TRANSPORT OF QUANTUM INFORMATION IN SPIN CHAINS

Joachim Stolze
Institut für Physik, Universität Dortmund, 44221 Dortmund

Göttingen, 7.6.2005

- Why quantum computing ?
- Why quantum information transfer ?
- Entangled states
- Spin chain dynamics
- Putting things together: Spin Chains as Perfect Quantum State Mirrors (Peter Karbach, JS, quant-ph/0501007)

Why Quantum Computing ?

www.usatoday.com

USA TODAY

Money

WEDNESDAY, JULY 14, 1999

COVER STORY

Beyond the PC: Atomic QC

Quantum computers could be a billion times faster than Pentium III

By Kevin Maney
USA TODAY

Around 2030 or so, the computer on your desk might be filled with liquid instead of transistors and chips. It would be

It wouldn't operate on anything so mundane as physical laws. It would employ quantum mechanics, which quickly gets into things such as teleportation and alternate universes and is, by all accounts, the weirdest stuff known to man.

Why Quantum Computing ?

Quantum computers can do anything...

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...but what can they do better ?

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Problems suitable for a quantum computer:

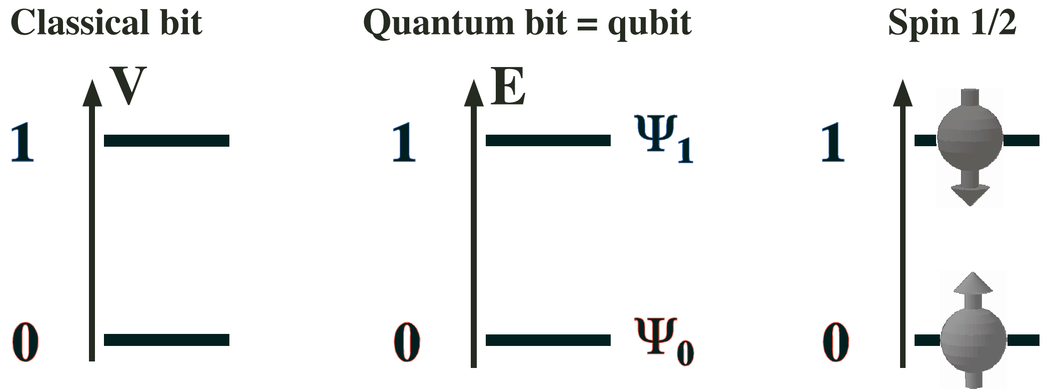
many possible states must be handled ($\rightarrow$ quantum parallelism) but

only few results are needed.

- Search in unstructured data basis $\rightarrow$ Grover's search algorithm
- Global property of a function ("Is $f(2l+1) > 0$?") $\rightarrow$ Shor's factoring algorithm

Quantum hardware

Quantum bits store information. Superpositions $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle \rightarrow$ quantum parallelism.



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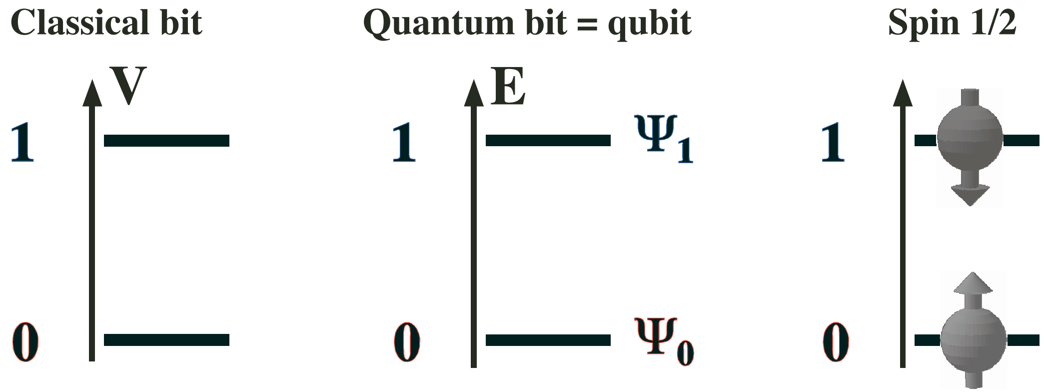

Quantum gates manipulate quantum bits

$$U_j = e^{i\frac{\pi}{2\sqrt{2}}(X_j+Z_j)} \qquad U_{jk} = e^{i\phi(Z_j+Z_k-Z_jZ_k)}$$

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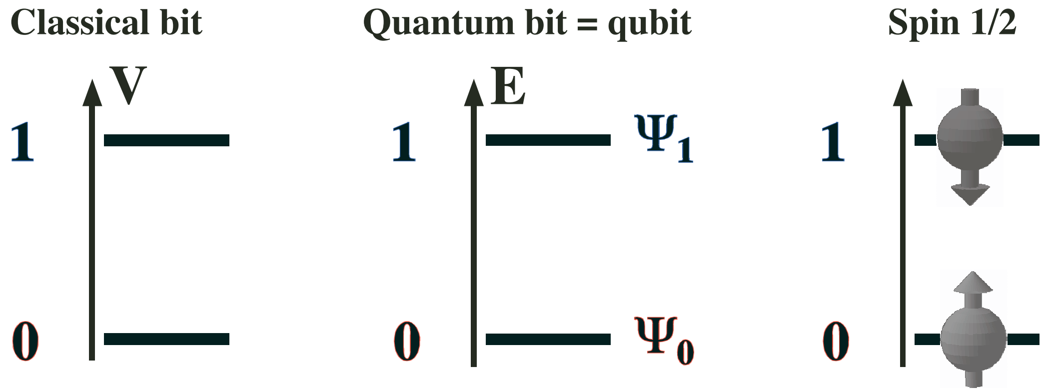

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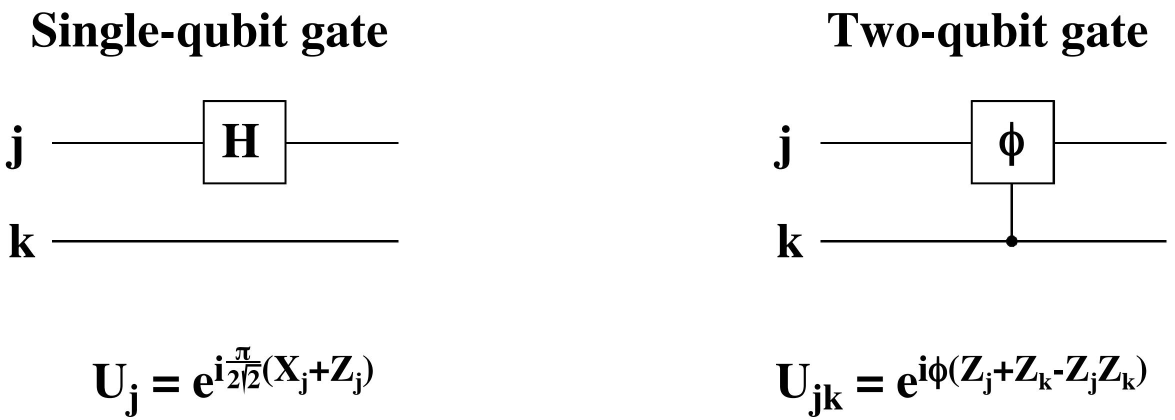


What about quantum lines to transmit information?

Long-range cryptographic information transfer à la BB84

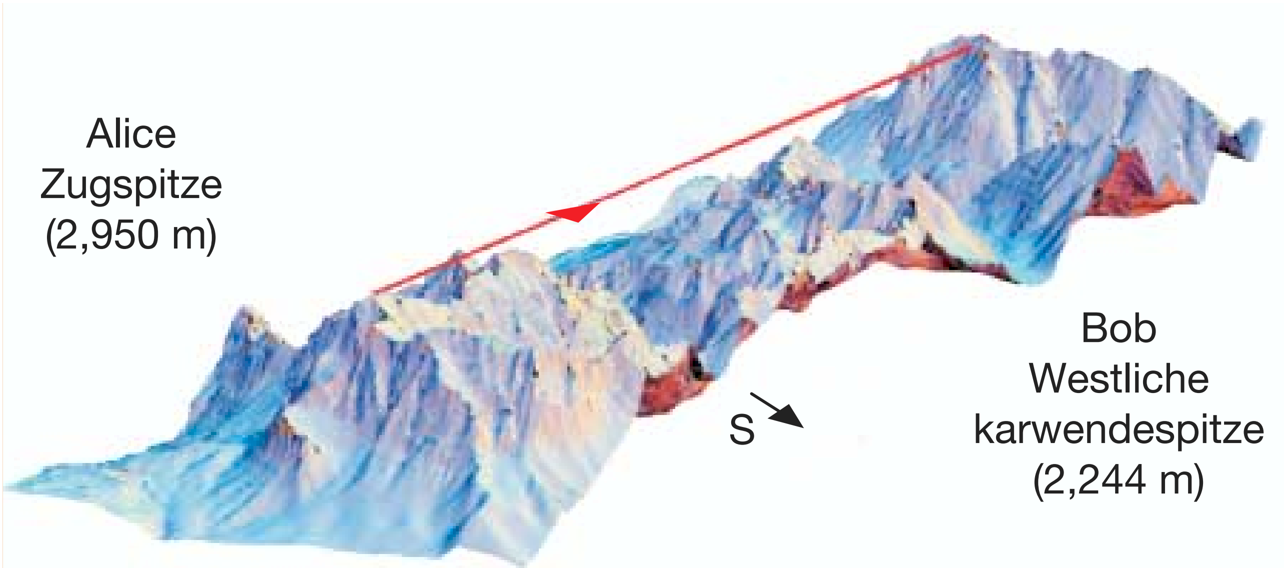
Information is encoded in the polarisation states $(\uparrow, \rightarrow, \nearrow, \nwarrow)$ of single photons.

IBM 1989 (Bennett et al.) : 30 cm, air.

University of Geneva 1997 (Gisin et al.): 23 km, telecom fiber optic cable.

Los Alamos National Lab 2002 (Hughes et al.): 10 km, air *during daytime* (New Mexico!): from Pajarito Mountain (3000 m) to TA 53 (2200 m).

LMU München 2002 (Kurtsiefer et al.): 23,4 km, air, at night: from Zugspitze (3000 m) to Westliche Karwendelspitze (2200 m).



Transfer of multi-qubit states?

Single photons carry no entanglement, but quantum algorithms must handle entangled states.

What *is* entanglement, actually?

Product states $|\psi\rangle_A \otimes |\phi\rangle_B$ are not entangled.

Many definitions and measures of entanglement: two / more subsystems, pure / mixed states,...

Some examples:

- The Bell states

$$\frac{1}{\sqrt{2}}\Big[|\uparrow\rangle_A \otimes |\uparrow\rangle_B \pm |\downarrow\rangle_A \otimes |\downarrow\rangle_B\Big] \qquad \frac{1}{\sqrt{2}}\Big[|\uparrow\rangle_A \otimes |\downarrow\rangle_B \pm |\downarrow\rangle_A \otimes |\uparrow\rangle_B\Big]$$

  In each Bell state measurement of any single-qubit observable leads to completely random results $\rightarrow$ the Bell states cannot be distinguished by any single-qubit measurement. However, they induce Einstein's famous *spukhafte Fernwirkungen*.

- Homogeneous n-qubit superposition state ($0 \equiv\uparrow$, $1 \equiv\downarrow$):
  $|0\rangle = |000...000\rangle, \quad |1\rangle = |000...001\rangle, \quad ... \quad , |2^n - 1\rangle = |111...111\rangle$ are the *computational basis* states of an n-qubit register; their equal-weight, equal-phase superposition

$$\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle$$

  is important in the Grover and Deutsch-Jozsa algorithms.

- The Greenberger-Horne-Zeilinger state (for ≥ 3 spins)

$$|GHZ\rangle = \frac{1}{\sqrt{2}} (|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle)$$

  collapses to a product state if S^z of one of the qubits is measured (but not so for S^x).

- The n-qubit W state

$$|W\rangle = \frac{1}{\sqrt{n}} (|000...001\rangle + |000...010\rangle + ... + |010...000\rangle + |100...000\rangle)$$

  is a more robust multipartite entangled state. Multiplication of the kth term with a phase factor $\exp iqk \rightarrow$ twisted W state, a.k.a single spin-wave state.

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The spin-$\frac{1}{2}$ Heisenberg-XXZ chain

Heisenberg exchange interaction between two $s = \frac{1}{2}$ spins

$$H_{\text{Heisenberg}} = -J\vec{S}_1 \cdot \vec{S}_2$$

chain of N spins with nearest-neighbor interactions, anisotropic in spin space:

$$\begin{aligned} H_{\text{XXZ}} &= -J\sum_{i=1}^{N-1}\left[(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y) + \Delta S_i^z S_{i+1}^z\right] - h\sum_{i=1}^{N} S_i^z \\ &= -J\sum_{i=1}^{N-1}\left[\frac{1}{2}(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) + \Delta S_i^z S_{i+1}^z\right] - h\sum_{i=1}^{N} S_i^z \end{aligned}$$

Jordan-Wigner mapping ▷

Spins		Fermions
S^+, S^-	$\longleftrightarrow$	$\pm a^\dagger, \pm a$
S^z	$\longleftrightarrow$	$a^\dagger a - 1/2$
$J(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y)$	$\longleftrightarrow$	$t\ (a_i^\dagger a_{i+1} + \text{h.c.})$ hopping
$\Delta J\ S_i^z S_{i+1}^z$	$\longleftrightarrow$	$V\ n_i n_{i+1}$ *interaction*
$h\ S_i^z$	$\longleftrightarrow$	$\mu\ n_i$ chemical potential

The spin-$\frac{1}{2}$ Heisenberg-XXZ chain: Eigenstates

The ferromagnetic ground state: $|\uparrow\uparrow\uparrow \dots \uparrow\uparrow\uparrow\rangle = |000\dots000\rangle$.

A single spin-flip state $S_2^-|\uparrow\uparrow\uparrow \dots \uparrow\uparrow\uparrow\rangle = |\uparrow\downarrow\uparrow \dots \uparrow\uparrow\uparrow\rangle = |010\dots000\rangle$ is no eigenstate of H_{XXZ}: $(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+)$ *moves* the inverted spin left or right.

How about coherent transport ?

A single spin-wave state

$$|q\rangle = \frac{1}{\sqrt{n}} \sum_{r=1}^{n} e^{iqr} S_r^- |\uparrow\uparrow\uparrow \dots \uparrow\uparrow\uparrow\rangle =: S^-(q)|\uparrow\uparrow\uparrow \dots \uparrow\uparrow\uparrow\rangle$$

is an eigenstate of H_{XXZ} with energy $\hbar\omega(q) = -J\cos q$. In the Jordan-Wigner picture this corresponds to a single fermion in a Bloch state in a tight-binding chain model.

However, a two spin-wave state

$$S(q_1)^- S(q_2)^- |\uparrow\uparrow\uparrow \dots \uparrow\uparrow\uparrow\rangle$$

is not an eigenstate of H_{XXZ}: the Jordan-Wigner fermions interact due to the $S_i^z S_{i+1}^z$ term.

Undistorted transfer of states with two or more flipped spins is probably difficult.

Spin wave packets

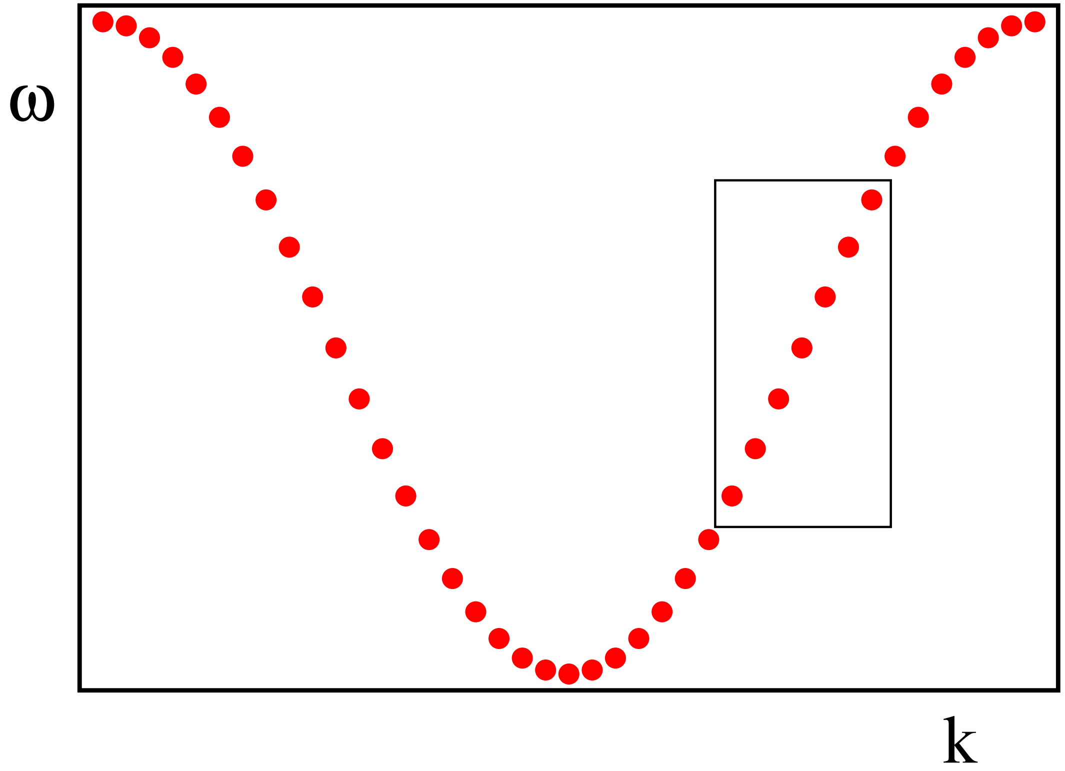
S. Bose: Quantum communication through an unmodulated spin chain. PRL **91**, 207901 (2003)
Prepare the first spin of a Heisenberg chain as desired.

$$(\alpha|\uparrow\rangle + \beta|\downarrow\rangle) \otimes |\uparrow\uparrow\uparrow \ldots \uparrow\uparrow\uparrow\rangle$$

is a superposition of the ground state and of single spin-wave states: a spin wave packet which may be received with reasonable fidelity at the other end of the chain after a certain time.

T.J. Osborne and N. Linden: Propagation of quantum information through a spin system. PRA **69**, 052315 (2004)

Instead of states localized at a single site, transfer Gaussian spin wave packets which occupy *only the least dispersive part of the dispersion relation*, and which are narrow in wavevector space rather than in real space.

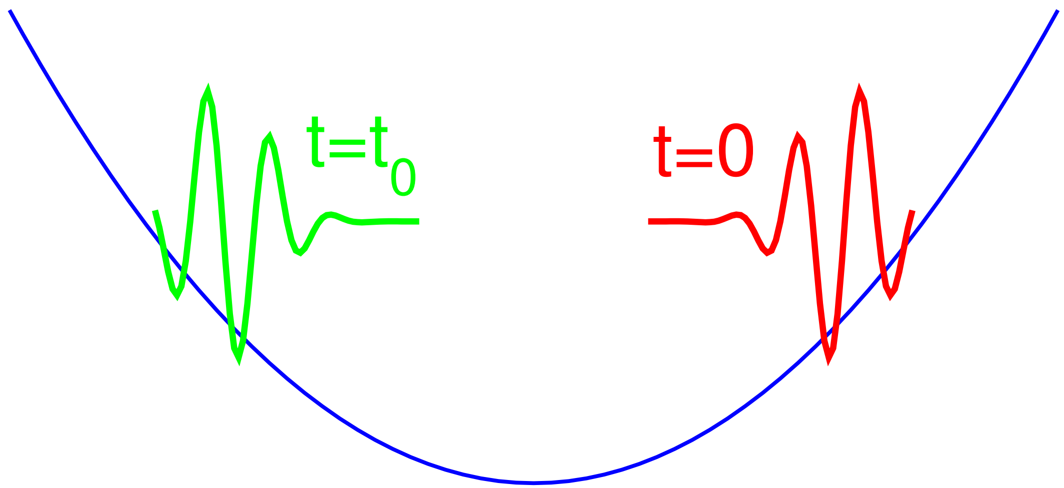


Note:
Least dispersive ≈ linear $\omega(k)$
≈ equidistant energy values

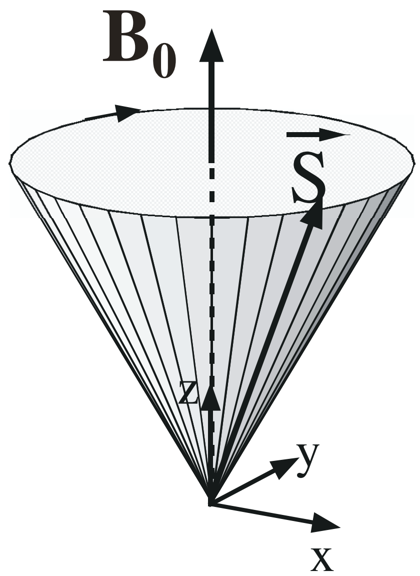
→ fairly good transfer of wave packets.

How about perfect transfer ?

Harmonic oscillator: Any wavepacket initially localized on the right develops into its perfect mirror image localized on the left.
Equidistant spectrum, but continuous degrees of freedom. Difficult to define qubits.



Angular momentum J: $|J_z = +J\rangle$ can develop into $|J_z = -J\rangle$ by rotation in a transverse field.
Equidistant spectrum, but zero-dimensional. No transport in space.



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M. Christandl et al.: PRL **92**, 187902 (2004); C. Albanese et al.: PRL **93**, 230502 (2004).

Single particle on a $(2J+1)$-site chain $\Longleftrightarrow$ Angular momentum J

State $|n\rangle$ localized at lattice site $n = 1, ..., 2J+1$ $\Longleftrightarrow$ J_z eigenstate $|m\rangle$

Transition amplitude (hopping matrix element) between two sites $\Longleftrightarrow$ (J_x or J_y) matrix element between $|m\rangle$ and $|m \pm 1\rangle$.

$$2J_x|m\rangle = (J_+ + J_-)|m\rangle = \sqrt{(J+m+1)(J-m)}|m+1\rangle + \sqrt{(J+m)(J-m+1)}|m-1\rangle$$

Find a lattice Hamiltonian H such that

$$H|n\rangle = \sqrt{n(N-n)}|n+1\rangle + \sqrt{(n-1)(N-n+1)}|n-1\rangle.$$

Solution

$$\begin{aligned} H &= \sum_{n=1}^{N-1} \sqrt{n(N-n)} \left[\frac{1}{2}\left(a^\dagger_{n+1} a_n + hc\right) + \Delta \left(a^\dagger_n a_n - \frac{1}{2}\right)\left(a^\dagger_{n+1} a_{n+1} - \frac{1}{2}\right)\right] \\ &\overset{JW}{=} \sum_{n=1}^{N-1} \sqrt{n(N-n)} \left[(S^x_n S^x_{n+1} + S^y_n S^y_{n+1}) + \Delta S^z_n S^z_{n+1}\right] \end{aligned}$$

Inhomogeneous XXZ chain; Δ is inactive as long as only one particle is present.

A state $|x\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$ of spin 1 is transferred to spin N by H after a time τ:

$$x\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow \quad\longrightarrow\quad \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow x$$

This is still just single-qubit transport; however, after the same time τ

$$\uparrow x\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow \quad\longrightarrow\quad \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow x\uparrow$$

and also

$$\uparrow\uparrow x\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow \quad\longrightarrow\quad \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow x\uparrow\uparrow$$

.... and so on.

Every state $xyz\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow$ of the spin chain is mapped to its mirror image $\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow zyx$ after τ, but only for $\Delta = 0$, so that "particles" (reversed spins) do not interact with each other, $\rightarrow$ inhomogeneous XX chain.

The mirror property of this spin-$\frac{1}{2}$ chain is due to
- the equidistant energy spectrum
- symmetry properties of the corresponding eigenvectors.

There is another spin-$\frac{1}{2}$ chain which acts as a perfect mirror for states (inhomogeneous XX with additional field in z direction).

Some simple questions

- Is that all or is there more?
- Can we engineer chains with perfect transfer/mirroring properties, plus other desirable features?
- How about mixed $(T > 0)$ states?
- What is really needed to achieve perfect transfer ?

The model

General inhomogeneous open-ended $(N+1)$-site $S=\frac{1}{2}$ XX chain:

$$H = 2\sum_{i=1}^{N} J_i(S_i^x S_{i-1}^x + S_i^y S_{i-1}^y) + \sum_{i=0}^{N} h_i \left(S_i^z + \frac{1}{2} \right).$$

Equivalent Hamiltonian of noninteracting spinless lattice fermions:

$$H = \sum_{i=1}^{N} J_i(c_{i-1}^\dagger c_i + c_i^\dagger c_{i-1}) + \sum_{i=0}^{N} h_i c_i^\dagger c_i$$

can be diagonalized,

$$H = \sum_{\nu=0}^{N} \varepsilon_\nu c_\nu^\dagger c_\nu.$$

$c_\nu^\dagger$ creates a fermion in a single-particle eigenstate $|\nu\rangle$ of energy ε_ν;
$c_i^\dagger$ creates a fermion at lattice site i.

The ε_ν and $|\nu\rangle$ determine the dynamics completely: every eigenstate of H is uniquely characterized by the fermion occupation numbers $n_\nu = c_\nu^\dagger c_\nu$.

Single-particle properties of a mirror Hamiltonian

ε_ν $(\nu = 0, ..., N)$ and $|\nu\rangle$ are eigenvalues and eigenvectors of the one-particle Hamiltonian matrix H_1.

Mirror symmetry: $h_i = h_{N-i}$ and $J_i = J_{N+1-i}$

$\Rightarrow$ the eigenvectors of H_1, have definite parity: either $\langle i|\nu\rangle = +\langle N-i|\nu\rangle$ or $\langle i|\nu\rangle = -\langle N-i|\nu\rangle$.

Parity alternates as ε_ν grows.

$$H_1 = \begin{pmatrix} h_0 & J_1 & & & & \\ J_1 & h_1 & J_2 & & & \\ & J_2 & h_2 & J_3 & & \\ & & J_3 & \ddots & & \\ & & & & \ddots & J_N \\ & & & & J_N & h_N \end{pmatrix}$$

(Discrete version of the "Knotensatz": *For a real symmetric tridiagonal matrix with only positive subdiagonal elements (i) all eigenvalues are real and nondegenerate, and (ii) the sequence of the components of the jth eigenvector ,in ascending order of the eigenvalues, $j = 0, 1, \ldots$ shows exactly j sign changes.*)

The eigenvectors of H_1, the single-particle eigenstates of H, are alternately even and odd.

Wanted: Operation M which maps an arbitrary many-particle state to its spatial mirror image.

Sufficient: M maps every single-particle state $|\nu\rangle$ to its mirror image: $M = \Pi(-1)^\nu$ (Π: parity).

Implement the extra sign for the odd states as a dynamical phase factor $\exp[i\pi(2n+1)]$ by designing the ε_ν appropriately.

Designing the spectrum

Evolution of the single-particle state $|i\rangle$ localized at site i: $e^{-iHt}|i\rangle = \sum_{\nu=0}^{N} e^{-i\varepsilon_\nu t}|\nu\rangle\langle\nu|i\rangle$.

Alternating parity $\Rightarrow$ $\langle N-i|\nu\rangle = (-1)^\nu \langle i|\nu\rangle$ $\Rightarrow$

$$|N-i\rangle = \sum_\nu |\nu\rangle\langle\nu|N-i\rangle = \sum_\nu (-1)^\nu |\nu\rangle\langle\nu|i\rangle.$$

Perfect quantum state mirroring at time τ occurs if

$$e^{-iH\tau}|i\rangle = e^{i\phi_0}|N-i\rangle,$$

for all i, for example if $e^{-i\varepsilon_\nu \tau} = e^{-i(\pi\nu+\phi_0)}$, or equivalently

$$\varepsilon_\nu \tau = (2n(\nu)+\nu)\pi + \phi_0,$$

where $n(\nu)$ is an *arbitrary* integer function.

Every system with such single-particle energies generates perfect mirror images of arbitrary input states!

Designing the Hamiltonian ?

The function $n(\nu)$ in $\varepsilon_\nu \tau = (2n(\nu) + \nu)\pi + \phi_0$ is completely arbitrary $\Rightarrow$ infinitely many single-particle spectra suitable for quantum state mirroring.
$n(\nu) \equiv 0$ and $n(\nu) = q\frac{\nu(\nu+1)}{2} + p\nu$ are the systems of Albanese et al.

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Which Hamiltonian (if any) yields a given / desired spectrum ε_ν ?

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Which Hamiltonian (if any) yields a given / desired spectrum ε_ν ?

Hald 1976: For a given nondegenerate single-particle spectrum there exists a unique symmetric tridiagonal Hamiltonian matrix with nonnegative subdiagonal elements and with the additional spatial symmetry properties discussed above.

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How to find that matrix ?

- Direct method; algorithm by Hochstadt (1974).

- Simulated annealing: optimizing the set of eigenvalues.

What do we have ?

Many proposals for quantum information transfer in spin chains are restricted: a single spin state is transported through the completely polarized (ground) state.

Here, states involving arbitrarily many sites are perfectly mirrored across the system.
No restriction to the ground state nor even to the set of pure states.
(All single-fermion eigenstates of the Hamiltonian and thus arbitrary many-fermion density operators are mirrored perfectly at the same instant of time τ.)

Mirroring twice reproduces the initial state.
$\Rightarrow$ Time evolution of the system is periodic with period 2τ.

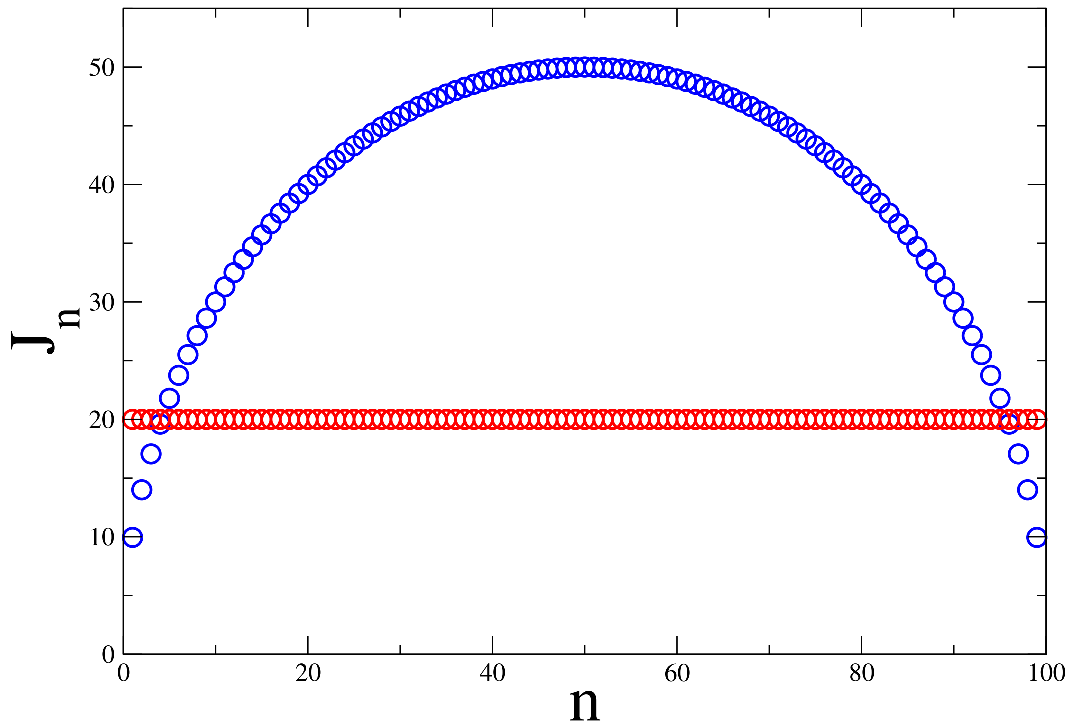
Proof: Time autocorrelation function of an arbitrary observable $A = A^\dagger$:

$$\langle A(t)A\rangle = Z^{-1}\sum_n \langle n|e^{-\beta H}e^{iHt}Ae^{-iHt}A|n\rangle = Z^{-1}\sum_{n,m} e^{-\beta E_n}e^{i(E_n-E_m)t}|\langle n|A|m\rangle|^2$$

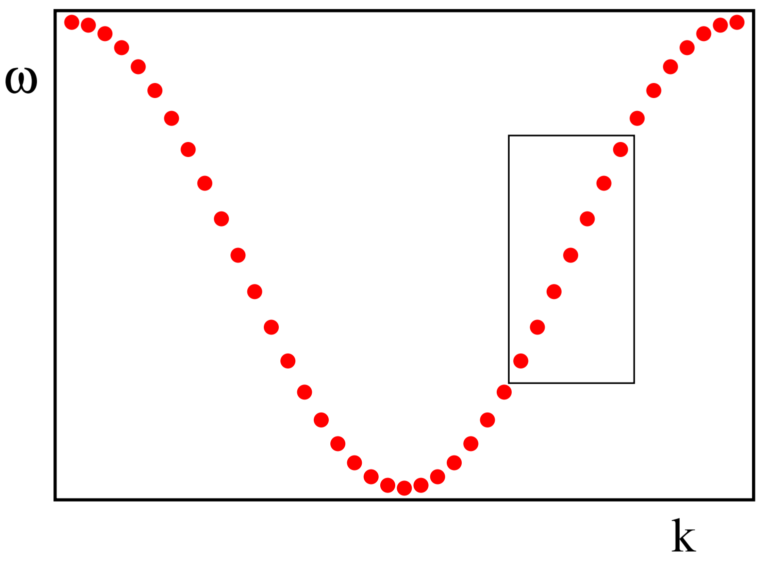
$(Z = \sum_n e^{-\beta E_n} \quad ; \quad \beta = (k_B T)^{-1} \quad ; \quad H|n\rangle = E_n|n\rangle)$
$(E_n - E_m)$ are all multiples of some energy, $\Rightarrow \langle A(t)A\rangle$ is a periodic function of t.

Quantum spin chain engineering

Homogeneous XX chain:
simple, but no perfect transport (dispersion).
Inhomogeneous chain:
Perfect transport, but awkward couplings.



Compromise ?

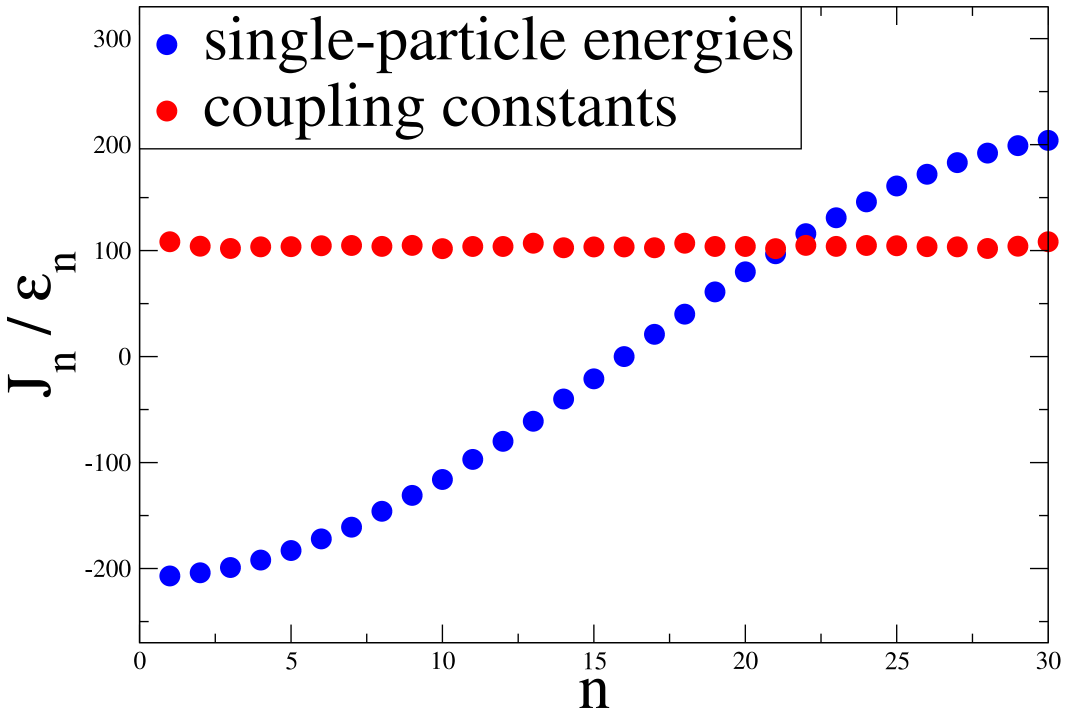


Idea:
Bring the old spin-wave dispersion relation into the right shape (all energy differences are suitable multiples of some-thing) by a little *tweaking*.

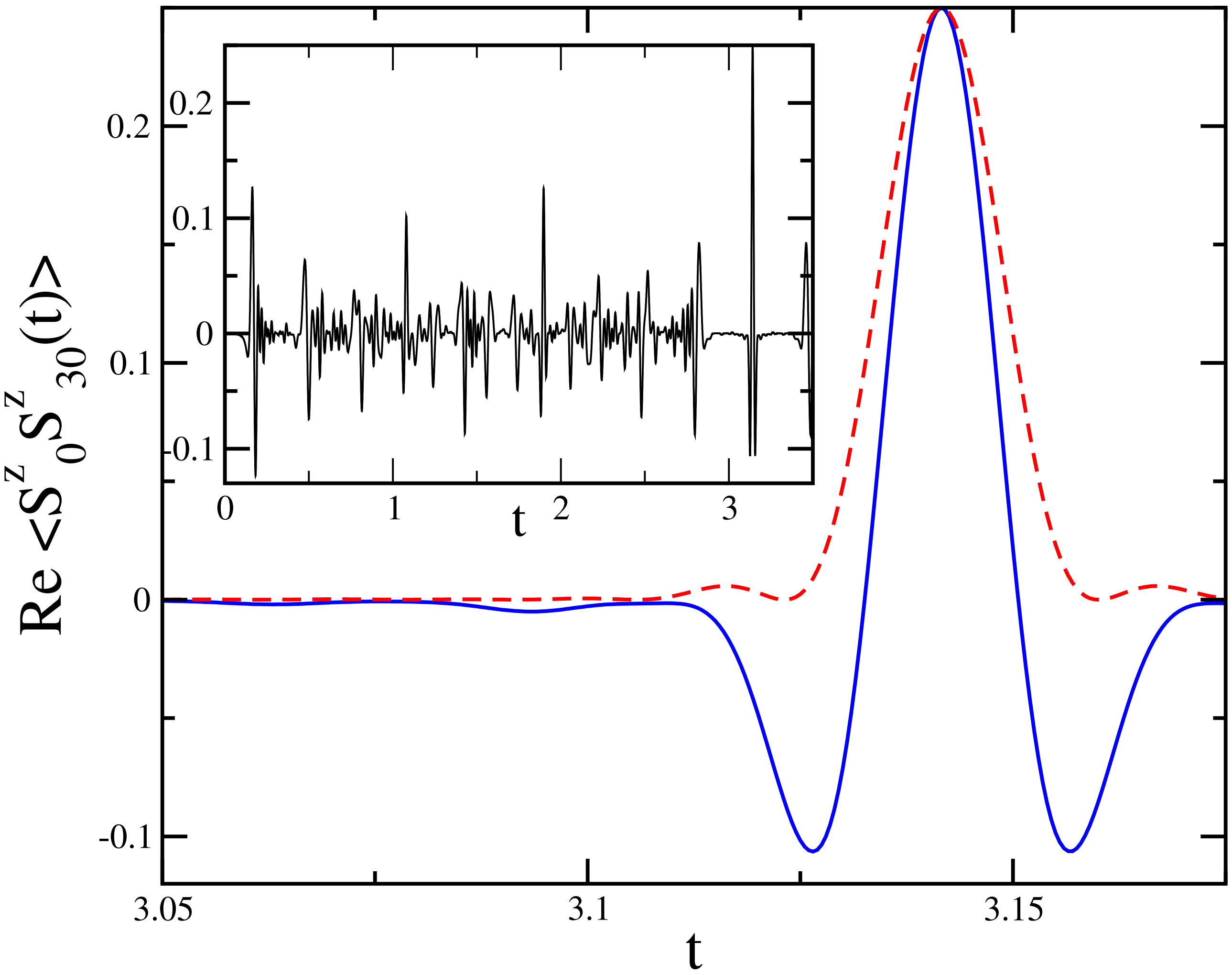
Results for a 31-spin chain:

- cosine-like dispersion
- almost constant ($\pm 3.3\%$ variation) couplings
- perfect transfer

(For 50 sites the coupling varies only by $\pm 1\%$.)



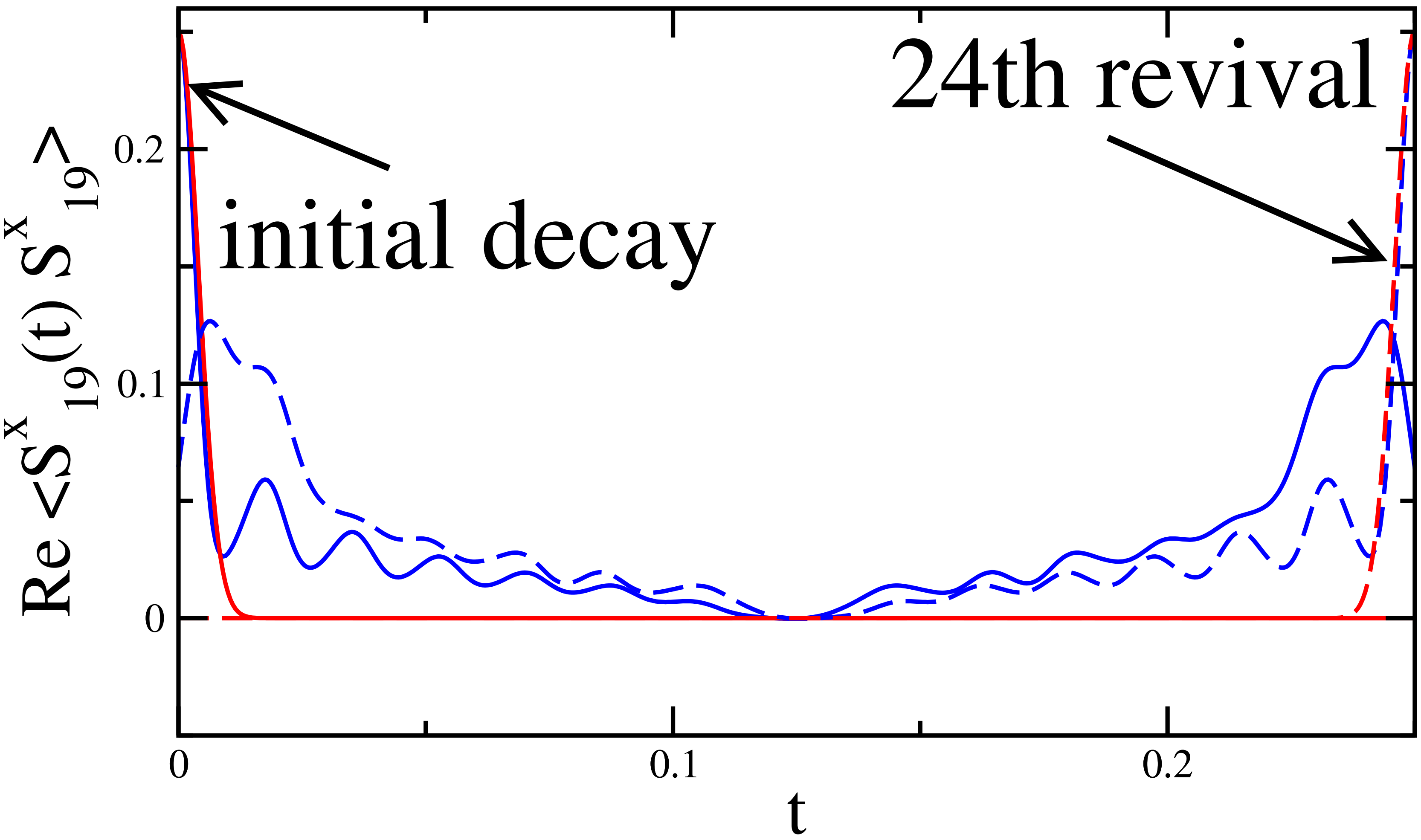
Safe transfer at any temperature



Real part of $\langle S_0^z S_{30}^z(t)\rangle$ in a 31-spin chain at $T = 0$ and $T = 1000$, near $t = \pi$. The maximum possible value $1/4$ of the correlation at $t = \pi$ demonstrates perfect state transfer.

Inset: same correlation for $T = 0$ over an extended time range shows somewhat irregular behavior.

Perfect long-time periodicity



Autocorrelation of the x spin component at site 19 in a 41-site chain, at times t (solid) and $t + 0.25 - 48\pi$ (dashed), at $T = 0$ and $T = 10^4$.

Jordan-Wigner $\rightarrow$ many-fermion correlation involving lattice sites 0 through 19.

Note the rapid decay and the absence of oscillations at high T. ($\rightarrow$ Gaussian).

Conclusions

- There is an infinitely large class of inhomogeneously coupled spin chain systems capable of perfect quantum information transfer.
- The freedom of choice within that class allows for some spin chain engineering.
- Perfect state transfer over fairly long distances in a chain with almost homogeneous exchange coupling and without external magnetic field.
- In contrast to many previous proposals, there is no restriction to the transfer of single-spin states at zero temperature. The systems discussed here can transfer genuinely entangled states involving several qubits, at arbitrary temperature.
- Sensitivity to perturbations like noise and imperfections will be the subject of further research.

Jordan-Wigner: The ugly details

Single-spin operators $\longrightarrow$ many-fermion operators

$$S_i^z = a_i^\dagger a_i - \frac{1}{2} = n_i - \frac{1}{2}$$

$$S_i^+ = (-1)^{\sum_{k<i} n_i} a_i^\dagger \quad ; \quad S_i^- = (-1)^{\sum_{k<i} n_i} a_i$$

$$(-1)^{a_k^\dagger a_k} = (a_k^\dagger + a_k)(a_k^\dagger - a_k)$$

$$\Longrightarrow S_i^+ = (a_1^\dagger + a_1)(a_1^\dagger - a_1)(a_2^\dagger + a_2)(a_2^\dagger - a_2)\ldots(a_{i-1}^\dagger + a_{i-1})(a_{i-1}^\dagger - a_{i-1})a_i^\dagger$$

◁