Saarland
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SFB 1027

Non-Markovian and Collective Search Strategies

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Overview:

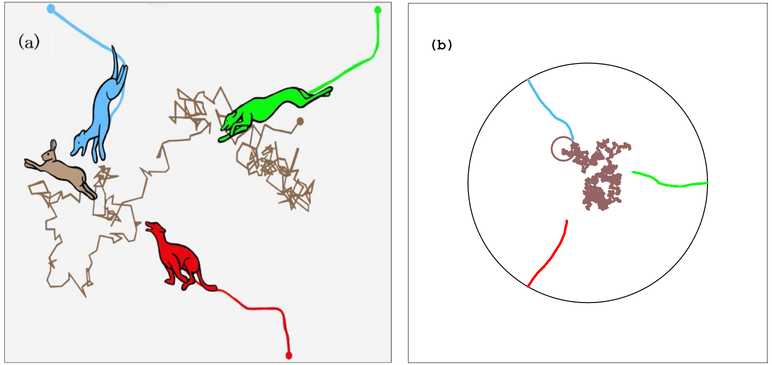
The Target Problem in Nature

The “target problem” in Nature

- Single searcher (stochastic), single target (immobile or stochastic) – typical FPT problem
- Group search: many searchers (stochastic), single target – independent / collective
- Group hunting: many searchers, single target – both with visibility
- Evasion strategies (predator-prey): single searcher, many targets – both with visibility
- Foraging: many searcher (communicating), many targets (immobile)

Group Hunting

Hunters see target:



$$\frac{d\boldsymbol{X}}{dt} = \sqrt{2D}\boldsymbol{\xi}(t); \qquad \frac{d\boldsymbol{Y}_n}{dt} = v_0 \frac{\boldsymbol{X}-\boldsymbol{Y}_n}{\|\boldsymbol{X}-\boldsymbol{Y}_n\|},$$

[Bernardi etal, PRL 128, 040601 (2022)]

+ Target sees hunter:

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \sqrt{2D}\boldsymbol{\xi}(t) + \boldsymbol{E}, \qquad \boldsymbol{E} = \frac{v_\mathrm{e}}{N}\sum_{n=1}^{N} g(r_n)\frac{\boldsymbol{X}-\boldsymbol{Y}_n}{\|\boldsymbol{X}-\boldsymbol{Y}_n\|}.$$

[Meng etal, NJP 25, 023033 (2023)]

Chasing a faster prey:

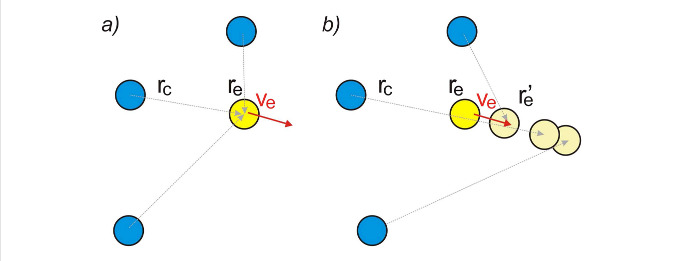


Figure 4. The chasers (blue) are chasing (a) the current position of the escaper (yellow) and (b) its position predicted according to (11).

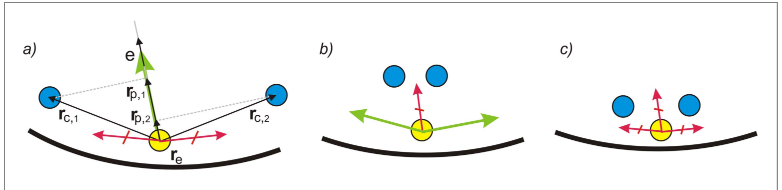
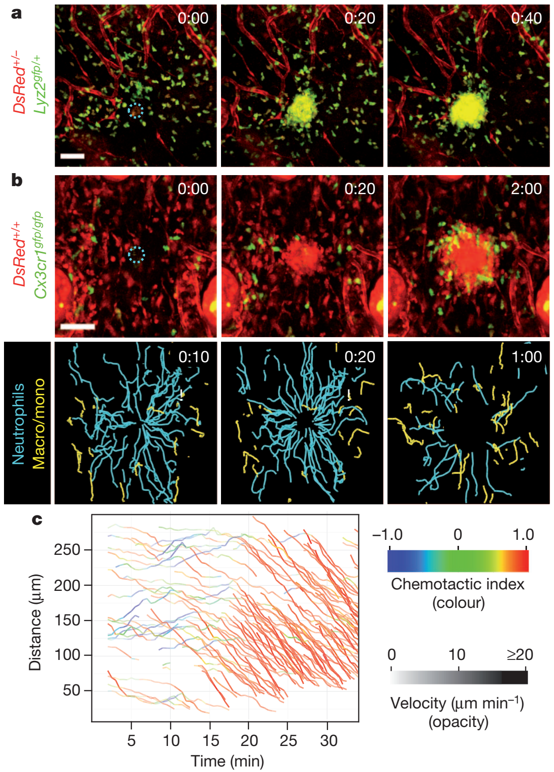


Figure 5. (a) The escaper (yellow) can slip through the gap (according to (26)) between the chasers (blue), therefore it escapes (green arrow), but if it stayed at the wall it would get caught (red arrows). (b) The escaper cannot slip through the gap between the chasers but it can escape by aligning to the wall. (c) The chasers are blocking all the possible escape directions.

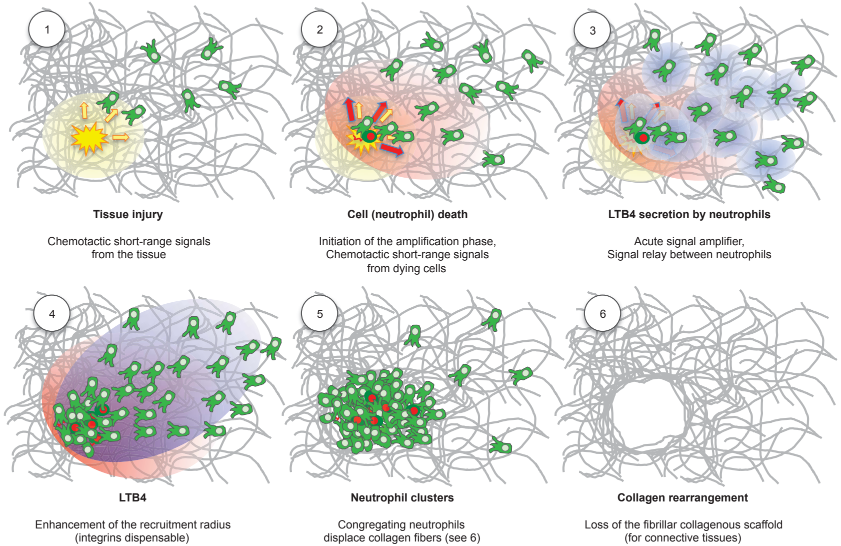
etc.

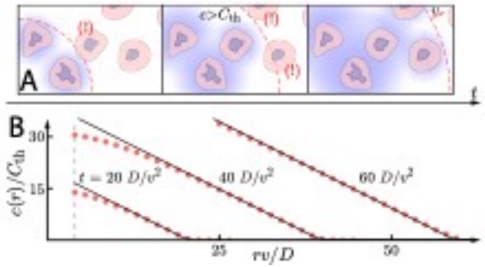
[Janosov etal, NJP 19, 053003 (2017)]

Neutrophil Swarming: Signaling Relay



[Lämmermann etal, Nature 498, 371 (2013)]



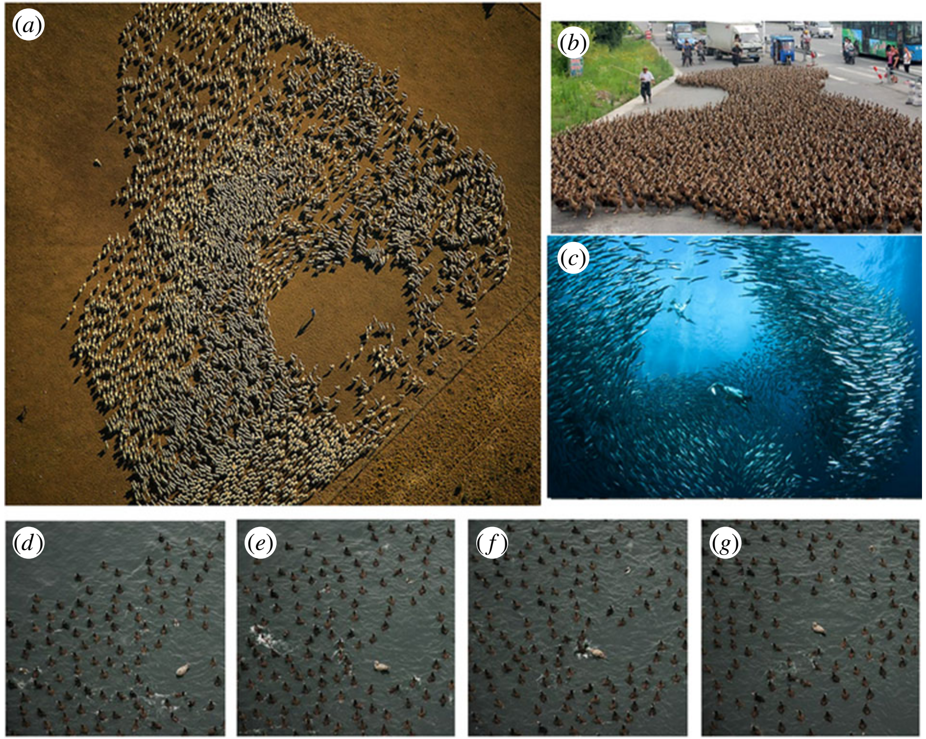


Models:

$$\frac{\partial c}{\partial t} = D\nabla^2 c + a\rho\delta(z)\Theta[c - C_{\rm th}]$$

[Dieterle etal, elife 9:e61771 (2020)]

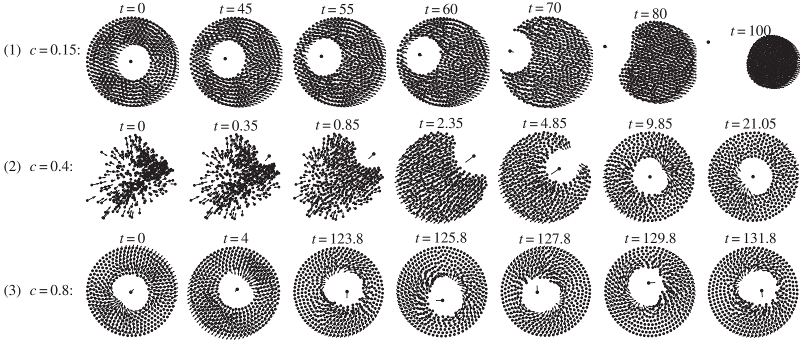
Evasion strategies (predator-swarm interaction)



Model:

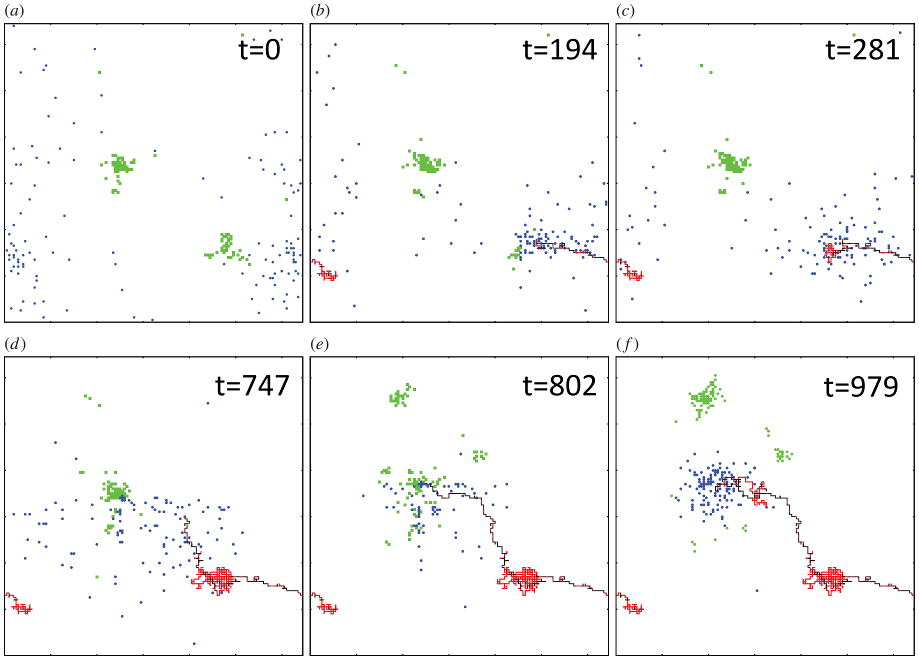
$$\frac{\mathrm{d}x_j}{\mathrm{d}t} = \frac{1}{N} \sum_{k=1, k \neq j}^{N} \left(\frac{x_j - x_k}{|x_j - x_k|^2} - a(x_j - x_k) \right) + b \frac{x_j - z}{|x_j - z|^2}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{c}{N} \sum_{k=1}^{N} \frac{x_k - z}{|x_k - z|^p}.$$



[Chen, Kolokolnikov: J. Royal Soc. Int. 11, 20131208 (2014)]

Collective Foraging (by sharing information)



foragers: blue; targets: green; example trajectory: red

[Bhattacharya et al, J. Royal Soc. Int. 11, 20140674 (2014)]

Continuum model::

$$\dot{\mathbf{r}}_i(t) = B_g \nabla g(\mathbf{r}_i) + B_C \nabla S(\mathbf{r}_i) + \eta_i(t),$$

g(r) environmental quality function

$$S(\mathbf{r}_i) = \sum_{j=1, j\neq i}^{N} A(g(\mathbf{r}_j)) \frac{\exp\left(-\frac{(\mathbf{r}_i - \mathbf{r}_j)^2}{2\sigma^2}\right)}{2\pi\sigma^2}$$

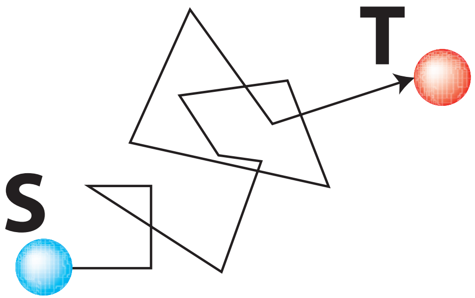
$$A(g(\mathbf{r})) = \Theta(g(\mathbf{r}) - \bar{\kappa})$$

-> Application to Mongolian Gazelles

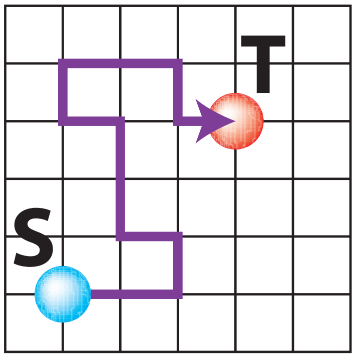
[Martinez-Garcia et al, PRL 110, 248106 (2013)]

Non-Markovian Search Strategies

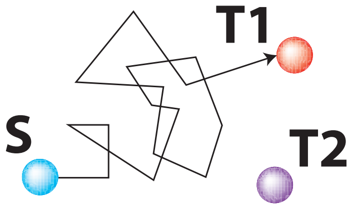
First passage problems



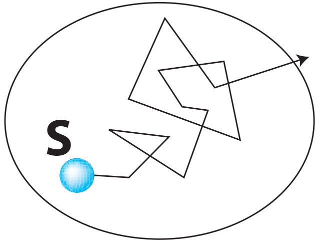
Scheme of a generic first-passage problem



Scheme of a discrete first-passage problem



Scheme of a two target first-passage problem



Scheme of a first-exit problem

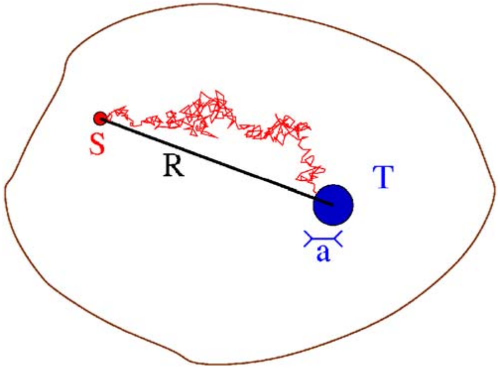
FPT(t) = probability density for the first-passage time, i.e the first time the random walker hits a target (start at position r_s, target at position r_t)

MFPT of Brownian motion in confined geometries

1d $\langle \mathbf{T}(\mathbf{r}_s) \rangle \sim \frac{L^2}{D}$

2d $\langle \mathbf{T}(\mathbf{r}_S) \rangle = \frac{A}{2\pi D} \ln \frac{R}{a}$

3d $\langle \mathbf{T}(\mathbf{r}_S) \rangle = \frac{V}{4\pi D} \left(\frac{1}{a} - \frac{1}{R} \right)$



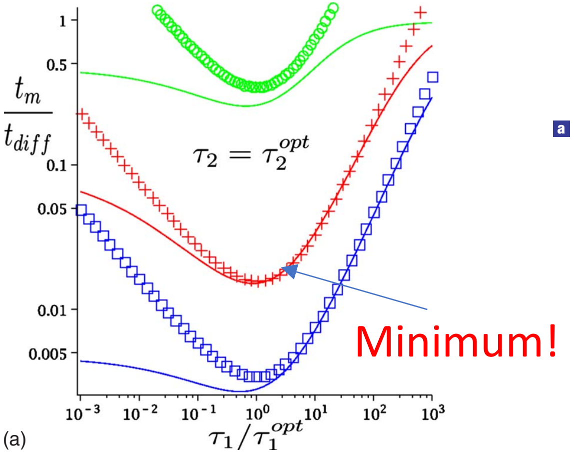
L, A, V system size (in 1,2,3d)
R initial searcher-target distance
a target size
D diffusion constant

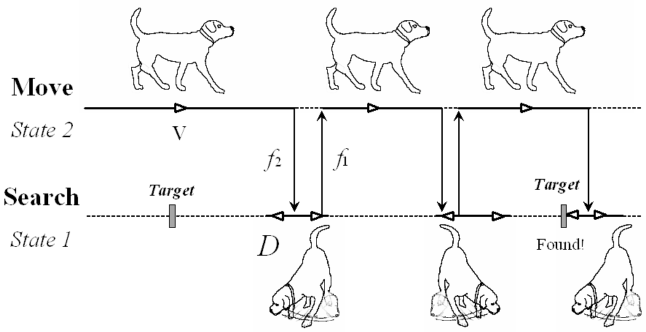
[Condamin etal, PRL 2005]

Optimizing search strategies

A **search strategy** is simply one parameter set
for the of a stochastic process / random walk under consideration

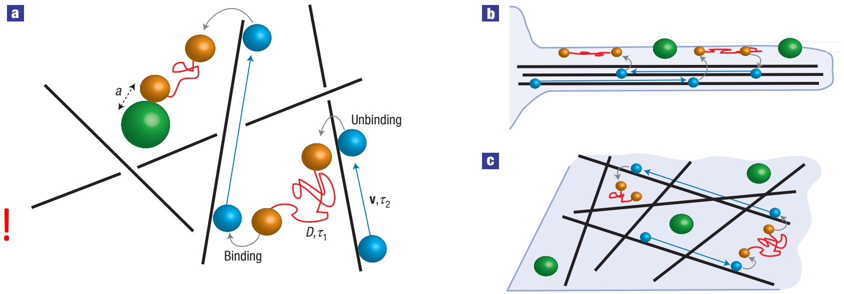
Example:
Intermittent search





Walker / searcher alternates stochastically between ballistic and diffusive behavior

[Bénichou etal, PRL 2005]



[Loverdo etal, Nat.Phys. 2008]

Optimizing search strategies (2)

Example: Persistent Walk

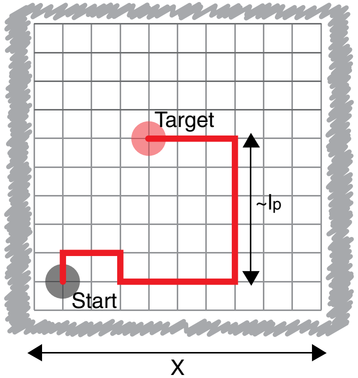
Probability

p_1 forward = $p_3+\varepsilon$

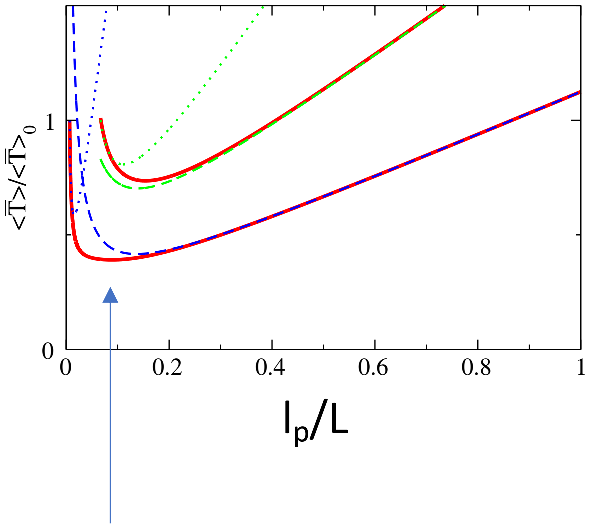
p_2 backward = $p_3-\delta$ ($\delta=0$)

p_3 orthogonal

⇒ persistence length $l_p \sim 1/(1-\varepsilon)$



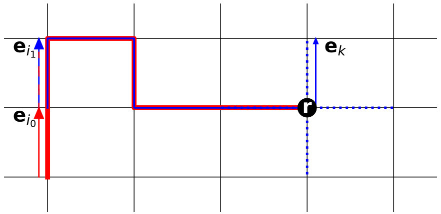
MFPT vs l_p



Minimum!

[Tejdor etal, PRL 2012]

Non-Markovian random walks with n-step memory



Conditional transition probabilities: $p(\mathbf{e}_k|\mathbf{e}_{i_0},\dots,\mathbf{e}_{i_{n-1}})$

$T_n(\mathbf{r},\mathbf{r}_T;\mathbf{e}_{i_0},\dots,\mathbf{e}_{i_{n-1}})$ = average first passage time to reach the target at position r_T with n last directions $\{\mathbf{e}_{i_0},\dots,\mathbf{e}_{i_{n-1}}\}$

Backward equation for T_n:

$$T_n(\mathbf{r},\mathbf{r}_T;\mathbf{e}_{i_0},\dots,\mathbf{e}_{i_{n-1}}) = 1 + \sum_k p(\mathbf{e}_k|\mathbf{e}_{i_0},\dots,\mathbf{e}_{i_{n-1}})\, T_n(\mathbf{r}+\mathbf{e}_k,\mathbf{r}_T;\mathbf{e}_{i_1},\dots,\mathbf{e}_{i_{n-1}},\mathbf{e}_k)$$

Solution:

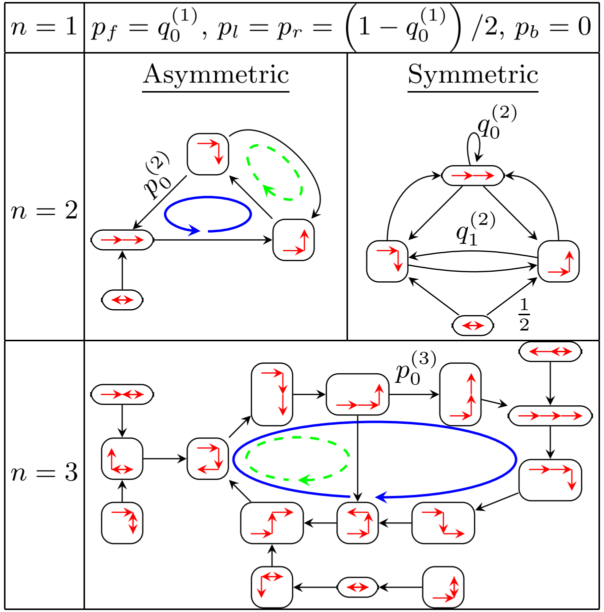
$$\tilde{t}_{n\alpha}(\mathbf{q},\mathbf{r}_T) = T_n(\mathbf{q},\mathbf{r}_T;s_{n\alpha})$$

$$\tilde{\mathbf{t}}_n(\mathbf{q},\mathbf{r}_T) = V[\delta(\mathbf{q}) - e^{-i\mathbf{q}\cdot\mathbf{r}_T}][\mathbb{I} - \mathbf{P}_n\mathbf{E}_n(\mathbf{q})]^{-1}\mathbf{u}_n$$

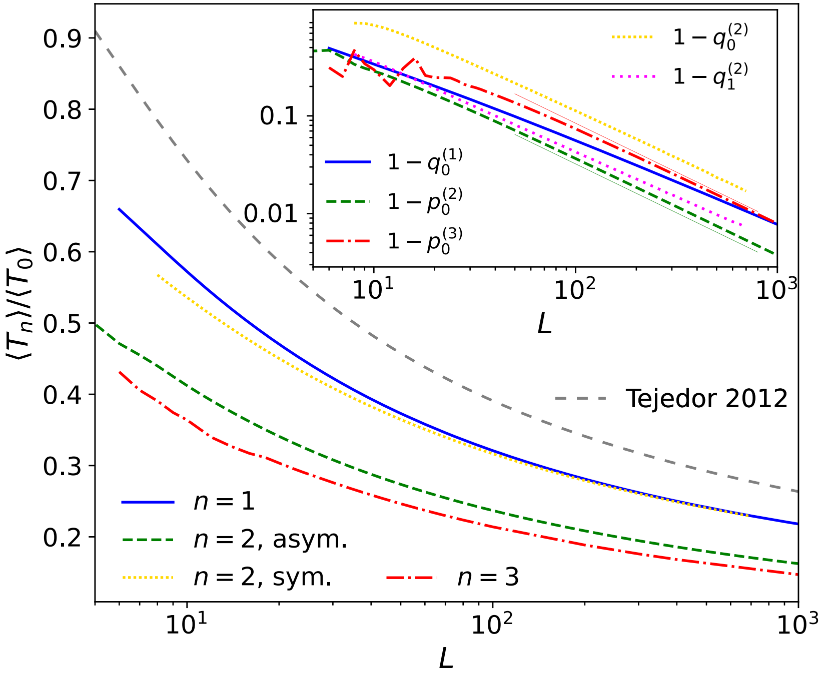
$$\langle \mathbf{t}_n \rangle = \sum_{\mathbf{q}\neq 0}[\mathbb{I} - \mathbf{P}_n\mathbf{E}_n(\mathbf{q})]^{-1}\mathbf{u}_n.$$

[Meyer, HR: PRL 2021]

Optimal search strategies with n-step memory



Optimal search strategies on a 2d square lattice for n = 1, 2, 3



The corresponding MFPT normalized by the MFPT for a blind random walk

[Meyer, HR: PRL 2021]

The auto-chemotactic searcher

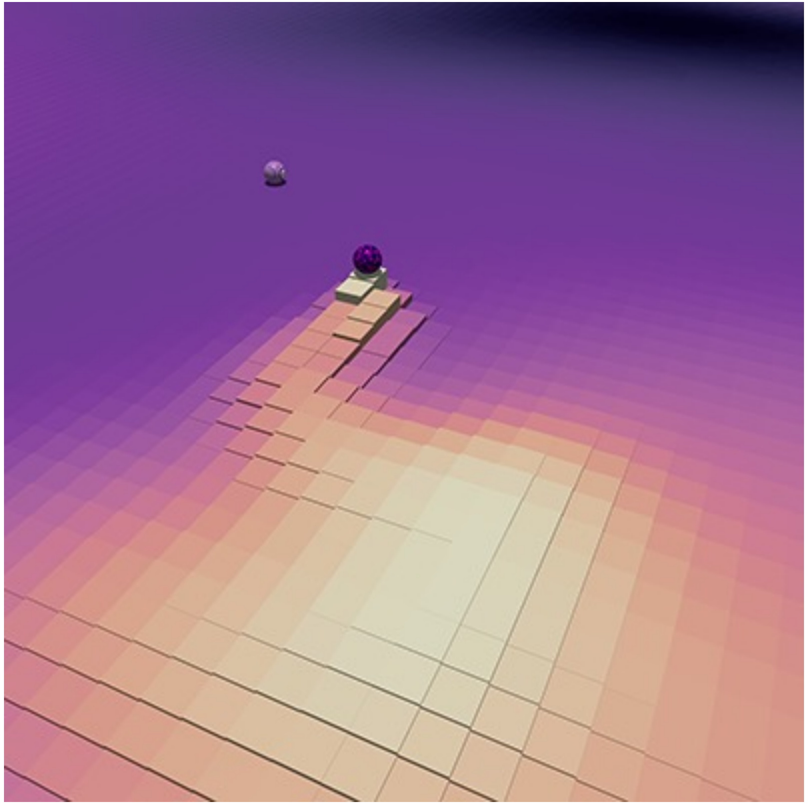
Philosophy:
try to avoid sites already visited!

Searcher releases chemo-repulsive **diffusive** cue c_i

Transition probabilities:

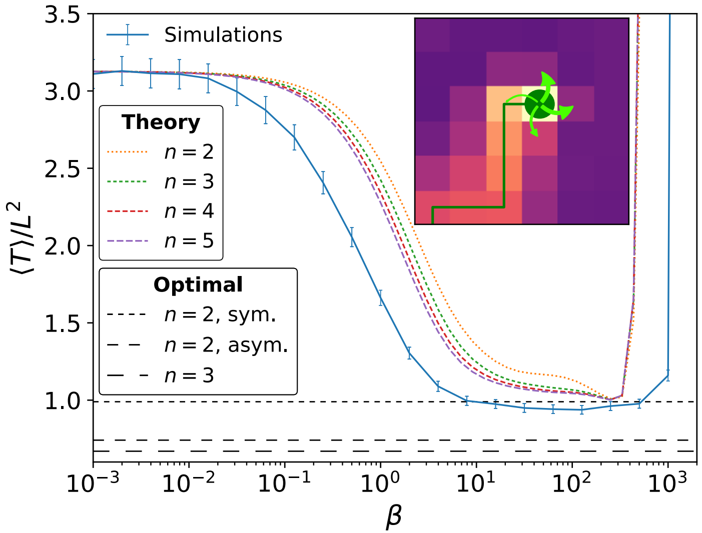
$$p_{i\to j} = \left[1 + \sum_{k\neq j} \exp\left[-\beta(c_k - c_j)\right]\right]^{-1}$$

$\beta \to 0$: conventional random walk
$\beta \to \infty$: jump to site with lowest c_i

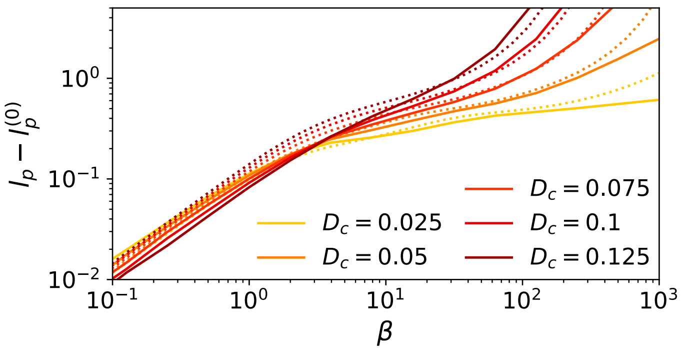


[Meyer, HR: PRL 2021]

Optimal search strategy f. auto-chemotactic searcher



MFPT of the auto-chemotactic searcher as function of β and comparison with prediction for n-step memory



Persistence length of the auto-chemotactic searcher as function of β and

[Meyer, HR: PRL 2021]

Collective Search:

Chemotactic Walker

N searcher / M targets

Consider confined 2d space, area A:

N independent, non-communicating random walkers, 1 target:

$FPT_1(t) \sim e^{-t/\tau} \Longrightarrow MFPT_1 \sim \tau \Longrightarrow MFPT_N \sim \tau/N$

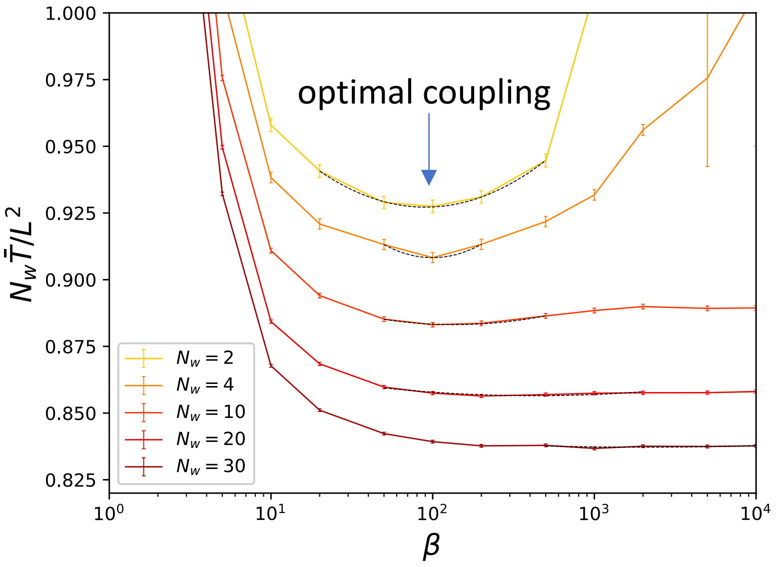
1 random walker, M randomly distributed targets:

$MFPT_{\text{first target}} \sim 1/M^{\beta}$, with $\beta \sim 1$.

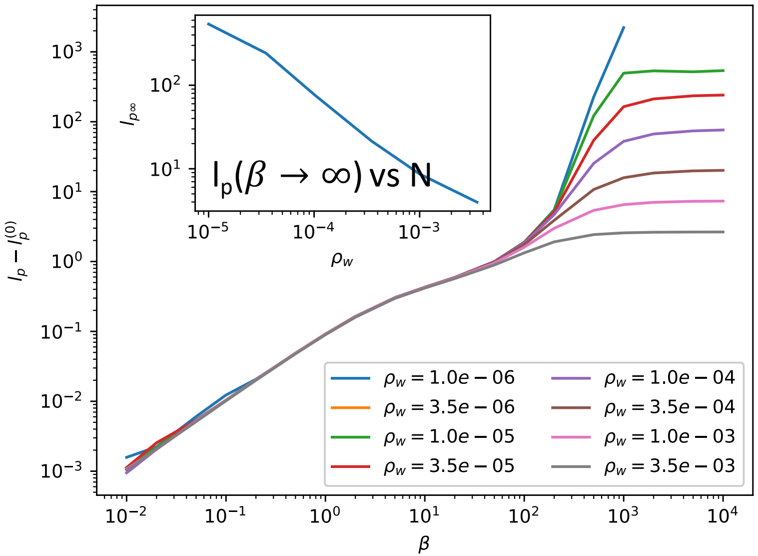
$MFPT_{\text{all targets}} \sim$ cover time(A) for M/A -> 1.

N chemotactic searchers, 1 target

Optimal coupling β to chemotactic field c



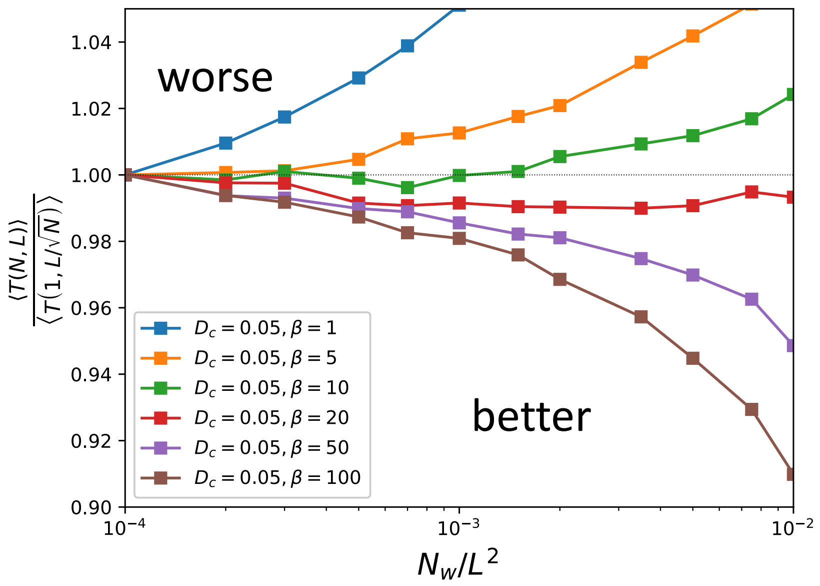
Persistence length vs. coupling β



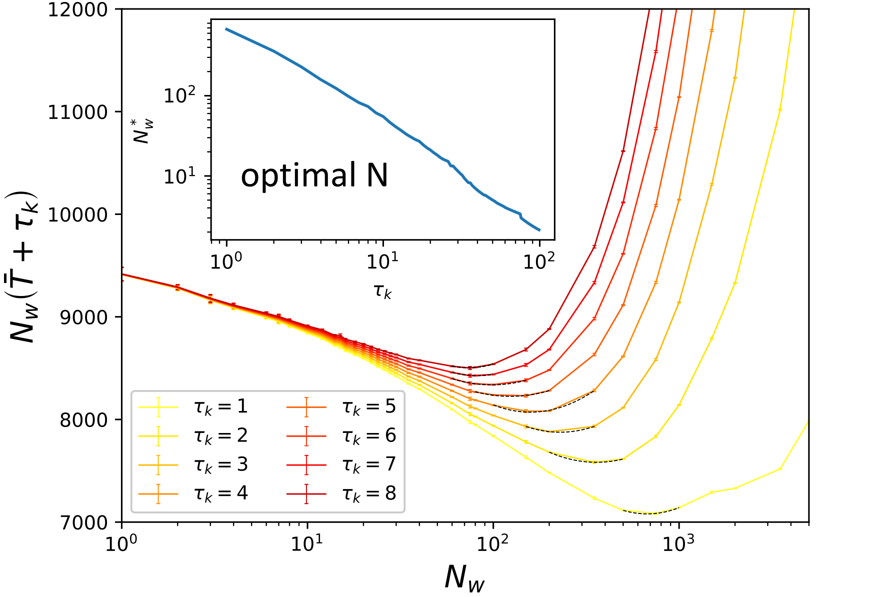
[Meyer, HR: t.b.p]

N chemotactic searchers, 1 target

Compare MFPT(N,L^2) (N searcher in area L^2) with MFPT(1,L^2/N) (1 searcher in area (L^2/N))



Cost for N-walker search:
N*MFPT + N*τ_k, τ_k=cost for 1 searcher



[Meyer, HR: t.b.p]

Optimal Searcher Number in a Costly Search

Total search cost: $$\mathcal{K} = J_T\bar{T} + J_N\bar{\mathcal{T}} + K_N\bar{N}$$

$\bar{T}$ mean search time (MFPT)
$\bar{\mathfrak{T}}$ sum of the times spent by all searchers
$\bar{N}$ number of searcher created

J_T target cost rate
J_N searcher support cost
K_N searcher creation cost

$\gamma = J_N/J_T$, $\beta = K_N/J_T$

searchers introduced at $t_1, t_2, \ldots, t_n, \ldots$.

Independent searcher result:

$$\mathcal{K} = \sum_{n=1}^{\infty}\left[(1+n\gamma)\int_{t_n}^{t_{n+1}} S_n(t)dt + \kappa S_n(t_n)\right]$$

$$S_n(t) = \prod_{k=1}^{n} s(t-t_k)$$

s(t) survival probability $s(t) = \int_t^{\infty}\rho(t)dt$

ρ(t) FPT distribution

Exponential survival probability
N_{opt} searcher must be introduced all at once

Algebraic survival probability
N_{opt} searcher must be introduced in fixed time intervals

[Meyer, HR, t.b.p.]

Collective Search:

Chemotactic ABPs

N active Brownian particles (ABPs), M targets

ABPs:

Disk-like particles with radius a

Motion with velocity v_0 along $\mathbf{e}_i = \begin{pmatrix} \cos\varphi_i \\ \sin\varphi_i \end{pmatrix}$

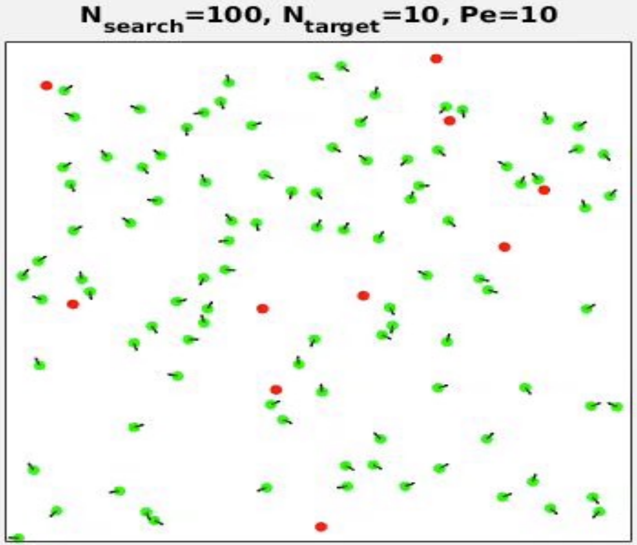
Position: $\dot{\mathbf{r}}_i(t) = V_0 \mathbf{e}_i(t) + \frac{1}{\gamma_t} \sum_{j \neq i} \mathbf{f}_{ji}$

excluded volume

Orientation:

$$\dot{\varphi}_i(t) = \sqrt{2D_r}\,\eta_i(t)$$

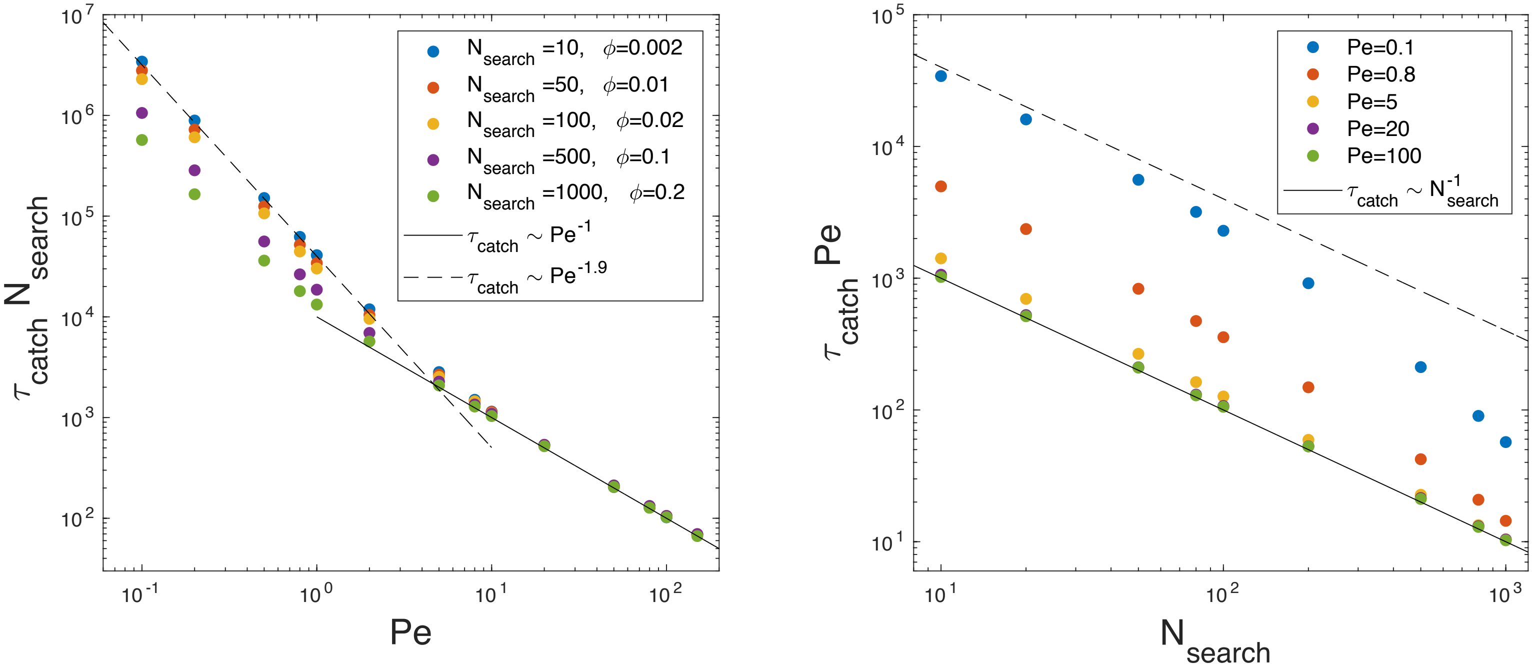
Gaussian white noise



Péclét number: $\mathrm{Pe} = \frac{V_0}{D_r a} = \frac{\text{persistence length}}{\text{particle radius}}$

[Wysocki, HR: t.b.p]

How long does it take (for ABPs) to catch **all** targets?



$$\mathrm{Pe} = \frac{V_0}{D_r a} = \frac{\text{persistence length}}{\text{particle radius}}$$

Diffusive motion: $\mathrm{Pe} \ll 1$

Ballistic motion: $\mathrm{Pe} \gg 1$

[Wysocki, HR: t.b.p]

N chemotactic active Brownian particles (ABPs), M targets

Position:

$$\dot{\mathbf{r}}_i(t) = V_0 \mathbf{e}_i(t) + \frac{1}{\gamma_t}\sum_{j\neq i} \mathbf{f}_{ji} \quad \text{with} \quad \mathbf{e}_i = \begin{pmatrix} \cos\varphi_i \\ \sin\varphi_i \end{pmatrix}$$

Orientation:

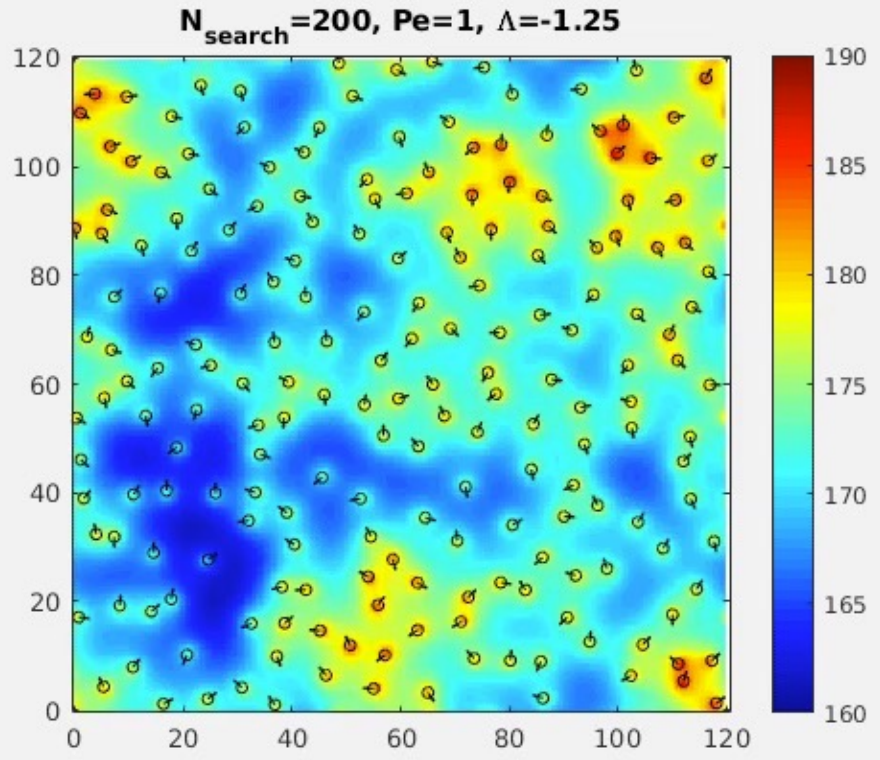
$$\dot{\varphi}_i(t) = \frac{1}{\gamma_r}\mathbf{e}_i(t) \times \kappa \nabla c(\mathbf{r}_i(t), t) + \sqrt{2D_r}\eta_i(t)$$

$\kappa < 0$ *align antiparallel to the gradient **(chemorepulsion)***

$\kappa > 0$ *align parallel to the gradient **(chemoattraction)***

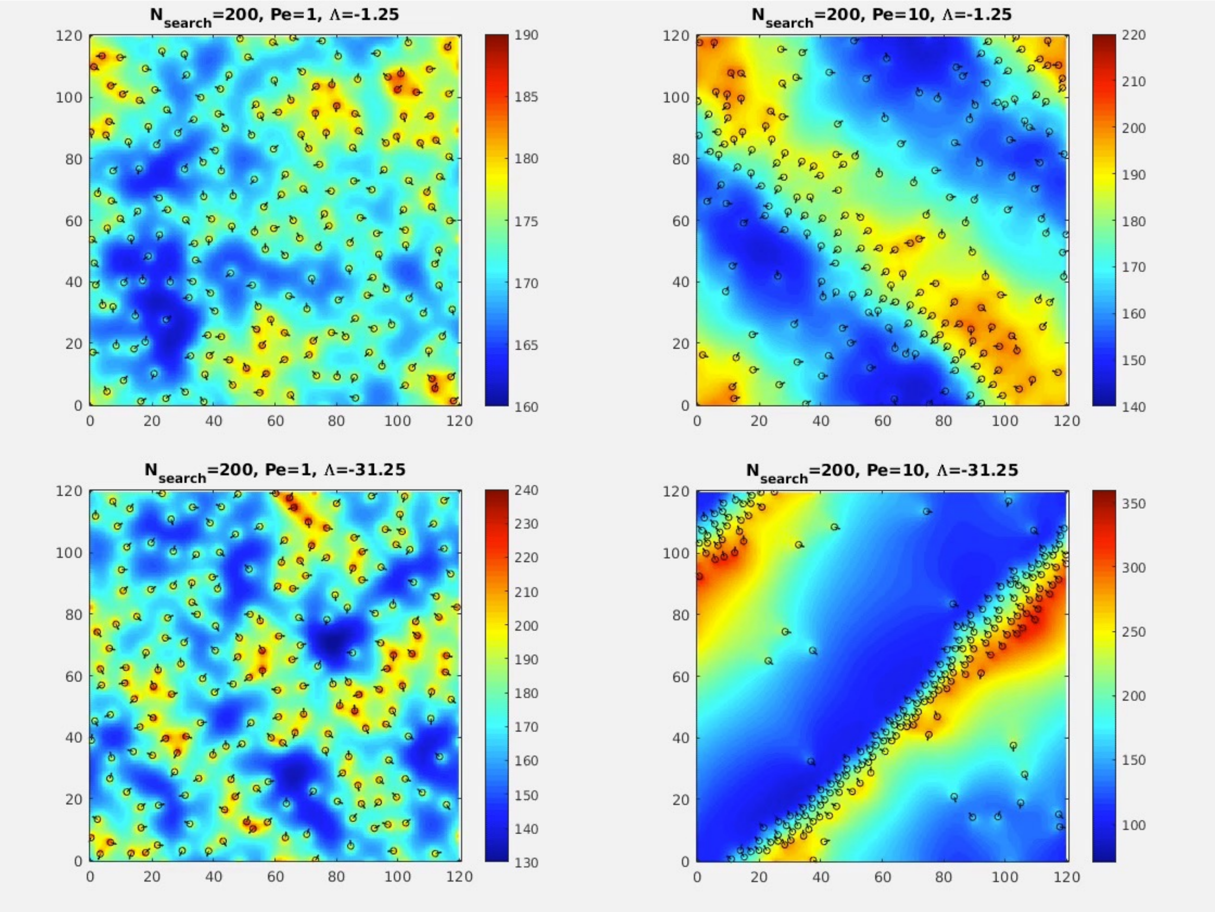
Chemical field

$$\frac{\partial}{\partial t} c(\mathbf{r}, t) = D\nabla^2 c - kc + h\sum_{i=1}^{N} \delta\left[\mathbf{r} - \mathbf{r}_i(t)\right]$$



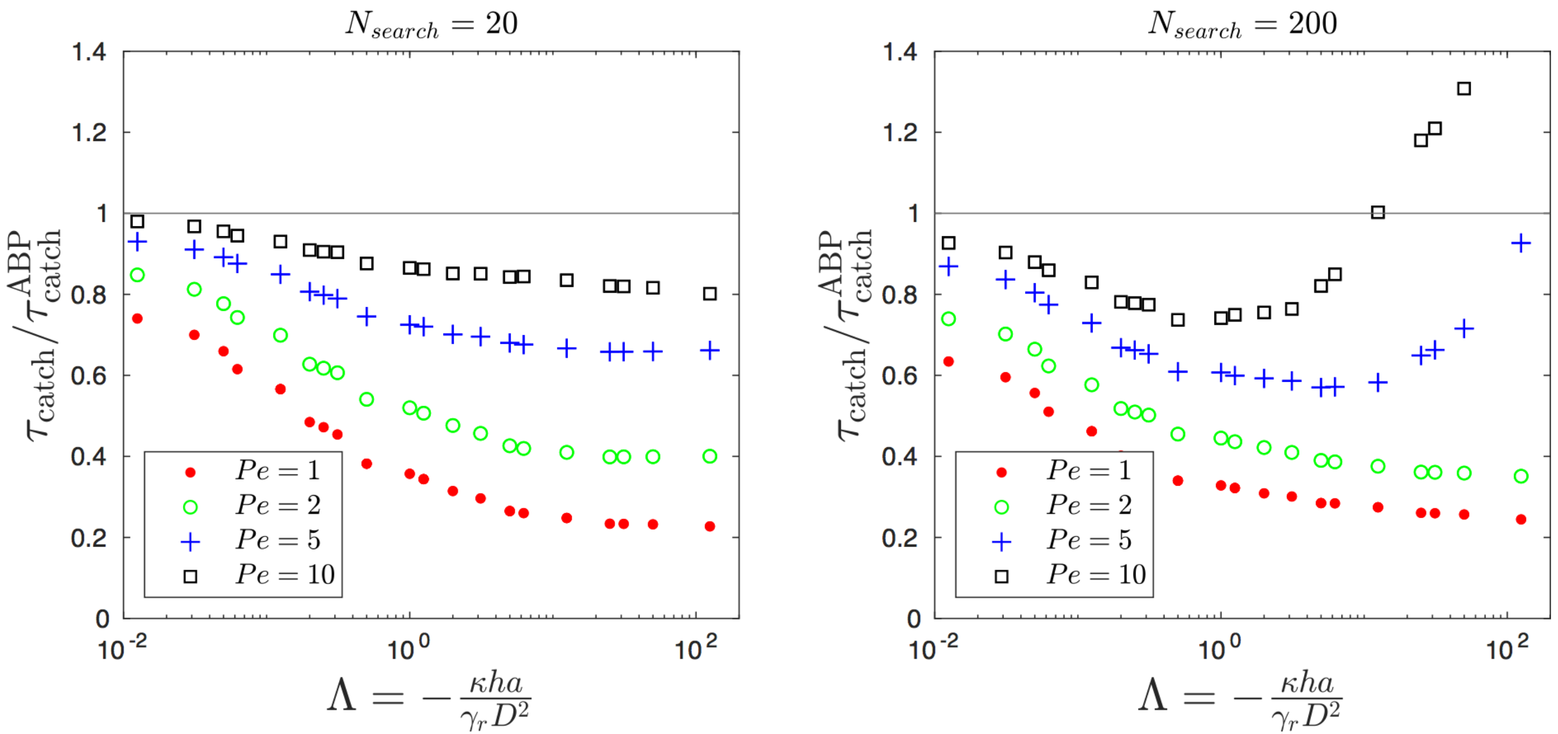
[Wysocki, HR: t.b.p]

Chemorepulsion



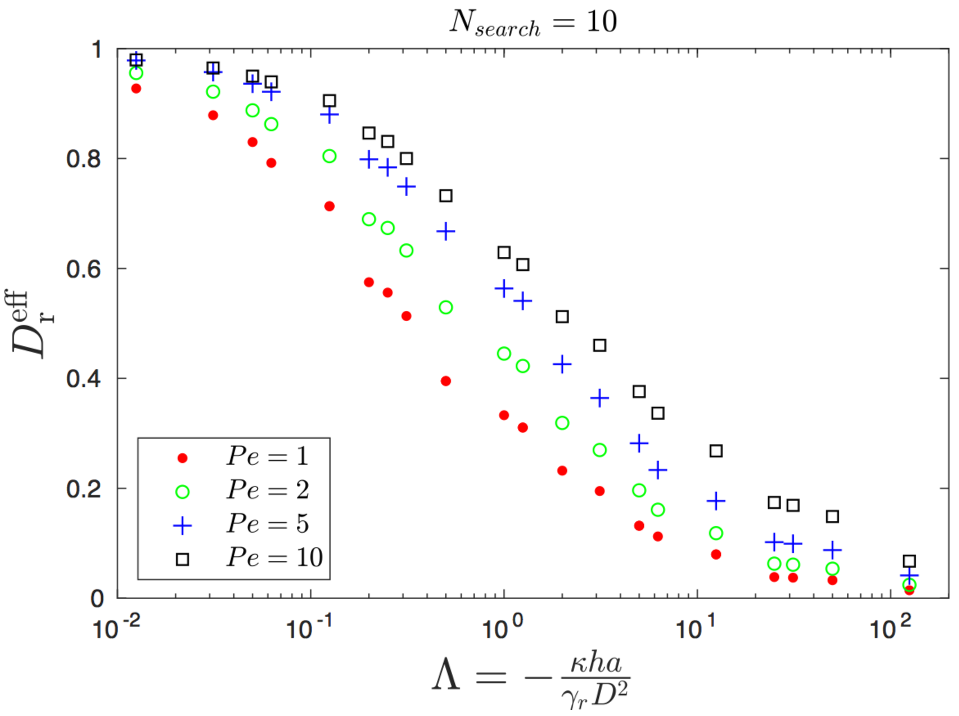
[Wysocki, HR: t.b.p]

Repulsive chemotactic particles search faster



[Wysocki, HR: t.b.p]

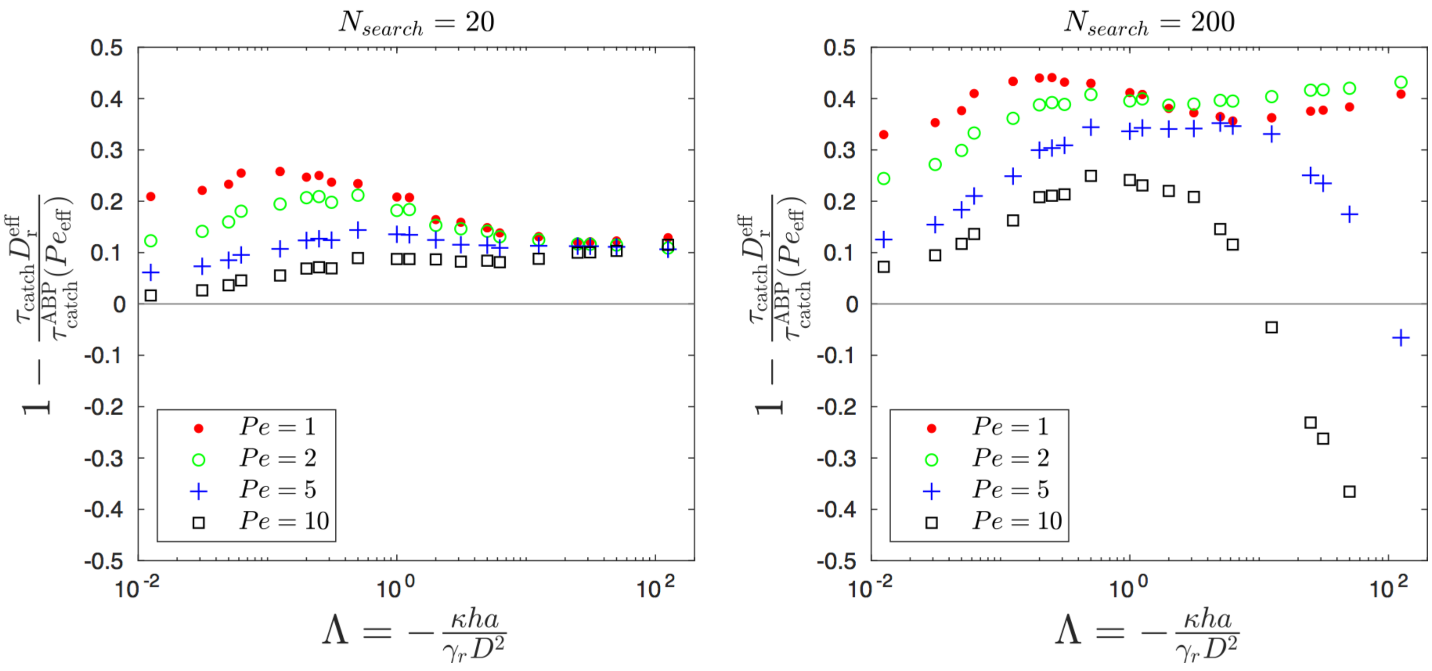
Chemo-repulsion leads to persistent motion



$$\langle \mathbf{e}(t) \cdot \mathbf{e}(0) \rangle \sim \exp\left(-D_r^{\mathrm{eff}} t\right) \Longrightarrow \mathrm{Pe}_{\mathrm{eff}} = \frac{V_0}{D_r^{\mathrm{eff}} a}$$

[Wysocki, HR: t.b.p]

Collective effects? Yes!



CTP's outperform ABP's if $\tau_{catch} D_r^{eff} < \tau_{catch}^{ABP}(Pe_{eff})$

[Wysocki, HR: t.b.p]

Thank you for your attention!

Title

Title

Mean first passage time MFPT

MFPT = <T> = $\int_0^{\infty} dt\, FPT(t)$

Markovian processes: $P(r_{n+1},t_{n+1} | r_n,t_n;\dots;r_1,t_1) = P(r_{n+1},t_{n+1} | r_n,t_n)$ for all $t_{n+1}>t_n>\dots>t_1$

Renewal equation: $$P(\mathbf{r}_T, t|\mathbf{r}_S) = \delta_{t,0}\delta_{\mathbf{r}_T,\mathbf{r}_S} + \int_0^t \mathrm{FPT}(t')P(\mathbf{r}_T, t-t'|\mathbf{r}_T)dt'$$

Def.: $$H(\mathbf{r}_T|\mathbf{r}_S) = \int_0^{\infty} \left(P(\mathbf{r}_T, t|\mathbf{r}_S) - P_{\mathrm{stat}}(\mathbf{r}_T)\right) dt$$

-> Express MFPT in terms of P:

$$\boxed{\langle \mathbf{T} \rangle = \frac{H(\mathbf{r}_T|\mathbf{r}_T) - H(\mathbf{r}_T|\mathbf{r}_S)}{P_{\mathrm{stat}}(\mathbf{r}_T)}}$$

Non-Markovian search

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Editors' Suggestion

Optimal Non-Markovian Search Strategies with *n*-Step Memory

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Stochastic search processes are ubiquitous in nature and are expected to become more efficient when equipped with a memory, where the searcher has been before. A natural realization of a search process with long-lasting memory is a migrating cell that is repelled from the diffusive chemotactic signal that it secretes on its way, denoted as an autochemotactic searcher. To analyze the efficiency of this class of non-Markovian search processes, we present a general formalism that allows one to compute the mean first-passage time (MFPT) for a given set of conditional transition probabilities for non-Markovian random walks on a lattice. We show that the optimal choice of the n-step transition probabilities decreases the MFPT systematically and substantially with an increasing number of steps. It turns out that the optimal search strategies can be reduced to simple cycles defined by a small parameter set and that mirror-asymmetric walks are more efficient. For the autochemotactic searcher, we show that an optimal coupling between the searcher and the chemical reduces the MFPT to $1/3$ of the one for a Markovian random walk.

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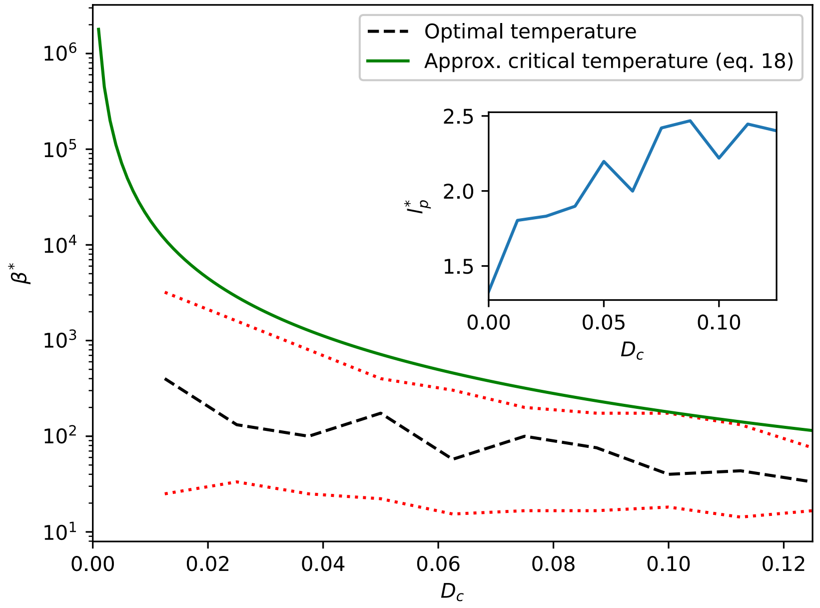
Auto-chemotactic searcher

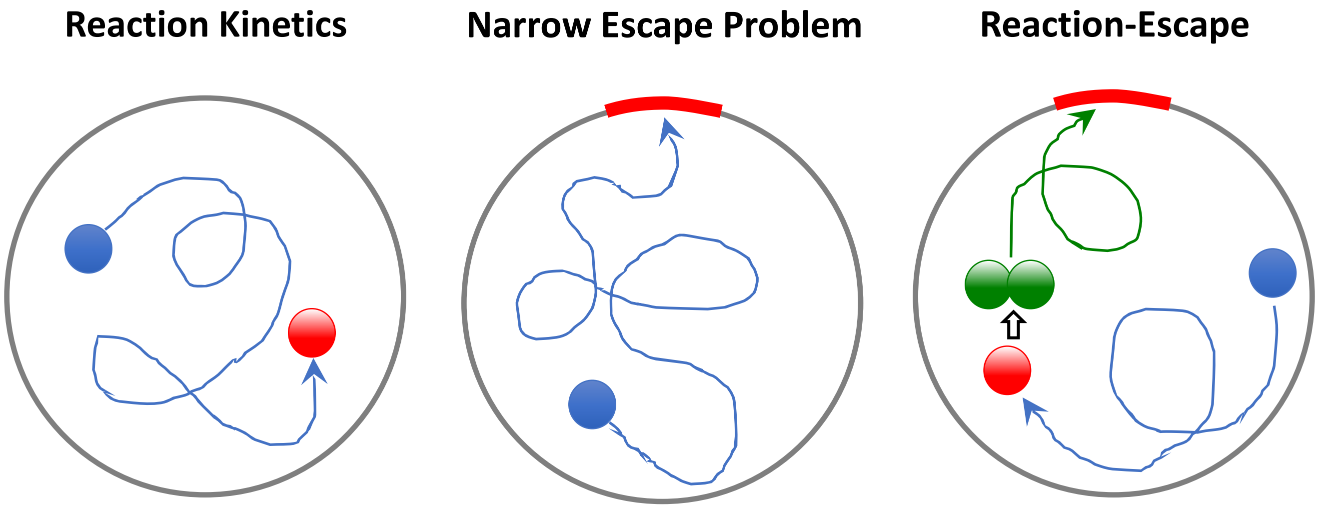


$$l_p \simeq 2 + \frac{e^{\beta D_c^2}}{2}$$

$$\beta^* = D_c^{-2} \ln\left(2l_p^* - 4\right)$$

Figure 3: Optimal inverse temperature range β^* as a function of D_c (black dashed line) with error range (red dotted line), together with equation (10) for $l_p^* = 5$. The inset shows the value of the persistence length at the optimal point (β^*, Dc).

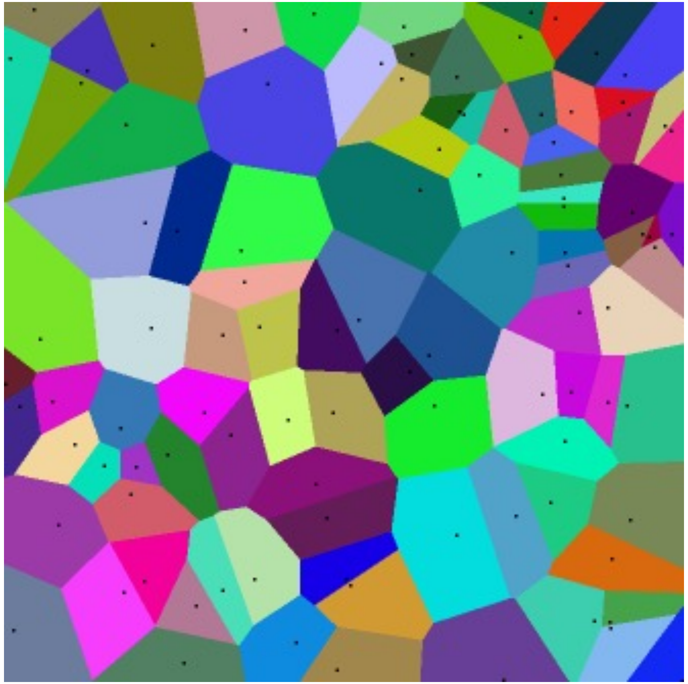
MFPT and Random Search Problems



Mean First Passage Time (MFPT)

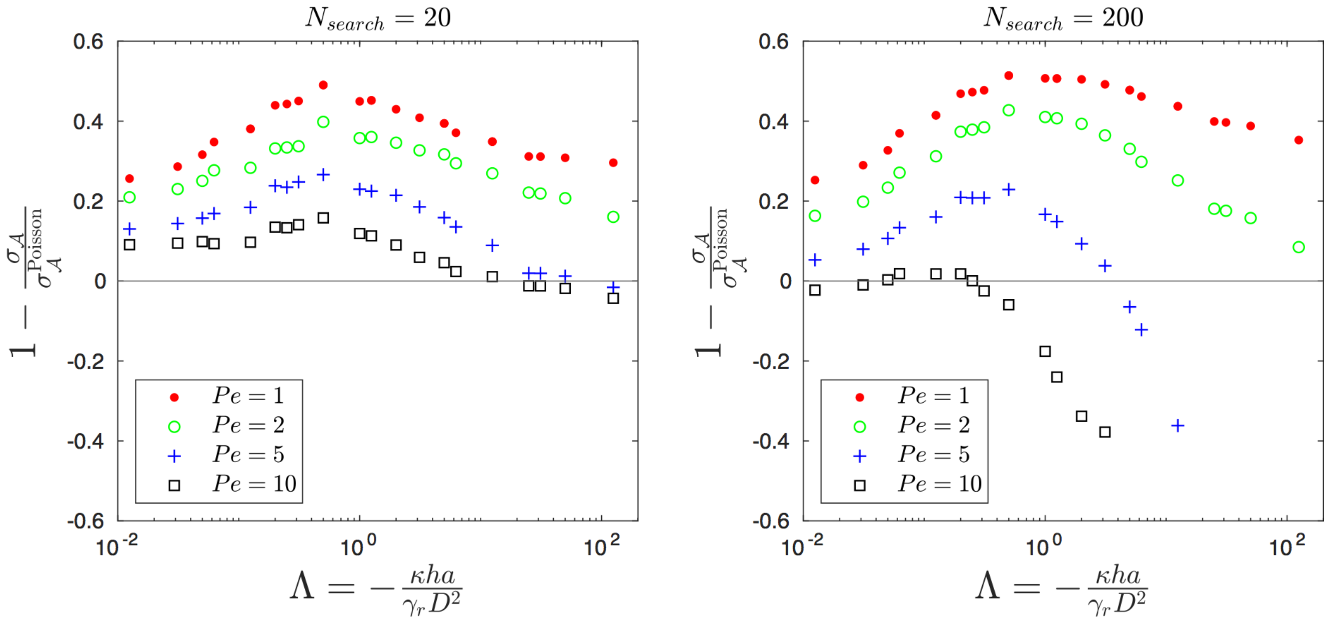
depends on geometrical and motility parameters

Are the search areas uniformly distributed ?



$\sigma_{\mathscr{A}}^{\text{Poission}} \approx 0.5292$ **is the standard deviation of the normalized areas** $\mathscr{A} = A/\langle A \rangle$ **of Poisson Voronoi cells.**

Spatial order correlates with search efficiency



Ordered if $\sigma_{\mathscr{A}} < \sigma_{\mathscr{A}}^{\text{Poisson}}$

Clustered if $\sigma_{\mathscr{A}} > \sigma_{\mathscr{A}}^{\text{Poisson}}$