Correlated Resetting Gas

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Collaborators

- Marco Biroli (LPTMS, Univ. Paris Saclay)
- Sergio Ciliberto (ENS, Lyon)
- Manas Kulkarni (ICTS, Bangalore)
- Hernan Larralde (UNAM, Mexico)
- Sanjib Sabhapandit (RRI, Bangalore)
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References:

M. Biroli, H. Larralde, S. N. Majumdar, G. Schehr, "Extreme Statistics and Spacing Distribution in a Brownian Gas Correlated by Resetting", Phys. Rev. Lett., **130**, 207101 (2023)

M. Biroli, H. Larralde, S. N. Majumdar, G. Schehr, "Exact extreme, order and sum statistics in a class of strongly correlated system", Phys. Rev. E **109**, 014101 (2024).

M. Biroli, M. Kulkarni, S. N. Majumdar, G. Schehr, "Dynamically emergent correlations between particles in a switching harmonic trap ", Phys. Rev. E **109**, L032106 (2024).

S. Sabhapandit & S. N. Majumdar, "Noninteracting particles in a harmonic trap with a stochastically driven center ", J. Phys. A: Math. Theor. **57**, 335003 (2024).

M. Kulkarni, S. N. Majumdar, S. Sabhapandit, "Dynamically emergent correlations in bosons via quantum resetting ", J. Phys. A: Math. Theor. **58**, 105003 (2025).

Plan

- Correlated gas in thermal equilibrium: examples and observables

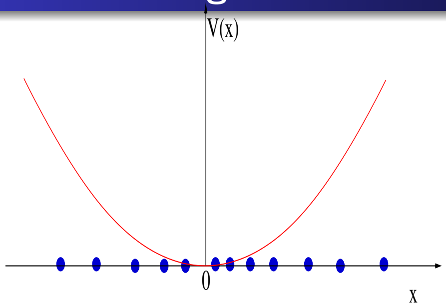
- Correlated gas in **nonequilibrium** stationary state created by **resetting**

- Exact results for various observables:
  - Average density
  - Extreme and Order statistics
  - Gap statistics
  - Full Counting statistics

- Summary and Conclusion

One dimensional Correlated Gas
In
Thermal Equilibrium

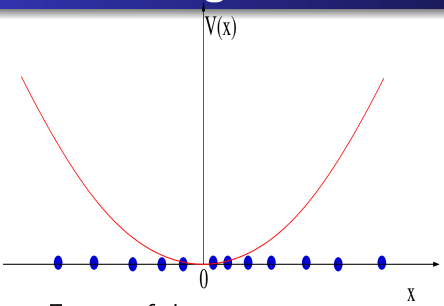
Correlated gas in thermal equilibrium



N particles on a line with coordinates $\Longrightarrow \{x_1, x_2, \ldots, x_N\}$

$V(x) \rightarrow$ external confining potential

Correlated gas in thermal equilibrium



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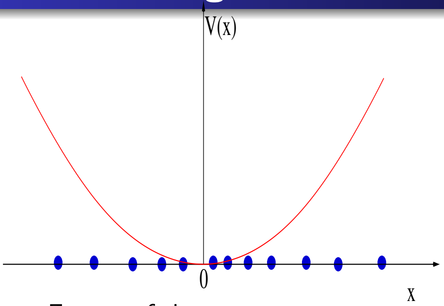
Energy of the gas:

$$E[\{x_i\}] = \sum_i V(x_i) + \sum_{i \neq j} V_2(x_i, x_j) + \sum_{i \neq j \neq k} V_3(x_i, x_j, x_k) + \ldots$$

Interactions: either short-ranged or long-ranged

Correlated gas in thermal equilibrium



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In thermal equilibrium, the joint distribution of the particle positions:

$$P(x_1, x_2, \ldots, x_N) = \frac{1}{Z} e^{-\beta E[\{x_i\}]}$$

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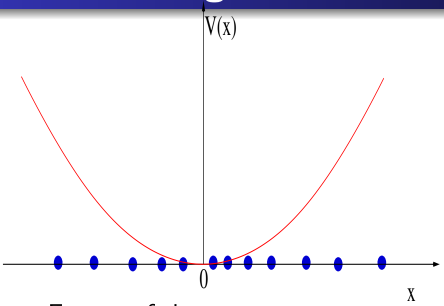

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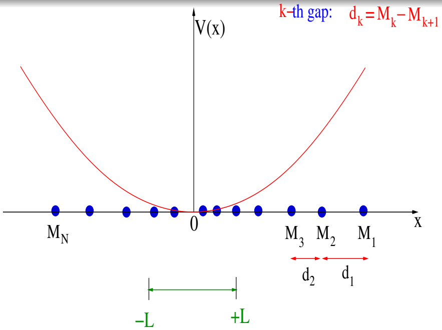
$$P(x_1, x_2, \ldots, x_N) = \frac{1}{Z}\, e^{-\beta\, E[\{x_i\}]} \neq p(x_1)p(x_2)\ldots p(x_N)$$

No **factorization** in the presence of **interactions**

Observables of interest



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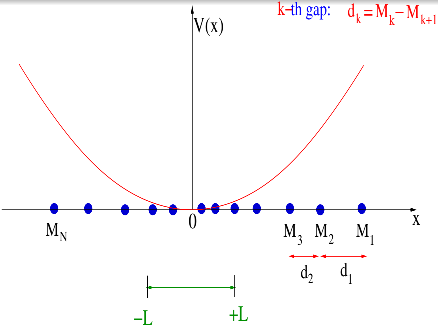

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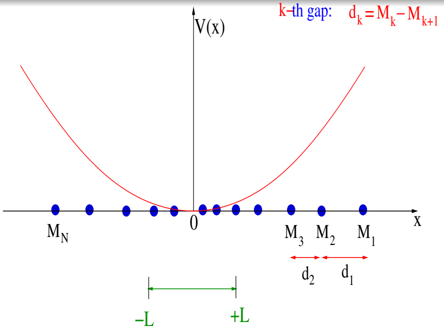

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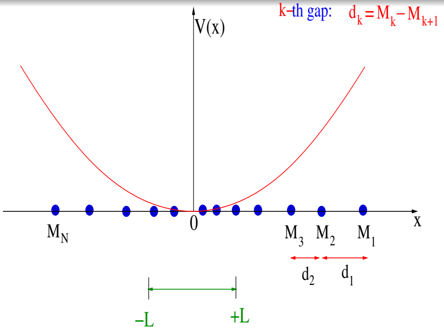

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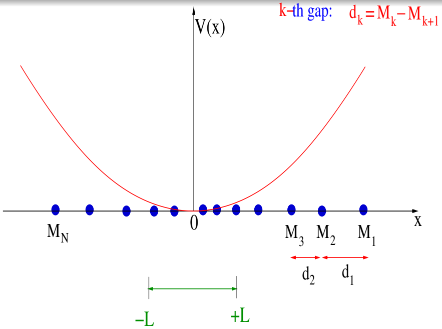

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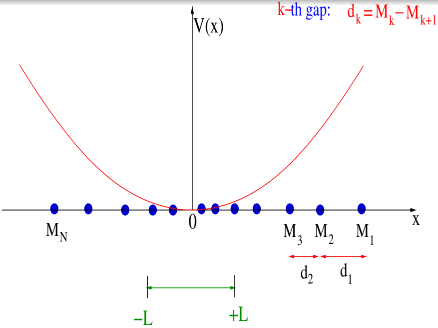

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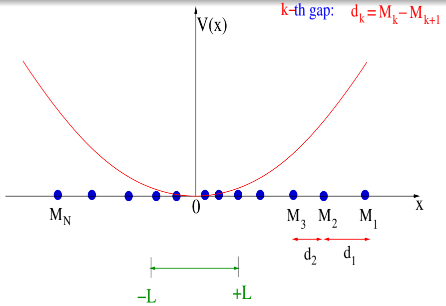
Generally hard to compute for a **correlated/interacting** gas !

Ideal gas: no interaction



In the absence of interactions

Energy: $E[\{x_i\}] = \sum_{i=1}^{N} V(x_i)$

Joint distribution **factorises** (i.i.d)

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where $p(x) = \frac{e^{-\beta\, V(x)}}{\int dx'\, e^{-\beta\, V(x')}}$

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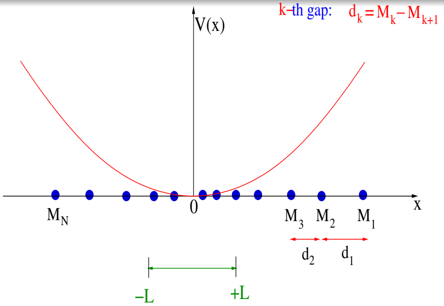

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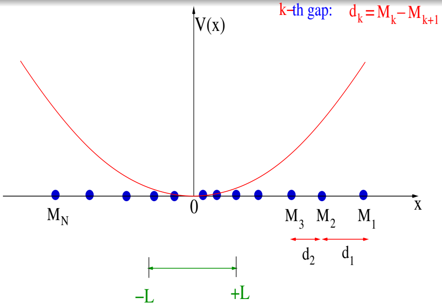

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- Distribution of the k-th maximum $M_k \Longrightarrow$ Order statistics
- Distribution of the k-th gap $d_k = M_k - M_{k+1}$
- Full counting statistics (FCS): $\mathrm{Prob.}[N_L, N]$

Exact results for observables in the Ideal gas

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$$\mathrm{Prob.}[M_k = w] = \frac{N!}{(k-1)!\,(N-k)!}\,p(w)\left[\int_w^\infty p(y)dy\right]^{k-1}\left[\int_{-\infty}^{w} p(y)dy\right]^{N-k}$$

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- **Full Counting Statistics:**

  $\mathrm{Prob.}[N_L, N] = \binom{N}{N_L}\,{q_L}^{N_l}(1-q_L)^{N-N_L}$ where $q_L = \int_{-L}^{L} p(y)\,dy$

  $N_L \Rightarrow$ no. of particles in the interval $[-L, L]$

Weakly and Strongly correlated gas

- Short-ranged gas $\longrightarrow$ Weakly correlated

  Observables can sometimes be computed using perturbation theory, renormalization group method etc.

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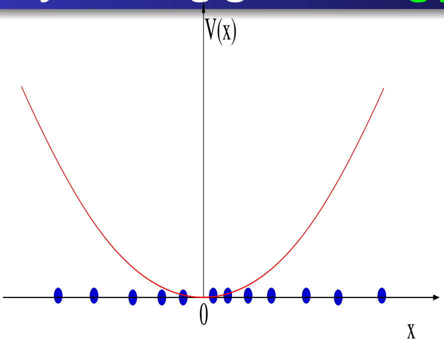
- Long-ranged gas $\longrightarrow$ Strongly correlated

  Observables $\longrightarrow$ much harder to compute **!**

S.M. & G. Schehr, "Statistics of Extremes and Records in Random Sequences" (Oxford Univ. Press, 2024)

Dyson's log-gas: Strongly correlated



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$$E[\{x_i\}] = \frac{N}{2}\sum_{i=1}^{N} x_i^2 - \frac{1}{2}\sum_{i\neq j} \log|x_i - x_j|$$

pairwise logarithmic repulsion Dyson, 1962

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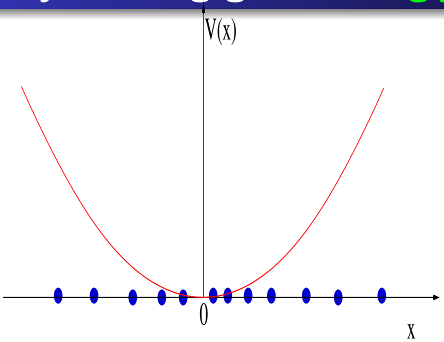

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Consider an $(N \times N)$ Gaussian Hermitian random matrix H_{ij} whose entries are distributed via:

$$\mathrm{Prob.}[H] \propto \exp\left[-N\sum_{i,j}|H_{ij}|^2\right] \propto \exp\left[-N\,\mathrm{Tr}\left(H^\dagger H\right)\right]$$

$\Longrightarrow$ invariant under unitary rotation (change of basis) (GUE)

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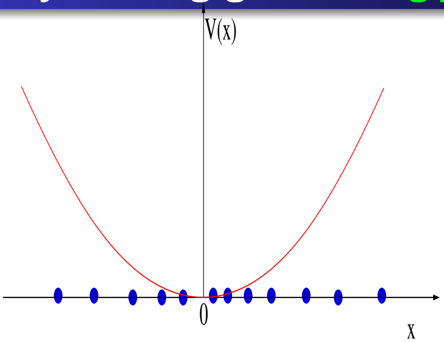

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N real eigenvalues: $\{\lambda_1, \lambda_2, \ldots, \lambda_N\} \longrightarrow$ **strongly correlated**

Dyson's log-gas

Joint distribution of eigenvalues of an $(N \times N)$ Gaussian Hermitian random matrix (Wigner, 1951):

$$P(\{\lambda_i\}) = \frac{1}{Z_N} \exp\left[-N\sum_{i=1}^{N}\lambda_i^2\right] \prod_{i<j}|\lambda_i - \lambda_j|^2$$

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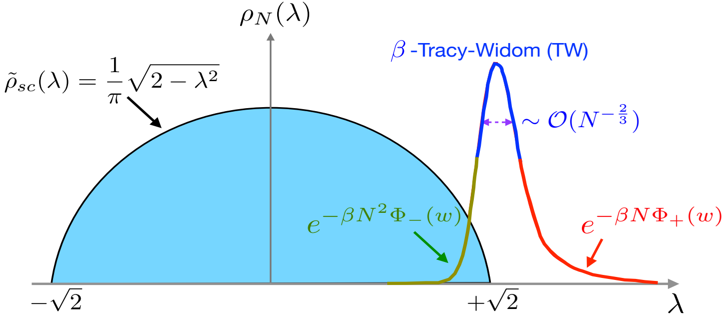
Hence one can identify the eigenvalues $\{\lambda_1, \lambda_2, \ldots \lambda_N\} \equiv \{x_1, x_2, \ldots, x_N\}$ as the positions of a 1-d gas of N particles with pairwise log-repulsion with $\beta = 2$ (Dyson, 1962)

Most of the observables can be computed exactly $\Longrightarrow$ not that **easy !**

Observables in the log-gas model



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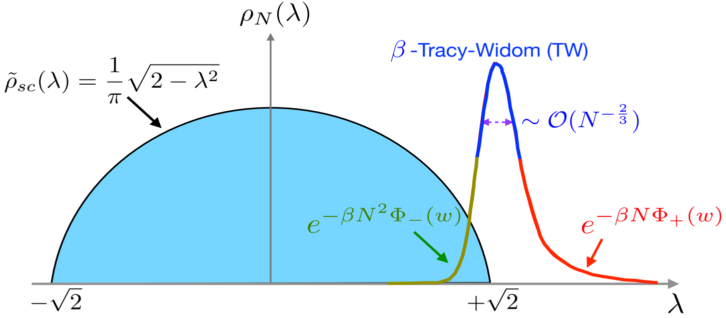

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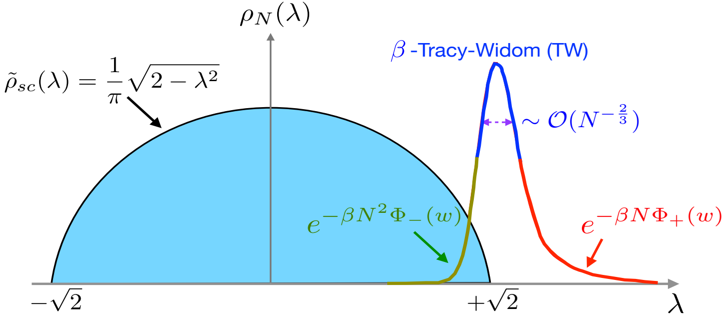

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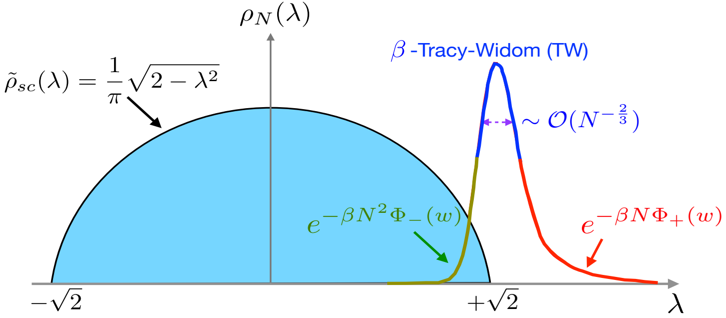

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Similarly, other observables are also known $\Longrightarrow$ huge literature

P. J. Forrester, "Log-gases and Random Matrices" (Priceton Univ. Press, 2010)

S.M. & G. Schehr, "Top eigenvalue of a random matrix: large deviations and third order phase transition", J. Stat. Mech. P01012 (2014)

Strongly correlated gas

in a Nonequilibrium Stationary State

Two major challenges in Nonequilibrium systems

- Unlike in equlibrium systems, the stationary state, if it exists, is determined by the dynamics itself that typically violates time-reversal symmetry (detailed balance)

$\Longrightarrow$ The joint distribution $P_{\rm st}(x_1, x_2, \ldots, x_N)$ in the nonequilibrum stationary state is not given by Gibb's measure and is typically very hard to obtain explicitly

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- Even if one can determine $P_{\mathrm{st}}(x_1, x_2, \ldots, x_N)$ explicitly, computing observables such as average density, extreme/order statistics, gap statistics, full counting statistics etc. are typically very hard due to the presence of strong correlations

In search of interacting many-body systems

- Are in a Nonequilibrium Stationary State

- Are Strongly Correlated

- Are still Exactly Solvable for different physical observables:
    - average density
    - extreme value statistics
    - gap statistics
    - full counting statistics

Strongly correlated gas in a Nonequilibrium Stationary State

Strongly correlated gas

in a Nonequilibrium Stationary State

generated by

Stochastic Resetting

Stochastic Resetting $\Longrightarrow$ explosion of activities

- Optimization of random search algorithms
- Diffusion processes
- Enzymatic reactions in biology (Michaelis-Menten reaction)
- Lévy flights, Lévy walks, fractional BM with resetting
- Space-time dependent resetting rate $r(x, t)$
- Search via nonequilibrium reset dynamics vs. equilibrium dynamics
- Resetting dynamics of extended systems
- Memory dependent reset
- Quantum dynamics with reset
- Active particles with reset
- Cost of resetting
- Optimal strategy for animal movements and navigations

... $\Longrightarrow$ a long list **!**

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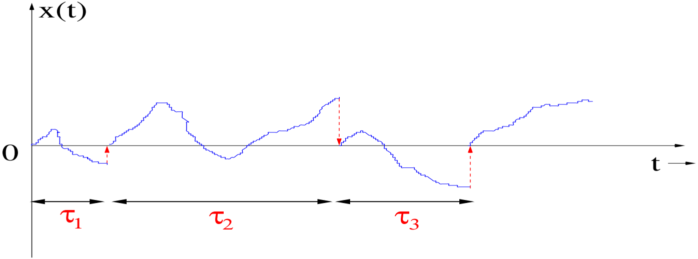
Reviews: "Stochastic resetting and applications",

M.R. Evans, S.M., & G. Schehr, J. Phys. A. : Math. Theor. 53, 193001 (2020)

"The inspection paradox in stochastic resetting",

A. Pal, S. Kostinski & S. Reuveni, J. Phys. A. : Math. Theor. 55, 021001 (2022)

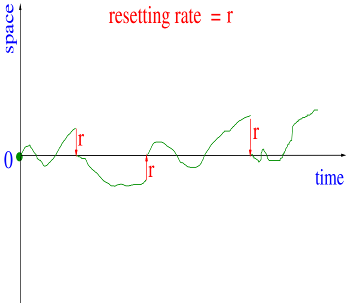
Stochastic Resetting in a nutshell



- Natural dynamics $\Longrightarrow$ deterministic/stochastic/classical/quantum
- Resetting at random times and then natural dynamics restarts afresh
- Intervals $\{\tau_1,\, \tau_2,\, \tau_3,\, \ldots\}$ between resettings $\Longrightarrow p(\tau)$ independently

  $\Longrightarrow$ renewal process
- If $p(\tau) = r\, e^{-r\,\tau} \Longrightarrow$ Poissonian resetting

Simplest Ex: Diffusion with stochastic resetting

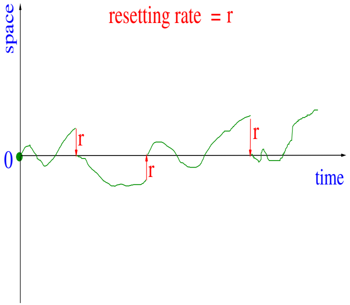


Poissonian resetting

Time intervals between successive resettings distributed as:

$$p(\tau) = r\, e^{-r\tau}$$

Simplest Ex: Diffusion with stochastic resetting



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Dynamics: In a small time interval Δt

$$x(t+\Delta t) = 0 \qquad \text{with prob. } r\Delta t \qquad \text{(resetting)}$$

$$= x(t) + \eta(t)\,\Delta t \qquad \text{with prob. } 1 - r\Delta t \qquad \text{(diffusion)}$$

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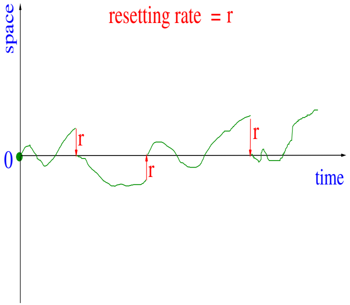

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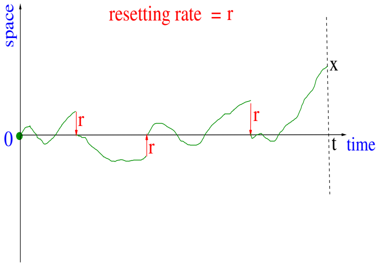
$\eta(t) \to$ Gaussian white noise: $\langle \eta(t) \rangle = 0$ and $\langle \eta(t)\eta(t') \rangle = 2\,D\,\delta(t-t')$

[M.R. Evans & S.M., PRL, 106, 160601 (2011)]

Prob. density $p_r(x,t)$ with resetting rate $r>0$



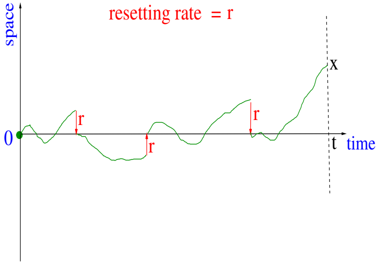
$p_r(x,t) \to$ prob. density at time t,

given $p_r(x,0)=\delta(x)$

- In the absence of resetting ($r=0$):

$$p_0(x,t) = \frac{1}{\sqrt{4\pi\, D\, t}}\, e^{-\frac{x^2}{4Dt}}$$

Prob. density $p_r(x,t)$ with resetting rate $r > 0$



$p_r(x,t) \to$ prob. density at time t,

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- In the absence of resetting ($r = 0$):

$$p_0(x,t) = \frac{1}{\sqrt{4\pi\, D\, t}}\, e^{-\frac{x^2}{4Dt}}$$

- In the presence of resetting ($r > 0$):

$$p_r(x,t) = ?$$

Fokker-Planck (Master) Equation

Fokker-Planck Equation:

$$\partial_t p_r(x,t) = D\,\partial_x^2 p_r(x,t) - r\,p_r(x,t) + r\,\delta(x)$$

Initial condition: $p_r(x,0) = \delta(x)$

Fokker-Planck (Master) Equation

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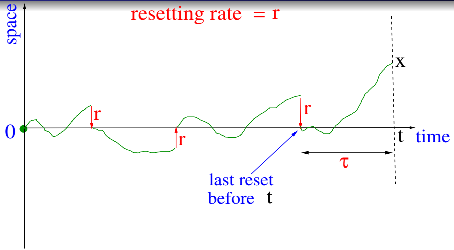
Initial condition: $p_r(x,0) = \delta(x)$

This linear equation can be solved at all t exactly by Fourier transform

Exact solution valid at all times t



- Exact solution at all times t:

$$p_r(x,t) = e^{-r\,t}\, p_0(x,t) + \int_0^t d\tau\, (r\, e^{-r\,\tau})\, p_0(x,\tau)$$

where $p_0(x,\tau) = \text{diffusion propagator} = \frac{1}{\sqrt{4\pi\, D\, \tau}}\, \exp[-x^2/4D\tau]$

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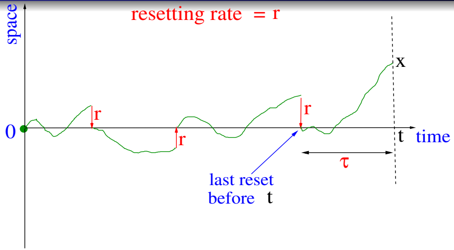

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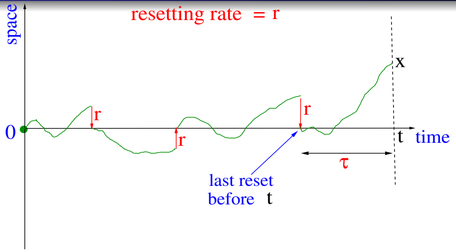
Renewal interpretation: $\tau \to$ time since the last resetting during which $\Longrightarrow$ free diffusion

Exact solution valid at all times t



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Renewal interpretation: $\tau \to$ time since the last resetting during which $\Longrightarrow$ free diffusion

- As $t \to \infty$, $p_r^{\rm st}(x) = r\, \int_0^\infty p_0(x,\tau)\, e^{-r\,\tau}\, d\tau = \frac{\alpha_0}{2}\, \exp[-\alpha_0\, |x|]$

where $\alpha_0 = \sqrt{r/D}$

Stationary State

Exact solution $\rightarrow$ $\boxed{p_r^{\mathrm{st}}(x) = \frac{\alpha_0}{2}\,\exp[-\alpha_0\,|x|]}$ with $\alpha_0 = \sqrt{r/D}$

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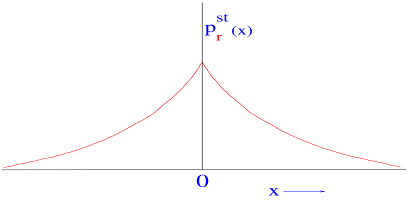

$\rightarrow$ nonequilibrium stationary state (NESS)

$\Rightarrow$ current carrying with detailed balance $\rightarrow$ violated

$p_r^{\rm st}(x) = \alpha_0\,\exp[-V_{\rm eff}(x)]$

effective potential: $\alpha_0|x|$

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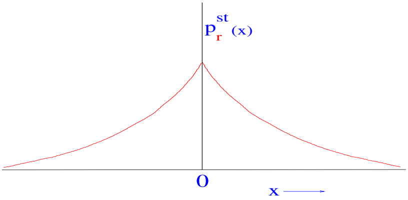

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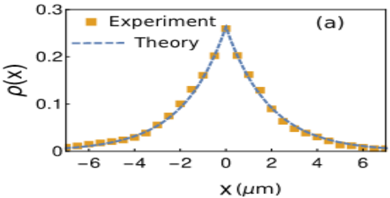

Experimental verificaion using holographic optical tweezers

Tal-Friedman, Pal, Sekhon, Reuveni, & Roichman

J. Phys. Chem. Lett. 11, 7350 (2020)

Optical Trap experiments on Stochastic Resetting

Besga, Bovon, Petrosyan, S.M., Ciliberto, Phys. Rev. Res. 2, 032029 (2020) $\longrightarrow$ 1-dimension

Faisant, Besga, Petrosyan, Ciliberto, S.M., J. Stat. Mech. 113203 (2021) $\longrightarrow$ 2-dimension

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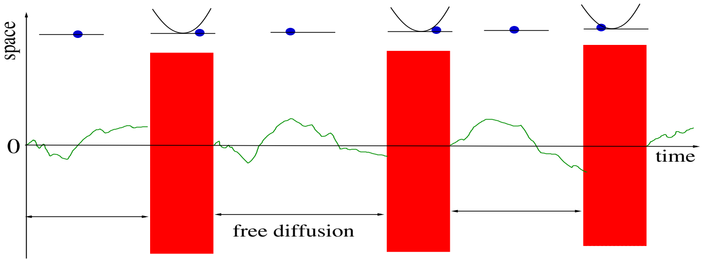
Experimental protocol for a single Brownian (colloidal) particle:

1. Free diffusion for a certain period (deterministic or random)
2. Switch on an optical harmonic trap and the let the particle relax to its equilibrium distribution using Engineered Swift Equilibration (ESE) technique $\Longrightarrow$ mimics instantaneous resetting

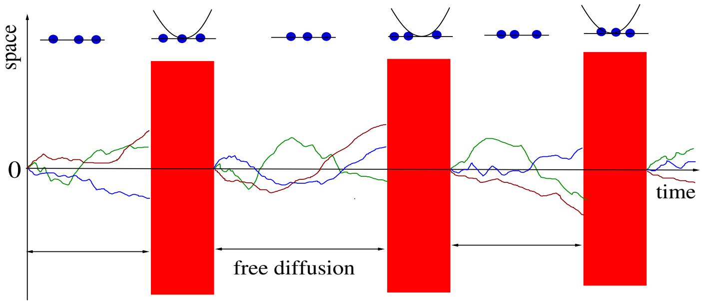
Steps 1 and 2 alternate ...



N non-interacting Brownian particles

1. Free diffusion of N noninteracting particles during an exponentially distributed period
2. Switch on an optical harmonic trap and the let the particles relax to their equilibrium distribution $\Longrightarrow$ mimics instantaneous resetting

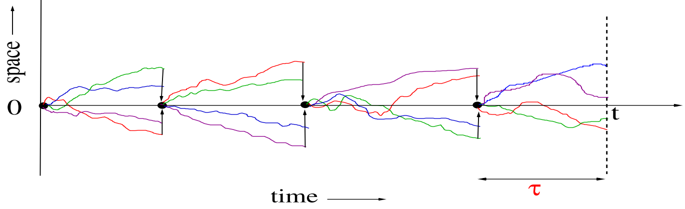
Steps 1 and 2 alternate ...



Resetting Brownian Gas

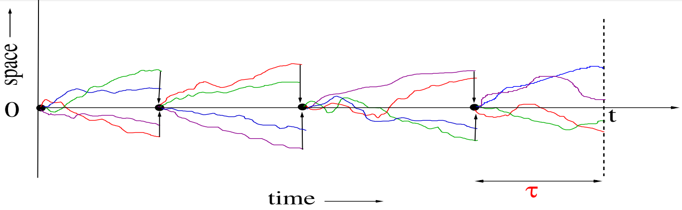
$\Longrightarrow$ **A simple model**

A simple model $\longrightarrow$ Correlated resetting gas



Consider N Brownian motions (independent) that are **simultaneously** reset with rate r to the origin

A simple model $\longrightarrow$ Correlated resetting gas



Consider N Brownian motions (independent) that are **simultaneously** reset with rate r to the origin

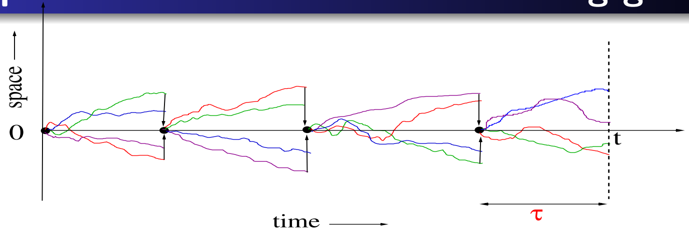
Joint distribution at any time t:

$$P_r(\{x_i\}, t) = e^{-r\,t} \prod_{i=1}^{N} p_0(x_i, t) + r \int_0^t d\tau\, e^{-r\,\tau} \prod_{i=1}^{N} p_0(x_i, \tau)$$

where $p_0(x, \tau) = \frac{1}{\sqrt{4\pi D\tau}}\, e^{-x_i^2/4D\tau}$

M. Biroli, H. Larralde, S. M., G. Schehr, PRL, 130, 207101 (2023)

A simple model $\longrightarrow$ Correlated resetting gas



Consider N Brownian motions (independent) that are **simultaneously** reset with rate r to the origin

The joint position distribution approaches a nonequilibrium stationary state (NESS) at long times $t \to \infty$

$$P_r^{\rm st}(\{x_i\}) = r \int_0^\infty d\tau\, e^{-r\,\tau} \prod_{i=1}^{N} \frac{1}{\sqrt{4\pi D\tau}}\, e^{-x_i^2/4D\tau}$$

The joint distribution does not **factorize** $\Longrightarrow$ **correlated** resetting gas

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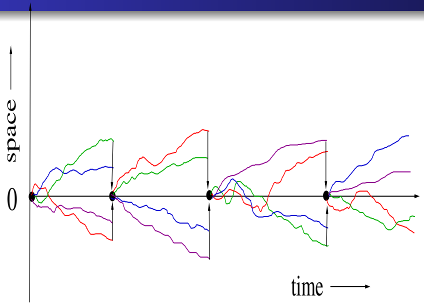
Solvable Correlated Gas



Joint distribution:

$$P_r^{\rm st}(\{x_i\}) = r\int_0^\infty d\tau\, e^{-r\,\tau} \prod_{i=1}^{N} p_0(x_i,\tau)$$

$$p_0(x,\tau) = \frac{1}{\sqrt{4\pi D\tau}}\, e^{-x_i^2/4D\tau}$$

In this model, interactions between particles are **not built-in**, but the correlations are generated by the dynamics (**simultaneous resetting**), that persist all the way to the stationary state

$\longrightarrow$ **dynamically emergent correlations**

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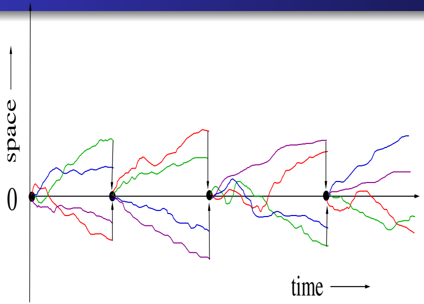

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$\longrightarrow$ **dynamically emergent correlations**

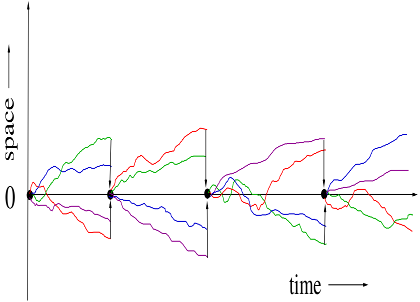
The gas is **strongly** correlated in the NESS

For any pair $i \neq j$:

While $\langle x_i x_j\rangle - \langle x_i\rangle\langle x_j\rangle = 0$ by symmetry

$\langle x_i^2 x_j^2\rangle - \langle x_i^2\rangle\,\langle x_j^2\rangle = 4\frac{D^2}{r^2} \implies$ **attractive all-to-all** interaction

Solvable Correlated Gas



Joint distribution:

$$P_r^{\rm st}(\{x_i\}) = r\int_0^\infty d\tau\, e^{-r\tau}\prod_{i=1}^{N} p_0(x_i,\tau)$$

$$p_0(x,\tau) = \frac{1}{\sqrt{4\pi D\tau}}\, e^{-x_i^2/4D\tau}$$

The stationary joint distribution has a **CIID** structure $\Longrightarrow$ Solvable

$$P_r^{\rm st}(x_1,x_2,\ldots,x_N) = \int_{-\infty}^{\infty} du\, h(u)\prod_{i=1}^{N} p(x_i|u)$$

CIID $\Longrightarrow$ Conditionally Independent and Identically Distributed

Solvable Correlated Gas

Joint distribution:

$$P_r^{\rm st}(\{x_i\}) = r \int_0^\infty d\tau\, e^{-r\,\tau} \prod_{i=1}^{N} \frac{1}{\sqrt{4\pi D\tau}}\, e^{-x_i^2/4D\tau}$$

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Despite the presence of **strong correlations**, several physical observables can be computed **exactly** in the NESS due to the **CIID** structure

- Compute any observable for the ideal gas $\Rightarrow$ I.I.D variables with distribution $p_0(x,\tau)$ parametrized by $\tau \Longrightarrow$ easy
- Average over the **random** parameter τ using $p(\tau) = r\, e^{-r\tau}$

Solvable Correlated Gas

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Examples:

- Average density
- Distribution of the k-th maximum: Order statistics
- Spacing distribution
- Full Counting Statistics

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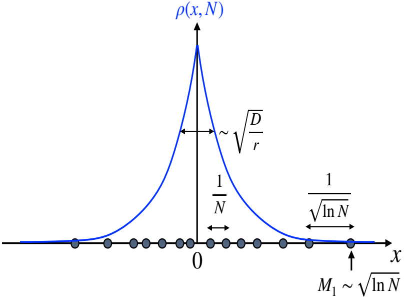
Explicit Results

Average Density



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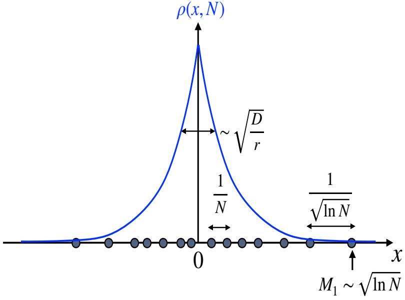
Average density:

$$\rho(x,N) = \frac{1}{N}\sum_{i=1}^{N}\langle\delta(x_i - x)\rangle = \int P_r^{\rm st}\left(x,\, x_2,\ldots,x_N\right)\, dx_2\, dx_3 \ldots dx_N$$

Average Density



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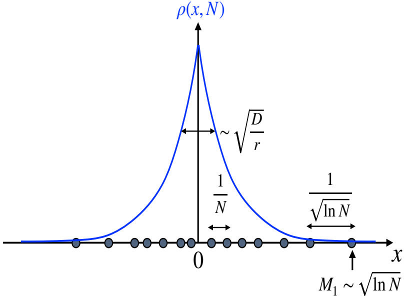
$$= r\int_0^\infty d\tau\, e^{-r\tau}\, p_0(x,\tau) = \frac{\alpha_0}{2}\,\exp[-\alpha_0\,|x|]$$

where $\alpha_0 = \sqrt{r/D}$

$\Longrightarrow$ same as the single particle position distribution

Order Statistics



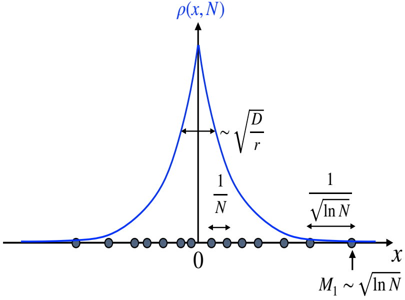
$M_k \Longrightarrow k$-th maximum

Set $k = \alpha\, N$

$\alpha \sim O(1) \Longrightarrow$ **bulk**

$\alpha \sim O(1/N) \Longrightarrow$ **edge**

Order Statistics



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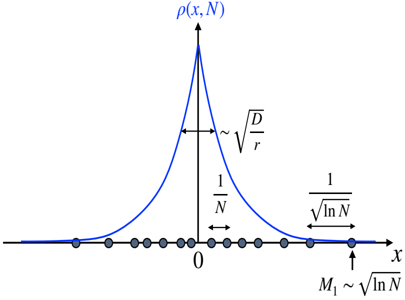
- Bulk: $\mathrm{Prob.}[M_k = w] \approx \frac{1}{\Lambda(\alpha)}\, f\left(\frac{w}{\Lambda(\alpha)}\right)$ where $\Lambda(\alpha) = \sqrt{\frac{4D}{r}}\, \mathrm{erfc}^{-1}(2\,\alpha)$

Order Statistics



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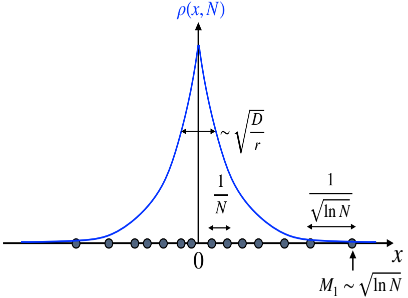
- Edge: $\mathrm{Prob.}[M_k = w] \approx \frac{1}{L_N}\, f\left(\frac{w}{L_N}\right)$ where $L_N = \sqrt{\frac{4D\, \ln N}{r}}$

Order Statistics



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The scaling function $\mathbf{f(z) = 2\,z\,e^{-z^2}\,\theta(z)} \Longrightarrow$ **universal** (indep. of α)

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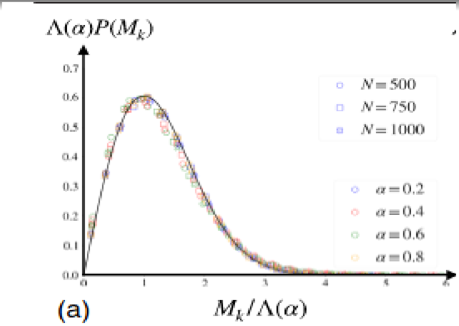
Order Statistics



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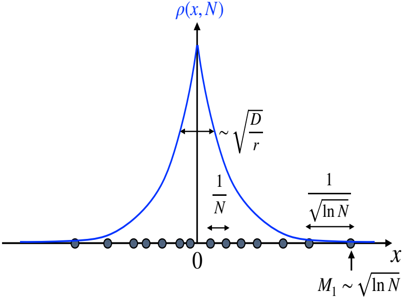
Gap/Spacing Statistics



$M_k \Longrightarrow k$-th maximum

k-th gap: $d_k = M_k - M_{k+1}$

Set $k = \alpha N$

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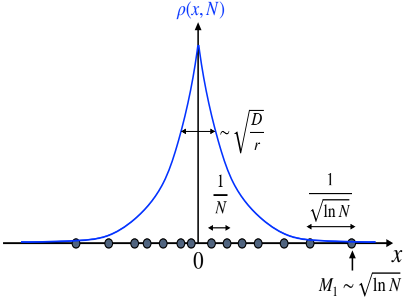
Gap/Spacing Statistics



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k-th gap: $d_k = M_k - M_{k+1}$

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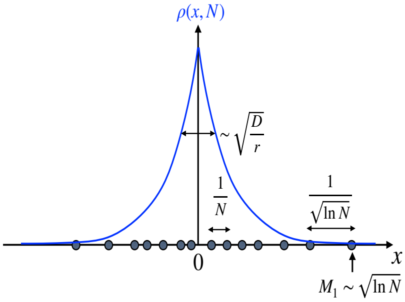
- Bulk: $\mathrm{Prob.}[d_k = g] \approx \frac{1}{\lambda_N(\alpha)}\, h\left(\frac{g}{\lambda_N(\alpha)}\right)$ where $\lambda_N(\alpha) = \frac{1}{b\sqrt{r}\,N}$ with $b = \exp\left(-[\mathrm{erfc}^{-1}(2\,\alpha)]^2\right)/\sqrt{4\pi D}$

Gap/Spacing Statistics



$M_k \Longrightarrow k$-th maximum

k-th gap: $d_k = M_k - M_{k+1}$

Set $k = \alpha N$

$\alpha \sim O(1) \Longrightarrow$ **bulk**

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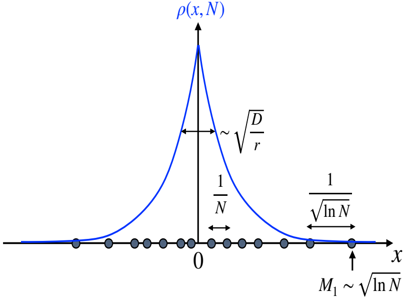

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The scaling function $h(z) = 2\int_0^\infty du\, e^{-u^2 - z/u}$ $\quad (z \geq 0)$

$\Longrightarrow$ **universal** (indep. of α)

M. Biroli, H. Larralde, S. M., G. Schehr, PRL, 130, 207101 (2023)

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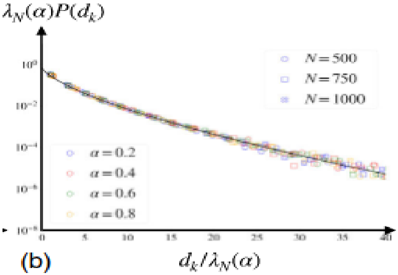

The gap scaling function:

$h(z) = 2\int_0^\infty du\, e^{-u^2 - z/u}$

$h(z) \to \sqrt{\pi}$ as $z \to 0$

$h(z) \sim \exp[-3(z/2)^{2/3}]$ as $z \to \infty$

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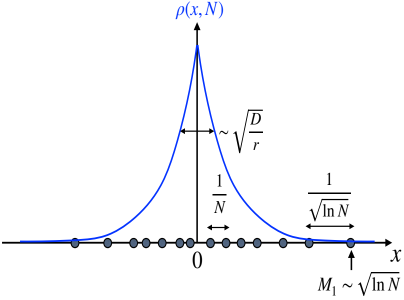
Full Counting Statistics

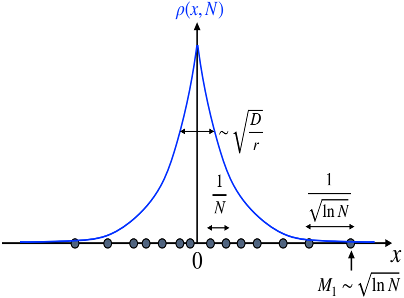


$N_L \Longrightarrow$ number of particles in $[-L, L]$

Clearly, $0 \leq N_L \leq N$

$P(N_L, N) = ?$

Full Counting Statistics



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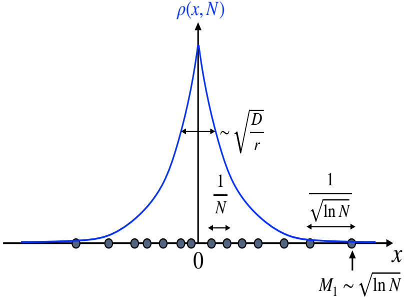

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where the scaling function:

$$H(\kappa) = \gamma\sqrt{\pi}\,[u(\kappa)]^{-3}\,\exp\left[-\gamma\, u^{-2}(\kappa) + u^2(\kappa)\right]$$

with $\gamma = r\,L^2/(4D)$ and $u(\kappa) = \mathrm{erf}^{-1}(\kappa)$

M. Biroli, H. Larralde, S. M., G. Schehr, PRL, 130, 207101 (2023)

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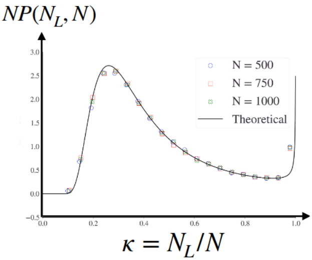

The scaling function $H(\kappa)$

$$H(\kappa) \to \frac{8\gamma}{\pi\,\kappa^3}\ \exp\left[-\frac{4\gamma}{\pi\,\kappa^2}\right] \quad \text{as } \kappa \to 0$$

$$H(\kappa) \to \frac{\gamma\,\sqrt{\pi}}{(1-\kappa)\,[\ln(1-\kappa)]^{3/2}} \quad \text{as } \kappa \to 1$$

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M. Biroli, H. Larralde, S. M., G. Schehr, PRL, 130, 207101 (2023)

Generalisations

The structure of the joint distribution for N independent particles driven by simultaneous resetting is very general:

$$P_r^{\rm st}\left(\{x_i\}\right) = r \int_0^\infty d\tau\, e^{-r\,\tau} \prod_{i=1}^{N} p_0(x_i, \tau)$$

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Ex: ballistic motion, Lévy flights etc.

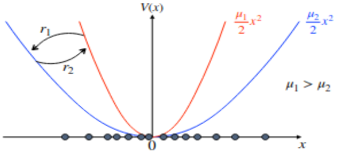
$\Longrightarrow$ a whole class of **solvable** correlated gases in their nonequilibrium stationary state

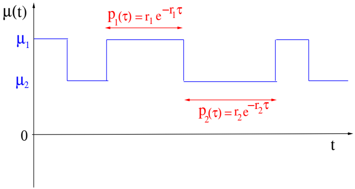
$\Longrightarrow$ a new application of stochastic resetting

M. Biroli, H. Larralde, S. M., G. Schehr, Phys. Rev. E **109**, 014101 (2024)

Other models with CIID stationary state

N Brownian particles in a switching harmonic trap





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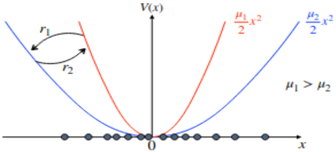

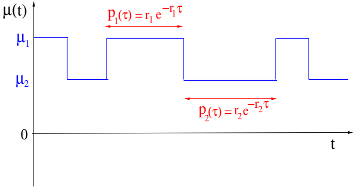

$$\frac{dx_i}{dt} = -\mu(t)\, x_i + \sqrt{2D}\, \eta_i(t)$$

$\eta_i(t) \longrightarrow$ Gaussian white noise with zero mean and correlator $\langle \eta_i(t)\eta_j(t')\rangle = \delta_{i,j}\, \delta(t-t')$

The stiffness $\mu(t)$ of the harmonic trap changes from $\mu_1 \to \mu_2 < \mu_1$ with rate r_1 and $\mu_2 \to \mu_1$ with rate $r_2 \Longrightarrow$ dichotomous telegraphic noise

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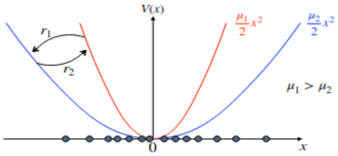

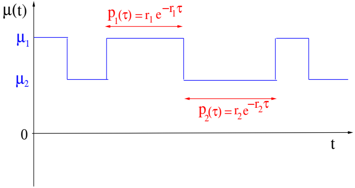

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$\Longrightarrow$ drives the system into a correlated NESS with a stationary joint distribution $P(x_1, x_2, \ldots, x_N, t \to \infty) = P(\vec{x}, t \to \infty) = ?$

Biroli, Kulkarni, S.M., Schehr, PRE, 109, L032106 (2024)

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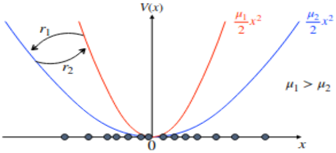

The limit

$\mu_1 \to \infty$, $\mu_2 \to 0$,
$r_1 \to \infty$ and $r_2 = r$

$\Longrightarrow$ simultaneous resetting model

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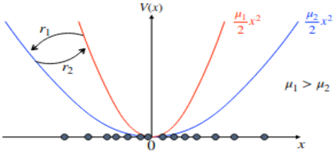

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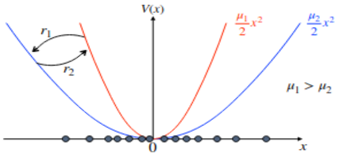

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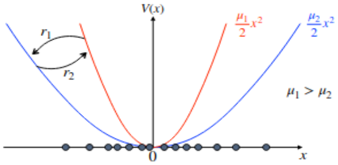
They satisfy a pair of coupled Fokker-Planck equations:

$$\partial_t P_1 = D \sum_{i=1}^{N} \partial_{x_i}^2 P_1(\vec{x}, t) + \mu_1 \sum_{i=1}^{N} \partial_{x_i} (x_i P_1) - r_1 P_1 + r_2 P_2$$

$$\partial_t P_2 = D \sum_{i=1}^{N} \partial_{x_i}^2 P_2(\vec{x}, t) + \mu_2 \sum_{i=1}^{N} \partial_{x_i} (x_i P_2) - r_2 P_2 + r_1 P_1$$

with initial conditions: $P_1(\vec{x}, 0) = \frac{1}{2}\delta(\vec{x})$ and $P_2(\vec{x}, 0) = \frac{1}{2}\delta(\vec{x})$

Exact stationary solution



Fourier transforms:

$$\tilde{P}_{1,2}(\vec{k}, t) = \int P_{1,2}(\vec{x}, t)\, e^{i\, \vec{k}\cdot\vec{x}}\, d\vec{x}$$

Rotational symmetry

$$\Longrightarrow \tilde{P}_{1,2}(\vec{k}, t) = \tilde{P}_{1,2}(k, t)$$

where
$k^2 = k_1^2 + k_2^2 + \ldots + k_N^2$

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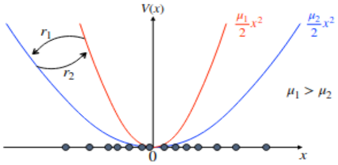

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$$\tilde{P}_1(k) = \frac{r_2}{r_1+r_2}\, e^{-D\,k^2/(2\mu_1)}\, M\left(R_1, 1+R_1+R_2, -\frac{D\,k^2\,(\mu_1-\mu_2)}{2\,\mu_1\,\mu_2}\right)$$
$$\tilde{P}_2(k) = \frac{r_1}{r_1+r_2}\, e^{-D\,k^2/(2\mu_2)}\, M\left(R_2, 1+R_1+R_2, -\frac{D\,k^2\,(\mu_2-\mu_1)}{2\,\mu_1\,\mu_2}\right)$$

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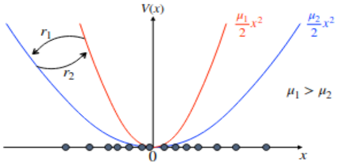

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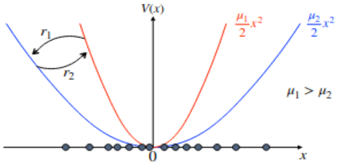

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Not **obvious** if the inverse Fourier transform $P(\vec{x})$ has a CIID structure

Hidden CIID structure of the stationary state

Using an integral representation of $M(a, b, z)$ one can express

$$\tilde{P}_1(k) = A_1 \int_0^1 du\, u^{R_1-1}\,(1-u)^{R_2}\, e^{-V(u)\,k^2/2}$$

where $A_1 = \frac{r_2}{r_1+r_2}\,\frac{\Gamma(1+R_1+R_2)}{\Gamma(R_1)\,\Gamma(1+R_2)}$ and $V(u) = D\left(\frac{u}{\mu_2} + \frac{1-u}{\mu_1}\right)$

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$$\boxed{P(\vec{x}) = \int_0^1 du\, h(u) \prod_{i=1}^{N} p(x_i|u)}$$

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Since $\int_0^1 h(u)du = 1$, the function $h(u)$ can be interpreted as the PDF of the random variable $u \in [0,1] \longrightarrow$ the fraction of time each particle spends in μ_2 phase

Biroli, Kulkarni, S.M., Schehr, PRE, 109, L032106 (2024)

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Using the explicit **CIID** structure of the stationary joint PDF

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all observables in the correlated **NESS** can be computed explicitly and they exhibit rich and interesting behaviors

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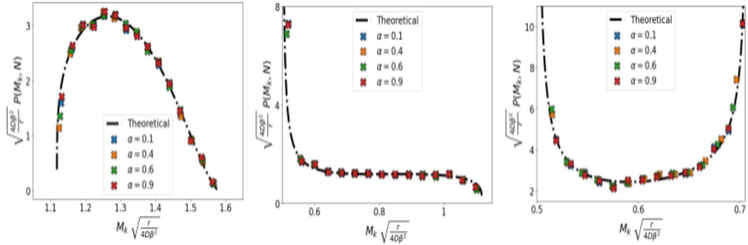
$$P(M_1 = w, N) \to \frac{1}{\sqrt{\ln N}}\, f\left(\frac{w}{\sqrt{\ln N}}\right)$$

where the exact scaling function (with $R_1 = \frac{r_1}{2\mu_1}$ and $R_2 = \frac{r_2}{2\mu_2}$):

$$f(z) = B\, z^3 \left(1 - \frac{z^2}{R_2}\right)^{R_2 - 1} \left(\frac{z^2}{R_1} - 1\right)^{R_1 - 1} \text{ with } \sqrt{R_1} \le z \le \sqrt{R_2}$$

$\longrightarrow$ a new extreme value distribution of **strongly** correlated random variables with a **finite** support

EVS with a finite support: Universality



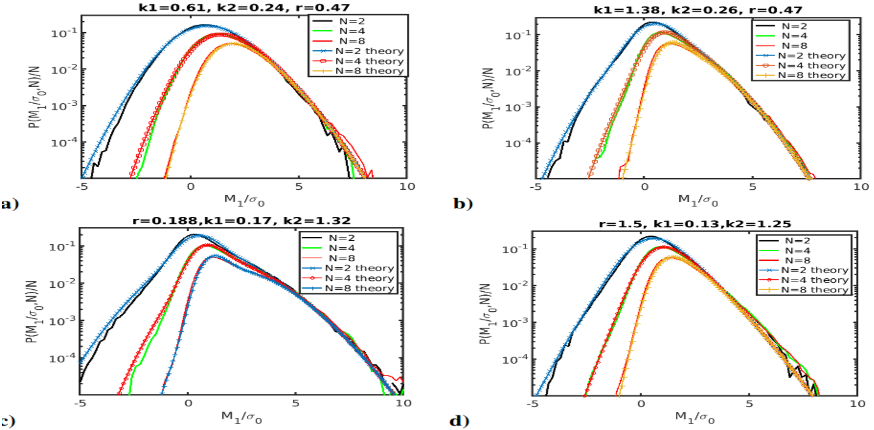
The exact scaling function for the distribution of the scaled k-th maximum M_k

$$f(z) = B\, z^3 \left(1 - \frac{z^2}{R_2}\right)^{R_2 - 1} \left(\frac{z^2}{R_1} - 1\right)^{R_1 - 1} \quad \text{with } \sqrt{R_1} \le z \le \sqrt{R_2}$$

The scaling function $f(z) \longrightarrow$ **universal**, i.e,., same for all M_k's in $d = 1$ and also for all $\mathbf{d} \geq \mathbf{1}$

Biroli, Kulkarni, S.M., Schehr, PRE, 109, L032106 (2024)

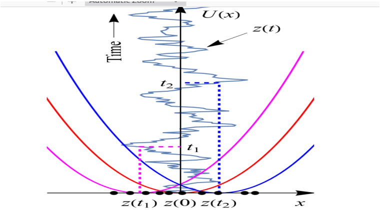
Experimental results:

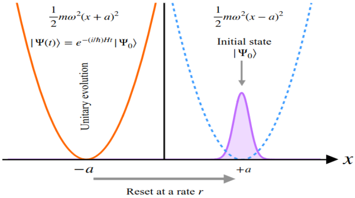


Experiments with a finite number of colloidal particles in an optical trap

$\Longrightarrow$ up to $N = 8$ particles [S. Ciliberto, unpublished data]

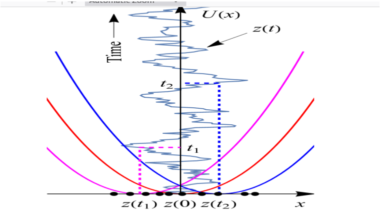
Two other models with CIID structure

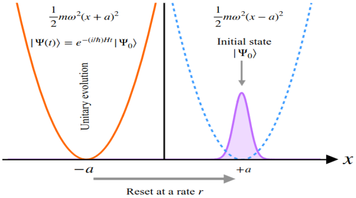




N noninteracting particles (bosons) in a harmonic trap

Two other models with CIID structure



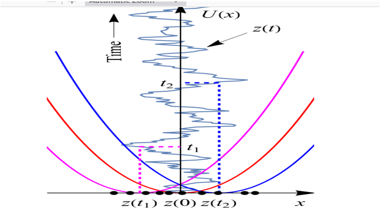


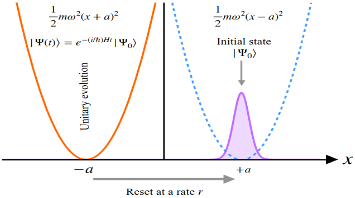
N noninteracting particles (bosons) in a harmonic trap

(1) Model 1 (Classical): The center of the harmonic trap performs a stochastic motion $\Longrightarrow$ drives the system into a correlated NESS

Sabhapandit & S.M. J. Phys. A.: Math. Theor. **57**, 335003 (2024)

Two other models with CIID structure





N noninteracting particles (bosons) in a harmonic trap

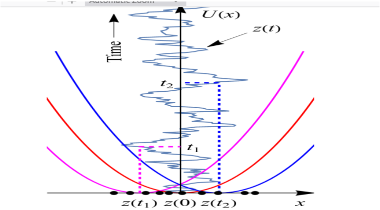
(1) Model 1 (Classical): The center of the harmonic trap performs a stochastic motion $\Longrightarrow$ drives the system into a correlated NESS

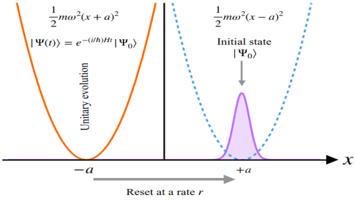
Sabhapandit & S.M. J. Phys. A.: Math. Theor. **57**, 335003 (2024)

(2) Model 2 (Quantum): N noninteracting bosons in the ground state of a harmonic trap whose center is quenched from $+a$ to $-a$, evolves unitarily for a random time and then the state is reset to the ground state with center at $+a$ $\Longrightarrow$ drives the system into a correlated NESS

Kulkarni, S.M. & Sabhapandit & S.M., J. Phys. A: Math. Theor. **58**, 105003 (2025).

Two other models with CIID structure





In both models, the NESS has the **CIID** (conditionally independent and identically distributed) structure

$$P_{\rm st}(x_1, x_2, \ldots, x_N) = \int_{-\infty}^{\infty} du\, h(u) \prod_{i=1}^{N} p(x_i|u)$$

This **CIID** structure makes the problem **solvable** for various observables such as average density, spacing distribution, extreme statistics, full counting statistics etc.

Summary and Conclusion

- A simple solvable model of a correlated gas of N diffusing particles in their nonequilibrium stationary state driven by simultaneous stochastic resetting

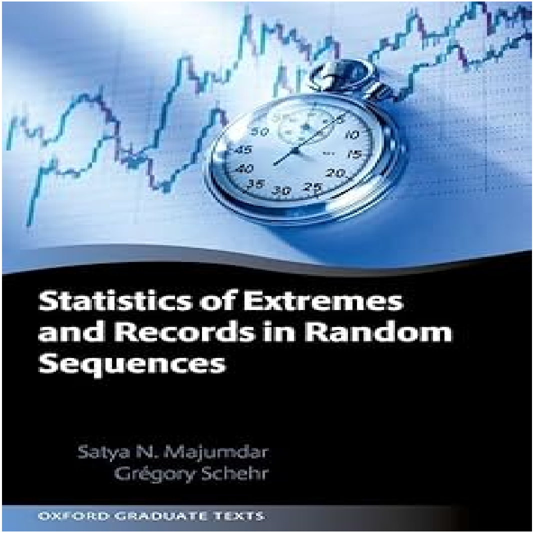
- The NESS has a **CIID** structure

  $\Longrightarrow$ Several physical observables are exactly computable and have rich interesting behaviors, despite being a **strongly correlated** system

- Easily generalisable to a whole new class of solvable correlated gases in their nonequilibrium stationary state $\longrightarrow$ ballistic particles, Lévy flights, particles in a switching harmonic potential etc.

  $\Longrightarrow$ all have this **CIID** structure $\Rightarrow$ Exactly solvable

Biroli, Larralde, S.M., Schehr, PRL, **130**, 207101 (2023); Phys. Rev. E **109**, 014101 (2024); Biroli, Kulkarni, S.M., Schehr, PRE, **109**, L032106 (2024); Sabhapandit, S.M., J. Phys. A: Math. Theor. **57**, 335003 (2024); Kulkarni, S. M. & S. Sabhapandit, J. Phys. A: Math. Theor. **58**, 105003 (2025).


Statistics of Extremes and Records in Random Sequences
Satya N. Majumdar
Grégory Schehr
OXFORD GRADUATE TEXTS