Understanding Search Trees Via Statistical Physics

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Sorting and Search

The Goal: Store data efficiently so that the search time is minimum

Ex: A random sequence of $N = 10$ integers: $\{6, 4, 5, 8, 9, 1, 2, 10, 3, 7\}$

Linear Sorting: Store the data sequentially onto a linear table

$$[6, 4, 5, 8, 9, 1, 2, 10, 3, 7]$$

Search for 7: Search proceeds sequentially by comparison

$$t_{\text{search}} = 10 \sim O(N) \to \text{BAD}$$

Tree Sorting: of $\{6, 4, 5, 8, 9, 1, 2, 10, 3, 7\}$

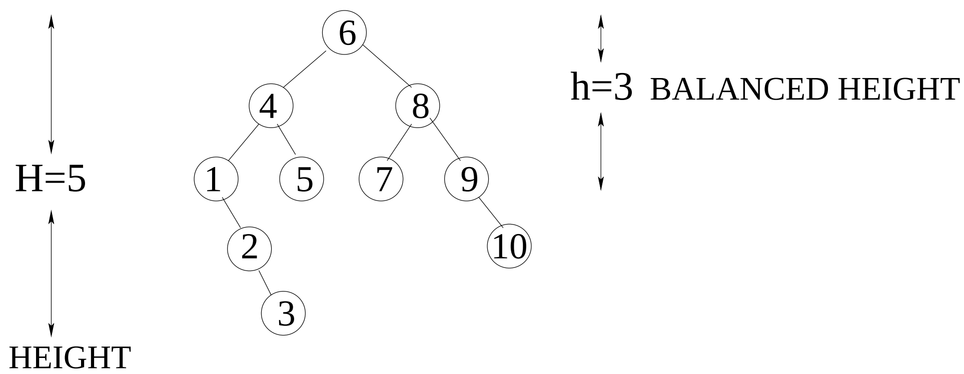


Figure 1: Binary Search Tree with $N = 10$ Elements.

$t_{\text{search}} = \text{Depth} = D$. Roughly $2^D \sim N$ implying: $t_{\text{search}} \sim O(\log N) \rightarrow$ BETTER

- HEIGHT $H = 5$: Distance of the farthest node from the root= Maximum possible time to search an element $\rightarrow$ WORST CASE SCENARIO

- BALANCED HEIGHT $h = 3$: Depth upto which the tree is balanced

Generalization to m-ary Search Trees

$m = 2 \rightarrow$ Binary Tree

Random Sequence: $\{6,\ 4,\ 5,\ 8,\ 9,\ 1,\ 2,\ 10,\ 3,\ 7\}$

Each node can contain atmost $(m-1)$ elements.

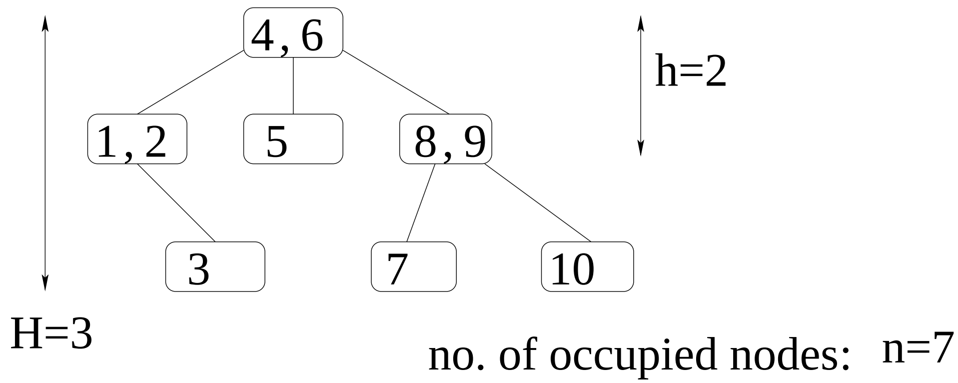


Figure 2: $m = 3$-ary Search Tree with $N = 10$ Elements

$H = 3$ is the HEIGHT. $h = 2$ is the BALANCED HEIGHT.

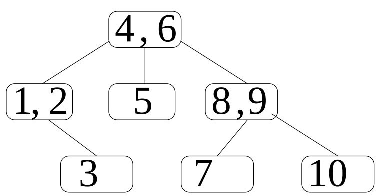
No. of NON-EMPTY nodes: $n = 7 \rightarrow$ No. of nodes required to store the data

Random m-ary Search Tree Model: RmST

$N = 10$ data elements: $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

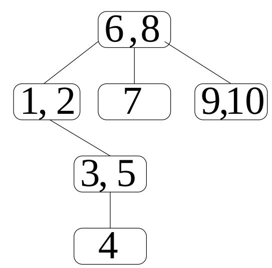
Each permutation $\rightarrow$ an m-ary tree.

{6, 4, 5, 8, 9, 1, 2, 10, 3, 7}



H=3, h=2, n=7

{8, 6, 9, 2, 1, 5, 3, 4, 7, 10}



H=4, h=2, n=6

In the RmST model: All $N!$ permuations are equally likely $\rightarrow$ RANDOM DATA.

Q: Statistics of HEIGHT H_N, BALANCED HEIGHT h_N and the no. of NON-EMPTY NODES n_N for RANDOM data of size N?

Asymptotic Results for RmST: for large data size N

(1) Height H_N:

- $\langle H_N \rangle \approx a_m \log(N) + b_m \log(\log(N))$ (??) $+ \ldots$
- $\mathrm{Var}(H_N) \approx O(1)$

(2) Balanced Height h_N: Depth upto which the tree is balanced.

- $\langle h_N \rangle \approx c_m \log(N) + d_m \log(\log(N))$ (??) $+ \ldots$
- $\mathrm{Var}(h_N) \approx O(1)$

Binary Tree ($m = 2$): $a_2 = 4.31107\ldots$ and $c_2 = 0.3733\ldots$ (Devroye, 87). The correction terms $\rightarrow$ conjectured by Hattori and Ochiai (simulations, 2001).

Other results by Robson (2001), Reed (2001), Drmota (2001-2003).

Asymptotic Results for RmST: for large data size N...continued

(3) No. of NON-EMPTY Nodes n_N: No. of nodes required to store the data of size N.

$$\langle n_N \rangle \approx \alpha_m N + \ldots$$

A striking PHASE TRANSITION occurs for the Variance: $\nu_N = \langle \left(n_N - \langle n_N \rangle\right)^2 \rangle$.

$$\begin{aligned} \nu_N &\sim N && \text{for } m \leq 26 \\ &\sim N^{2\theta(m)} && \text{for } m > 26 \text{ (Chern \& Hwang, 2001).} \end{aligned}$$

Q: Why 26? What is the mechanism of this Phase Transition and how generic is it? Can one calculate $\theta(m)$ exactly ?

Our Results:

• Mapping to a FRAGMENTATION Process $\rightarrow$ Dynamical Process

• Analysis of the FRAGMENTATION process using a variety of statistical physics techniques such as the Travelling Front method (for HEIGHTS and BALANCED HEIGHTS) and a Backward Fokker-Planck approach (for the no. of NON-EMPTY Nodes).

$\rightarrow$ A number of asymptotically EXACT results.

Ex: we calculate the constants a_m, b_m, c_m, d_m EXACTLY for all m as roots of transcendental equations. Scaling Relation between a_m and b_m: $b_2 = -3a_2/[2(a_2-1)]$.

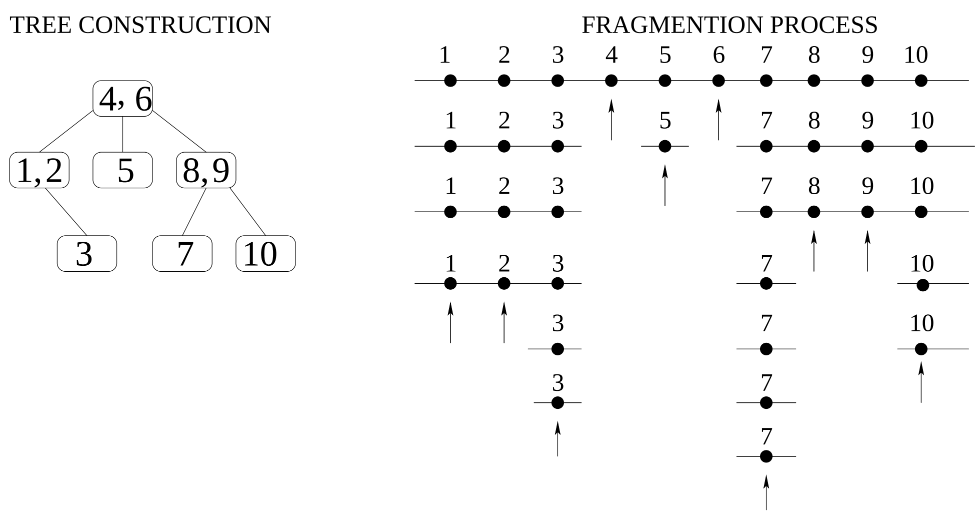
We show that $m_c = 26.0461...$: Find $\lambda(m)$ from $m(m-1)\mathrm{B}(\lambda+1, m-1) = 1$. The critical value m_c is obtained by setting, $Re[\lambda(m) = 1/2]$. For $m > m_c = 26.0461...$, $\theta(m) = \lambda(m)$. (D. Dean and S.M., 2002).

Various other generalizations: Vector Data

The Mapping to a Fragmentation Process

Construction of the Tree $\rightarrow$ Dynamical Fragmention Process: Split an interval into $(m-1)$ pieces with the break points chosen randomly. An interval can split iff it contains atleast one point.

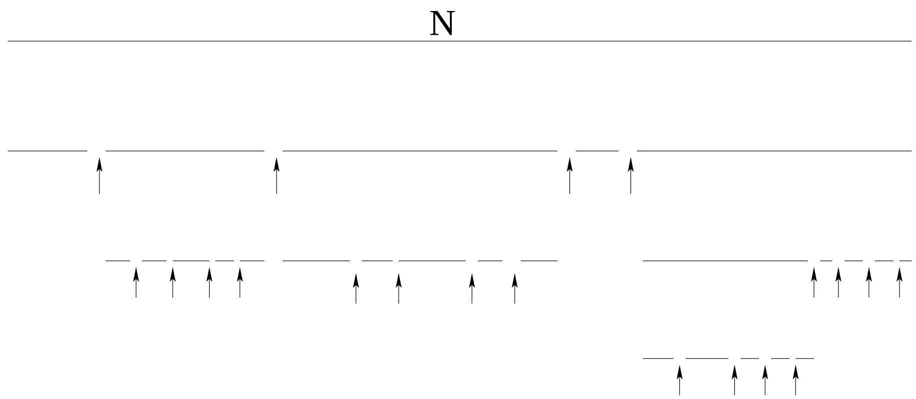
Ex: Consider the data: $\{6, 4, 5, 8, 9, 1, 2, 10, 3, 7\}$



NOTE:

No. of NONEMPTY nodes n=7= No. of SPLITTING EVENTS

Fragmentation Process:



1. Start with a stick of length N.
2. Choose $(m-1)$ break points randomly and split the stick into m pieces.
3. Examine each piece and if its length $> N_0 = 1$, again split it randomly into further m pieces. Stop splitting if length < 1.
4. Repeat the process till all pieces have length < 1 and then STOP.

DICTIONARY Between the Search Tree and the Fragmentation Process:

Height H_N:

- Prob[$H_N < n$]= Prob[$l_1 < 1$, $l_2 < 1$, ... after n steps]

Balanced Height h_N:

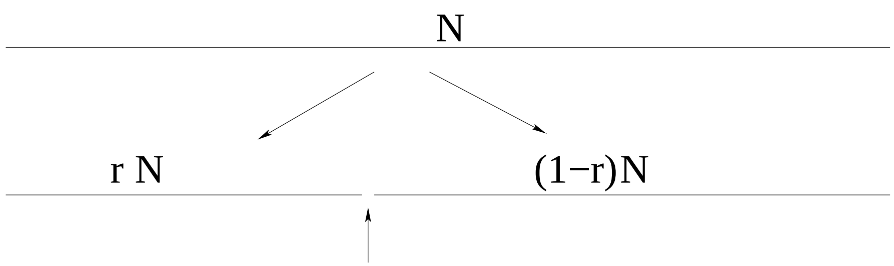
- Prob[$h_N > n$]=Prob[$l_1 > 1$, $l_2 > 1$, ... after n steps]

Number of Nonempty Nodes n_N ($m > 2$):

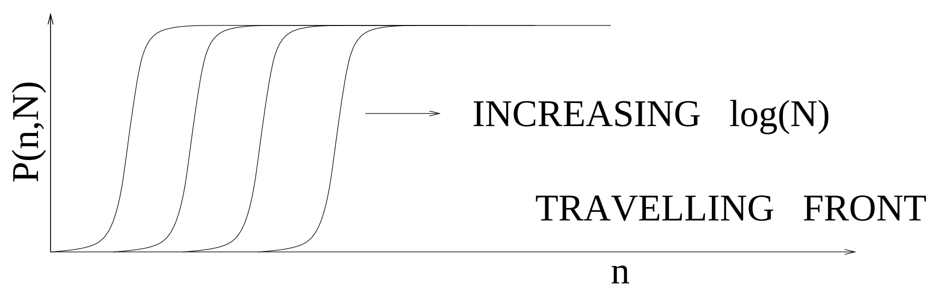
- Prob[$n_N = n$]= Prob[there are n SPILLITING EVENTS till the end of the Fragmentation process].

Analysis of HEIGHT H_N

P(n,N)= Prob[$H_N < n$]= Prob[$l_1 < 1$, $l_2 < 1$, $\ldots$ after n steps]



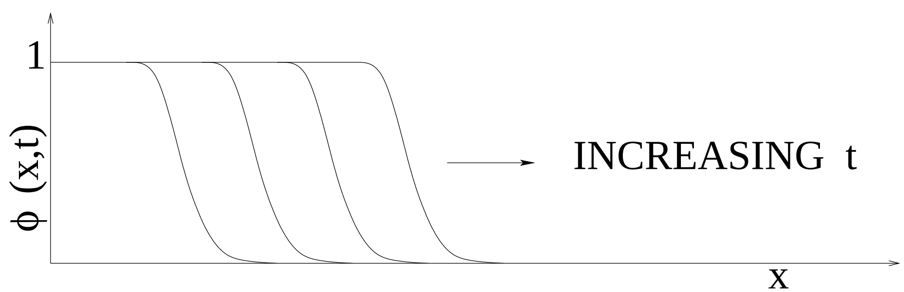
Recursion: $P(n, N) = \int_0^1 P\,(n-1, rN)\,P\,(n-1, (1-r)N)\,dr$ starting with $P(n, 1) = \theta(n-1)$.



Travelling Front in Fisher Equation

$\partial_t \phi(x,t) = \partial_x^2 \phi(x,t) + \phi - \phi^2$.

$\phi(x) = 1 \rightarrow$ STABLE Fixed point. $\phi(x) = 0 \rightarrow$ UNSTABLE Fixed point.

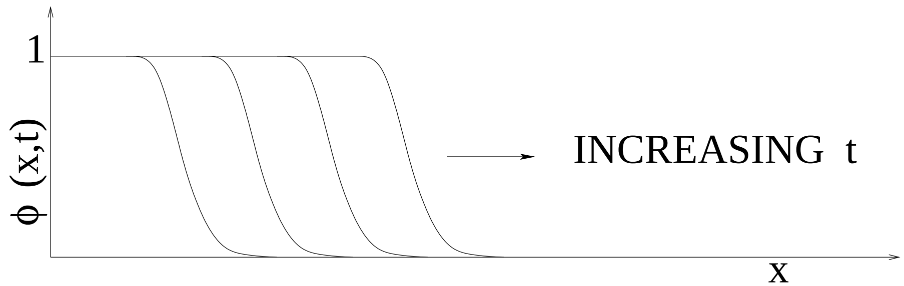


Travelling Front: $\phi(x,t) = f\,(x - x_f(t))$ for large t, where the front position

$$x_f(t) \sim \mathrm{v}\; t + \ldots .$$

Q: How to determine the Front Velocity v?

Kolmogorov's Velocity Selection Principle:



Linearize near the tail $\rightarrow \phi(x,t) \sim \exp[-\lambda(x-vt)]$

DISPERSION RELATION: $v(\lambda) = \lambda + \frac{1}{\lambda}$

$\rightarrow$ minimum at $\lambda^* = 1$. For sharp initial condition, $v = v(\lambda^*) = 2$.

More generally,

$x_f(t) \approx v(\lambda^*)t - \frac{3}{2\lambda^*}\log t + \ldots$ (Bramson, Brunet & Derrida, van Saarloos, ...)

Travelling Front Solution to Search Tree Height:

$P(n,N) = Prob[H_N < n] \approx f[n - n_f(N)]$ asymptotically. $t \equiv \log N \rightarrow$ correct variable.

Linearize near the tail: $P(n,N) \approx 1 - \exp[-\lambda\,(n - v(\lambda))\log N]$

$\rightarrow$ DISPERSION RELATION: $v(\lambda) = \frac{2e^{\lambda}-1}{\lambda}$ for $m = 2$.

Minimize $v(\lambda) \rightarrow \lambda^* = 0.76804\ldots$.

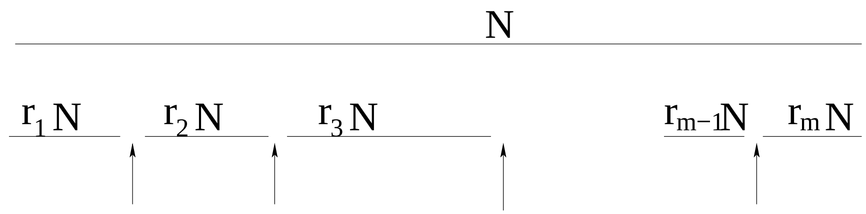
$$\langle H_N \rangle \approx n_f(N) \approx \; v(\lambda^*)\,\log(N) - \frac{3}{2\lambda^*}\log\left(\log(N)\right) + \ldots$$

$\rightarrow$ $a_2 = v(\lambda^*) = 4.31107\ldots$ and $b_2 = -\frac{3}{2\lambda^*} = -1.95303\ldots$

Similarly one gets a_m and b_m for all m.

Same strategy holds for the Balanced Height h_N.

No of Non-Empty Nodes:



$r_1 + r_2 + r_3 + \ldots\ldots + r_m = 1$

No. of Non-empty nodes $n(N)$ in the tree $\equiv$ Total no. of Splitting Events in the fragmentation process till the end, starting with the initial length N

Recursion:

$$n(N) \equiv n(r_1 N) + n(r_2 N) + n(r_3 N) + \cdots + n(r_m N) + 1; \qquad \sum_i^n r_i = 1$$

The marginal distribution of any fragment: $\eta(r) = (m-1)(1-r)^{m-2}$

Integral Equations for Average and Variance:

Average: $\mu(N) = \langle n(N) \rangle$ satisfies an integral equation:

$$\mu(n) = m \int_{1/N}^{1} \mu(rN)\eta(r)dr + 1$$

Variance: $\nu(N) = \langle (n(N) - \mu(N))^2 \rangle$ satisfies another integral equation:

$$\nu(n) = m \int_{1/N}^{1} \nu(rN)\eta(r)dr + \langle (S - \langle S \rangle)^2 \rangle$$

where the Source Function $S = \sum_{i=1}^{n} \mu(r_i N)$.

These integral equations can be solved analytically: for large N,

$$\begin{aligned} \nu_N &\sim N && \text{for } m \leq m_c \\ &\sim N^{2\theta(m)} && \text{for } m > m_c \end{aligned}$$

where m_c is determined as:

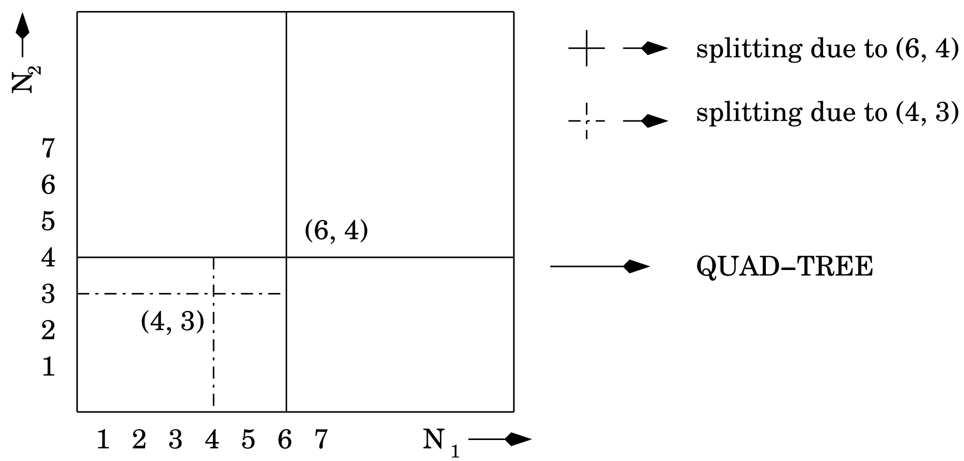
Find $\lambda(m)$ from $m(m-1)\mathrm{B}(\lambda+1, m-1) = 1$. The critical value m_c is obtained by setting, $Re[\lambda(m) = 1/2]$. For $m > m_c = 26.0461...$, $\theta(m) = \lambda(m)$. (D. Dean and S.M., 2002).

Generalization to Vector Data:

Scalar Sequence: $\{6, 4, 5, 8, 9, 1, 2, 10, 3, 7\}$

Vector Sequence: $\{(6, 4), (4, 3), (5, 2), (8, 7) \ldots\} \rightarrow D = 2$ vector.

Mapping to the Fragmentation Process:



Q: What are the statistics of Height H_N, Balanced Height h_N and the no. of Non-empty nodes n_N for a given vector data of N D-tuples?

Is there a PHASE TRANSITION in the variance of n_N?

Exact Results for Vector Data of N D-tuples for Large N:

Height H_N:

- $\langle H_N \rangle \approx 4.31107\ldots \log(N)$ - $\frac{1.95303...}{D} \log\left(D \log(N)\right) + \ldots$

Balanced Height h_N:

- $\langle h_N \rangle \approx 0.37336\ldots \log(N) + \frac{0.89374...}{D} \log\left(D \log(N)\right) + \ldots$

No. of Non-empty Nodes n_N: $\langle n_N \rangle \approx \frac{2}{D} V$ where $V = N^D$.

Variance ν_N has a Phase Transition

$$\begin{aligned} \nu_N &\sim V && \text{for } D \leq D_c \\ &\sim V^{2\theta(D)} && \text{for } D > D_c \end{aligned}$$

$$D_c = \frac{\pi}{\arcsin\left(\frac{1}{\sqrt{8}}\right)} = 8.69362\ldots$$

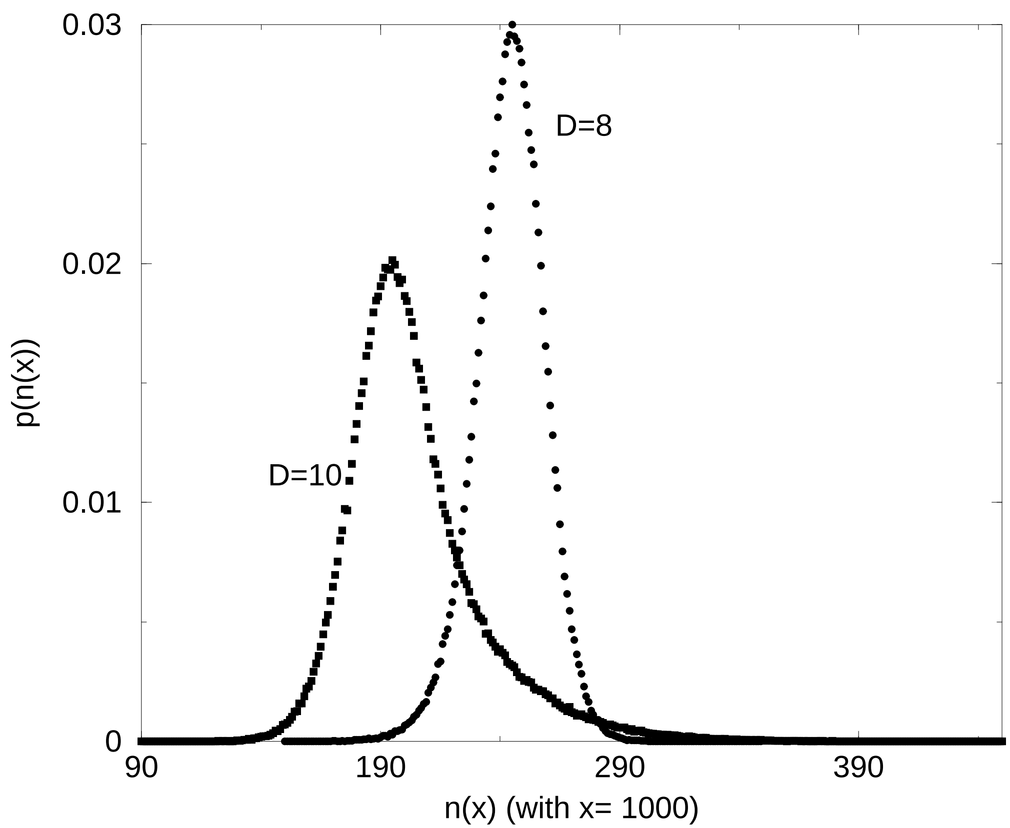
$$\theta(D) = 2\cos\left(\frac{2\pi}{D}\right) - 1 \rightarrow \text{increases continuously with } D$$

for $D > D_c$

Probability Distribution of the no. of Non-Empty Nodes n_V:

$$P[n_V] \to \text{ GAUSSIAN for } D < D_c = 8.69362\ldots$$

$$P[n_V] \to \text{NON-GAUSSIAN for } D > D_c = 8.69362\ldots$$



Summary and Conclusion:

- Analysis of m-ary search trees via techniques of statistical physics → Exact asymptotic results.

- Going beyond Random m-ary search trees...Digital Search Trees.. interesting connections to Diffusion Limited Aggregation (DLA) on the Bethe lattice and also to the Lempel-Ziv Data Compression Algorithm (S.M., 2003).

- Application of the Travelling Front technique in computer science problem.

- A simple mechanism for the peculiar Phase Transition in the fluctuation of the number of non-empty nodes

→ A rather Generic phase transition → New Exact Results for Vector Data.

The same mechanism is also responsible for the phase transition in a Growing Tree Model of Aldous & Shields (1988)...Explicit Results (S.M. and D.S. Dean, 2004).

Perspectives: Lots of beautiful open problems in Sorting and Search that may be possible to handle by using statistical physics techniques.

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