperature in quantum mechanics
- Determinantal process in probability
- ... the KPZ equation.

F. Bornemann *On the numerical evaluation of Fredholm determinants* Mathematics of Computation (2010)

Introduces a Matlab toolbox to evaluate Fredholm determinants.

Known results for all times

1. Full-space problem :
   - Droplet: Sasamoto, Spohn ('10), Calabrese, Le Doussal ('10), Dotsenko ('10), Amir, Corwin, Quastel ('10)
   - Flat: Calabrese, Le Doussal ('11)
   - Brownian: Imamura, Sasamoto ('12)

2. Half-space problem : only Droplet initial condition
   - $A = \infty$: Gueudré, Le Doussal ('12)
   - $A = 0$: Borodin, Bufetov, Corwin ('15)
   - $A = -\frac{1}{2}$: Barraquand, Borodin, Corwin, Wheeler ('17)

Example : droplet initial condition in full-space

Take $H(t) = h(0,t) - \langle h(0,t) \rangle$, the moment generating function of $e^{H(t)}$ is given by

$$\mathbb{E}_{\mathrm{KPZ}} \left[\exp\left(-z e^{H(t)} \right) \right] = \mathrm{Det}\left[I - \bar{K}_{t,z} \right]$$

The Fredholm determinant is associated to the kernel

$$\bar{K}_{t,z}(u,u') = \sigma_{t,z}(u) K_{\mathrm{Ai}}(u,u')$$

defined in terms of the Airy kernel and the weight function

$$K_{\mathrm{Ai}}(u,u') = \int_0^{+\infty} \mathrm{d}r \; \mathrm{Ai}(r+u)\mathrm{Ai}(r+u')$$

$$\sigma_{t,z}(u) = \frac{z}{z + e^{-t^{1/3}u}}$$

Determinantal point process : quantum mechanics 101

Take a simple problem in quantum mechanics where you consider n fermionic particles. If the eigenfunctions of the one particle Hamiltonian are $\psi_j(x)$ for $j \geq 1$, then the ground state of the n-particles problem is

$$\Psi(x_1, \ldots, x_n) = \frac{1}{\sqrt{n!}} \det \left(\psi_i(x_j)\right)_{i,j=1}^{n}$$

The probability density is then

$$|\Psi(x_1, \ldots, x_n)|^2 = \frac{1}{n!} \det \left[K(x_i, x_j)\right]_{i,j=1}^{n}$$

where

$$K(x, y) = \sum_{j=1}^{n} \psi_j(x) \psi_j^*(y)$$

Note the determinantal structure!

Duality between KPZ and determinantal point process

From the theory of determinantal processes

$$\mathbb{E}_K\left[\prod_{i=1}^{n}\left[1-\sigma(x_i)\right]\right]=\mathrm{Det}\left[1-\sigma K\right]$$

Specifying to $\sigma=\sigma_{t,z}$ and $K=K_{\mathrm{Ai}}$, we have

$$\mathbb{E}_{\mathrm{KPZ}}\left[\exp\left(-ze^{H(t)}\right)\right]=\mathbb{E}_{K_{\mathrm{Ai}}}\left[\prod_{i=1}^{\infty}\left[1-\sigma_{t,z}(x_i)\right]\right]$$

This duality is at the core of the link between the KPZ equation, fermions in quantum mechanics. The eigenvalues of random matrices also form a determinantal process, so there is also a connexion.

Known results for short times

Take $H(t) = h(0,t) - \langle h(0,t) \rangle$, then for all known initial conditions, the probability density function is given by a Large Deviation Principle at short time $t \ll 1$

$$P(H,t) \sim \exp(-\frac{\Phi(H)}{\sqrt{t}})$$

where the rate function Φ has the universal properties

$$\Phi(H) \simeq \begin{cases} c_- |H|^{5/2}, H \to -\infty \\ c_2 H^2, \quad |H| \ll 1 \\ c_+ H^{3/2}, \ H \to +\infty \end{cases}$$

the coefficients c_-, c_2, c_+ depend on the initial condition.

Cumulant expansion

- For a random variable X

$$\log \mathbb{E}[e^{zX}] = \sum_{n=1}^{\infty} \kappa_n \frac{z^n}{n!} \qquad \kappa_1 = \mathbb{E}[X],\ \kappa_2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

- For a determinantal point process $\{a_i\}_{i\in\mathbb{N}}$ with kernel K

$$\log \mathbb{E}_K[e^{-\lambda \sum_{i=1}^{\infty} \varphi(x_i)}] = \sum_{n=1}^{\infty} \kappa_n(\varphi) \frac{t^n}{n!}$$

$$\kappa_1(\varphi) = -\mathrm{Tr}(\varphi K), \qquad \kappa_2(\varphi) = \mathrm{Tr}(\varphi^2 K) - \mathrm{Tr}(\varphi K \varphi K)$$

where the trace is defined as $\mathrm{Tr}(\varphi K) = \int \mathrm{d}v\, \varphi(v) K(v, v)$

First cumulant approximation

At short time, we find for all studied initial conditions

$$\kappa_1(\varphi) = \mathcal{O}(\frac{1}{\sqrt{t}}), \quad \kappa_2(\varphi) = \mathcal{O}(1), \quad \dots$$

The higher the cumulant, the higher the power in time, therefore we truncate the sum at the first cumulant

$$\log \mathbb{E}_K[e^{-\sum_{i=1}^{\infty} \varphi(x_i)}] \underset{t \ll 1}{\simeq} \kappa_1(\varphi)$$

For example, for droplet initial condition,

$$\kappa_1(\varphi) = -\frac{\mathrm{Li}_{5/2}(-z)}{\sqrt{4\pi t}} + \mathcal{O}(1)$$

Final expression of the rate function

The rate function of the droplet IC is given by

- For $H \leq H_c = \log \zeta(\frac{3}{2})$

$$\Phi(H) = -\frac{1}{\sqrt{4\pi}} \min_{z \in [-1,+\infty[} [z e^H + \mathrm{Li}_{5/2}(-z)]$$

- for $H \geq H_c$

$$\Phi(H) = -\frac{1}{\sqrt{4\pi}} \min_{z \in [-1,0[} [z e^H + \mathrm{Li}_{5/2}(-z) - \frac{8\sqrt{\pi}}{3} [-\log(-z)]^{3/2}]$$

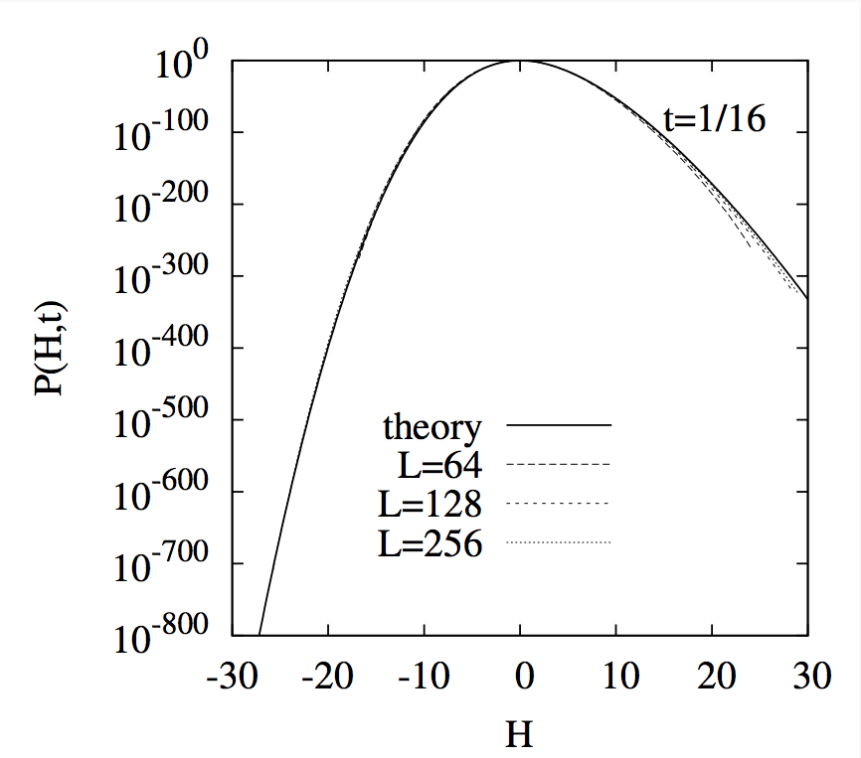
$\Phi(H)$ is analytic, the left tail is $\Phi(H) \simeq_{H \to -\infty} \frac{4}{15\pi} |H|^{5/2}$ and the right tail is $\Phi(H) \simeq_{H \to +\infty} \frac{4}{3} H^{3/2}$.

Recent progress on numerics for droplet IC

A. K. Hartmann, P. Le Doussal, S. N. Majumdar, A. Rosso, G. Schehr, *High-precision simulation of the height distribution for the KPZ equation* arXiv:1802.02106

Simulation of a directed polymer on a square lattice of size L at temperature $T^4 = \frac{2L}{t}$ using *importance sampling*.

$$H(x = 0, t) = \log(\frac{Z_{0,L}}{\bar{Z}})$$



Recent progress on numerics for droplet IC

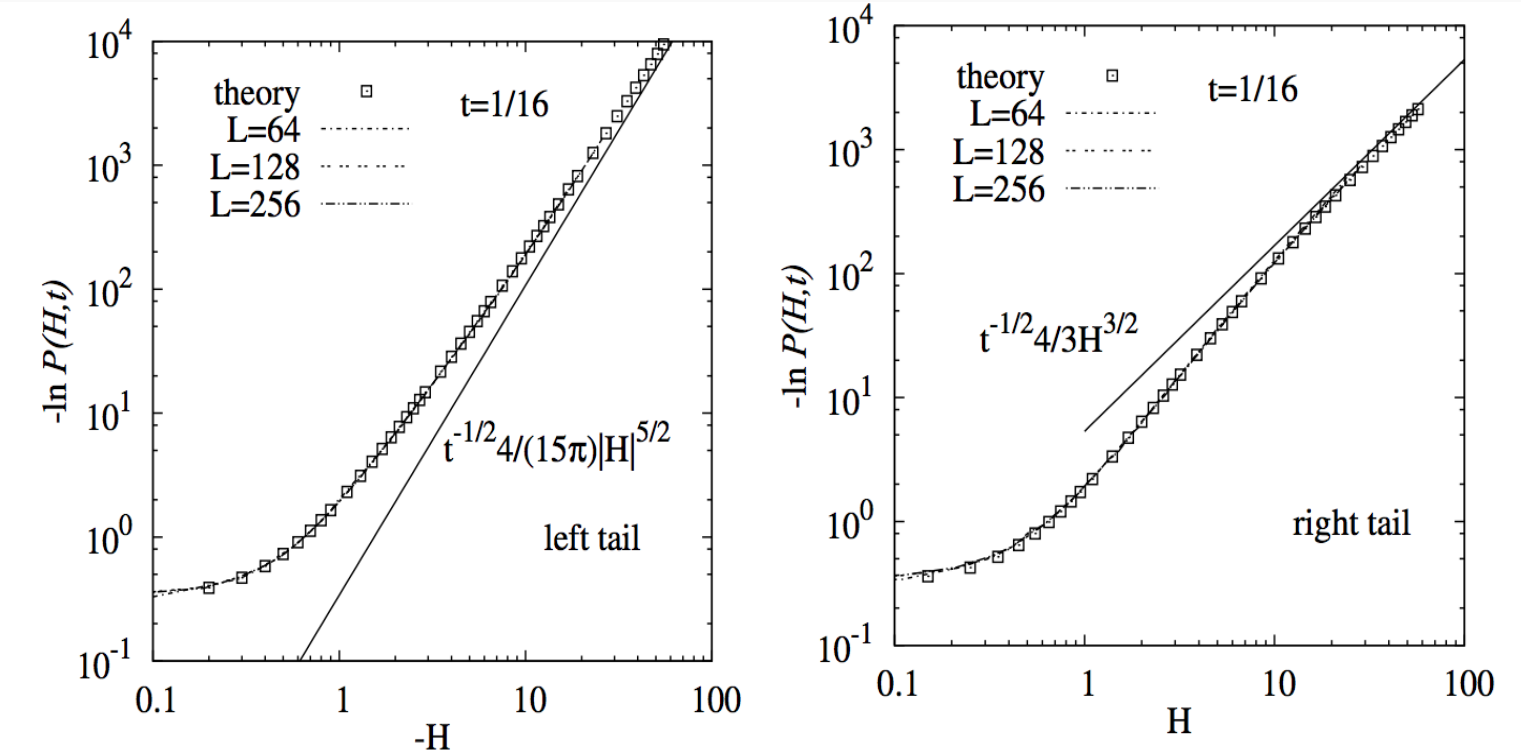


Figure: Details of the left and right tails compared to the analytical prediction at short time.

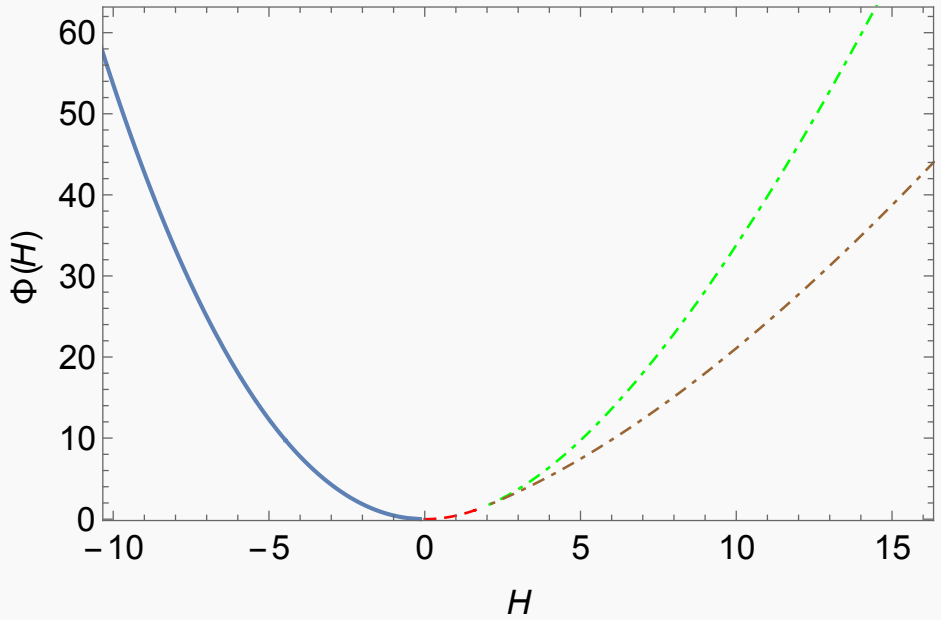
Exact short-time height distribution for the Brownian IC

The distribution of $H(t)$ at $x = 0$, time t is given by

$$P(H,t) \sim \exp\left(-\frac{\Phi(H)}{\sqrt{t}}\right)$$

$$\Phi(H) \simeq \begin{cases} \frac{4}{15\pi}|H|^{5/2}, H \to -\infty \\ \frac{\sqrt{\pi}}{4}H^2, \ |H| \ll 1 \\ c_+ H^{3/2}, H \to +\infty \end{cases}$$

where $c_+ = \frac{4}{3}$ (resp. $c_+ = \frac{2}{3}$) for the analytic (resp. non-analytic) branch.



$H_c = 2\ln(2e - \mathcal{I}) - 1$, with
$\mathcal{I} = \frac{1}{\pi}\int_0^\infty \mathrm{d}y \left[1 + \frac{1}{y}\right] \frac{\sqrt{y}}{e^{-1}+ye^y}$

Note that for the flat:
$\Phi_{\mathrm{flat}}(H) = 2^{-3/2}\Phi_{\mathrm{brown-analytic}}(2H)$

Singularity and dynamical phase transition

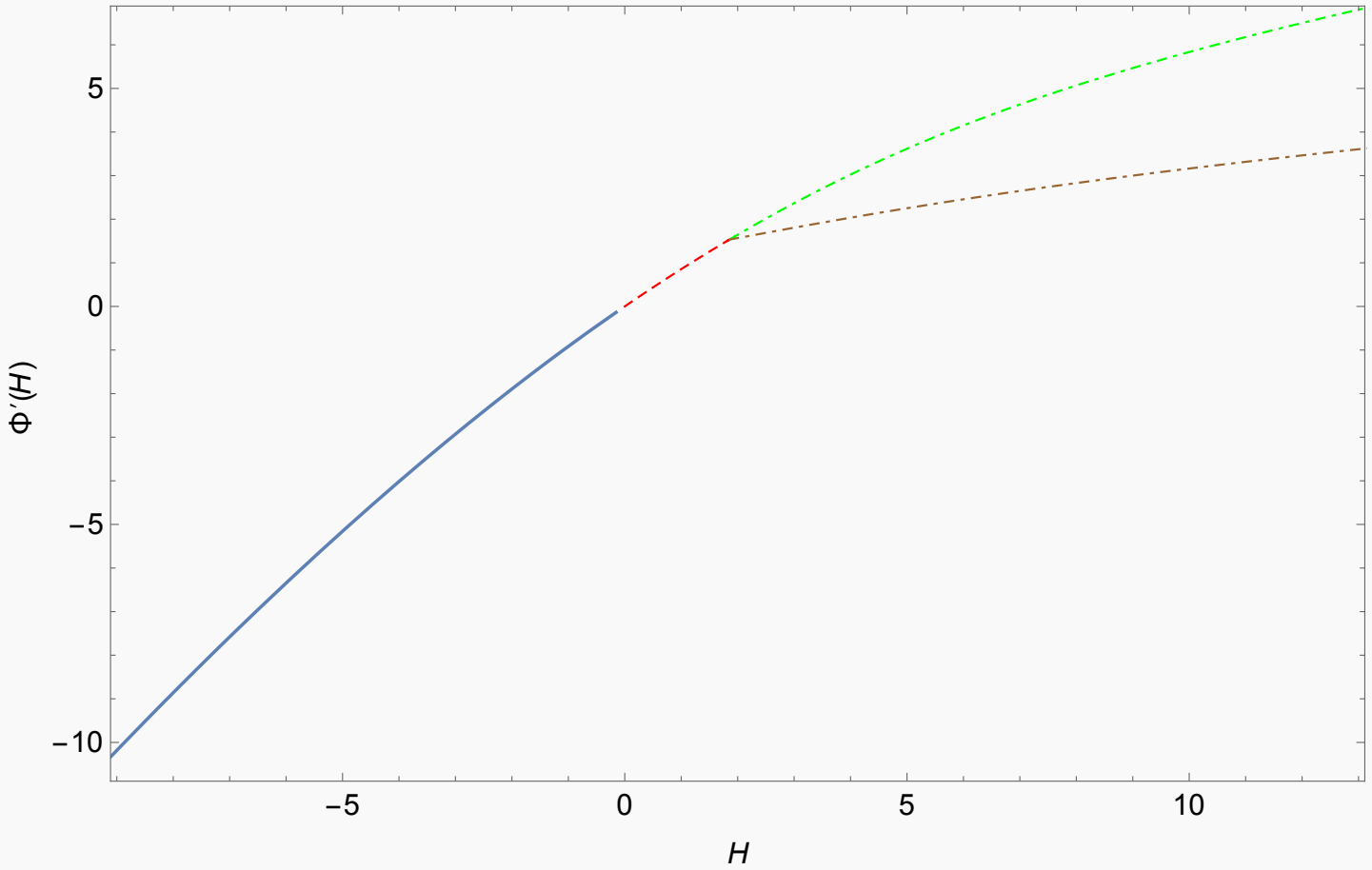


Figure: The function $\Phi'(H)$. The blue line corresponds to the $H < 0$ solution, the red line to the first continuation for $0 < H < H_c$, the green line to the analytic branch $H_c < H$ and the brown line to the non-analytic branch for $H_c < H$. Note the singularity for the brown line.

Bonus

Physics motivation - experiment on chemical reaction fronts

S. Atis, A. K. Dubey, D. Salin, L. Talon, P. Le Doussal, K. J. Wiese, *Experimental Evidence for Three Universality Classes for Reaction Fronts in Disordered Flows* Phys. Rev. Lett. (2015)

(videos on the PRL webpage)

Physics motivation - experiment on chemical reaction fronts

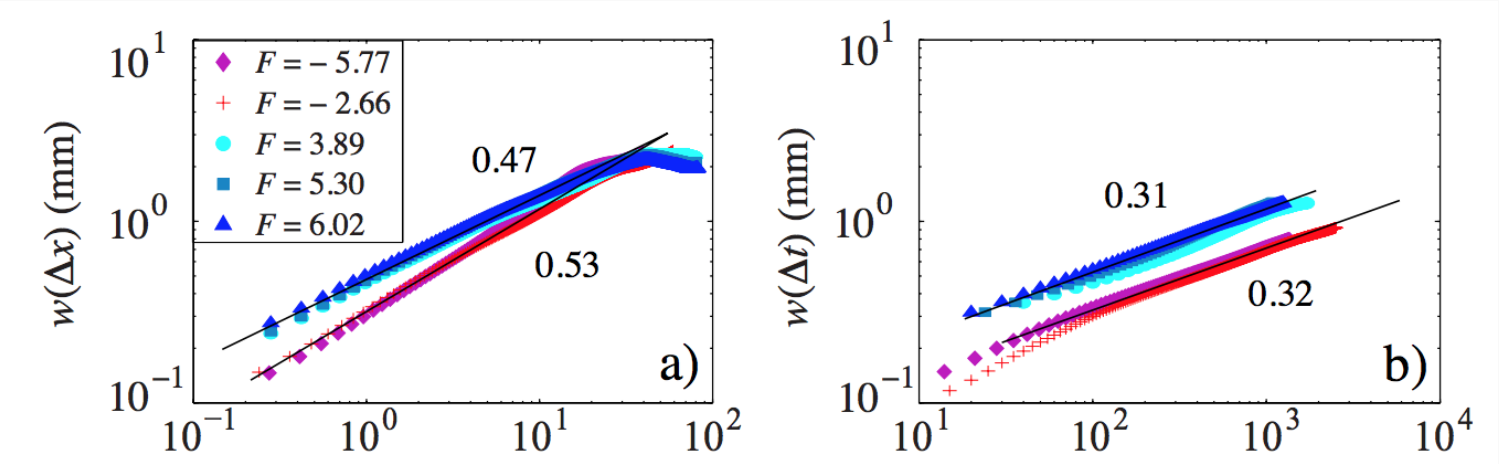


Figure: For the above video, $F = 4$

Define the temporal fluctuations
$w(\Delta t) = \sqrt{\langle [h(x,t) - \langle h \rangle]^2 \rangle} \propto \Delta t^{1/3}$

Similarly, take the roughness
$w(\Delta x) = \sqrt{\langle [h(x + \Delta x, t) - h(x,t)]^2 \rangle} \propto \Delta x^{1/2}$

How to obtain the cumulant expansion ?

$$\log \mathbb{E}_K[e^{-\lambda \sum_{i=1}^{\infty} \varphi(x_i)}] = \log \mathrm{Det}[I - (1 - e^{-\lambda\varphi})K]$$

$$= \mathrm{Tr} \log[I - (1 - e^{-\lambda\varphi})K]$$

Expand the logarithm and the exponential as a series.

The n-th cumulant $\kappa_n(\varphi)$ is defined as $n!$ times the term of order λ^n in this expansion.