Improved Linear Programming applied to the Vertex Cover Problem

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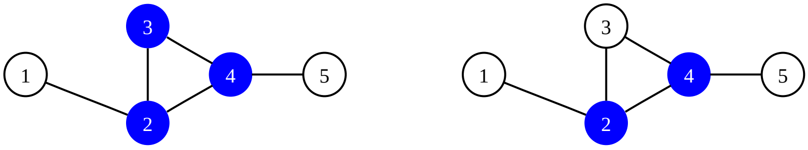
Outline

The Vertex Cover Problem

- Undirected graph $G = (V, E)$ with nodes V and edges E
- $N = |V|$ and $M = |E|$

Definition [GareyTheoCompSc76]:
A Vertex Cover (VC) is a subset $V_C \subset V$ of vertices, such that each edge $\{i, j\} \in E$ is at least incident to one node of V_C
$\rightarrow i \in V_C$ or $j \in V_C$.

Minimum VC: minimum cardinality $X_C = |V_C|$



(a) VC (b) minimum VC

VC Problem $\rightarrow$ NP-hard optimization problem

VC as Linear Programming Problem (LP)

- VC studied in physics with B&B algorithm or stochastic methods → here: Linear Programming
- Each node i of graph is represented by variable $x_i \in [0, 1]$:
  $x_i = 1 \leftrightarrow$ covered
  $x_i = 0 \leftrightarrow$ uncovered
  $x_i \in]0, 1[\leftrightarrow$ undecided
- Each of the M edges $\{j, k\} \rightarrow$ constraint $x_j + x_k \geq 1$
- Objective function: $x \rightarrow$ min

VC as LP:

Minimize $x = \sum_{i=1}^{N} x_i$

Subject to $0 \leq x_i \leq 1 \quad \forall\, i \in V$

$x_j + x_k \geq 1 \quad \forall\, \{j, k\} \in E$

Use Simplex algorithm to solve LP [DantzigBullAmerMathSoc48], [http://lpsolve.sourceforge.net/5.5/].

Example

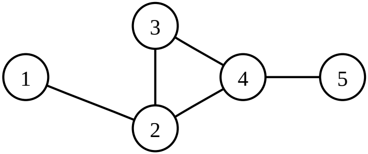


Figure: Example graph with $N = M = 5$

Corresponding LP:

Minimize $x = x_1 + x_2 + x_3 + x_4 + x_5$

Subject to $0 \leq x_i \leq 1 \quad \forall\, i \in V$

$$x_1 + x_2 \geq 1$$
$$x_2 + x_3 \geq 1$$
$$x_2 + x_4 \geq 1$$
$$x_3 + x_4 \geq 1$$
$$x_4 + x_5 \geq 1$$

Example

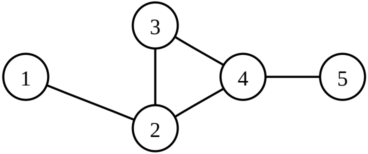


Figure: Example graph with $N = M = 5$

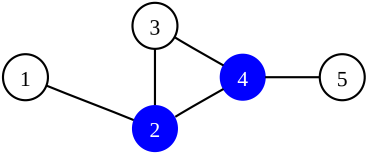


Figure: Minimum VC

Corresponding LP:

Minimize $x = x_1 + x_2 + x_3 + x_4 + x_5$

Subject to $0 \leq x_i \leq 1 \quad \forall\, i \in V$

$$x_1 + x_2 \geq 1$$
$$x_2 + x_3 \geq 1$$
$$x_2 + x_4 \geq 1$$
$$x_3 + x_4 \geq 1$$
$$x_4 + x_5 \geq 1$$

Solution:

$$x_1 = 0,$$
$$x_2 = 1,$$
$$x_3 = 0,$$
$$x_4 = 1,$$
$$x_5 = 0.$$

$\rightarrow$ Minimum VC with cardinality: $X_c = x = 2$

Cutting Plane (CP) approach

Aim: Reduce number of undecided variables $x_i \in]0,1[$

Idea: Limit solution space by adding extra constraints (CPs)

Two algorithms:

Loops: [arXiv:1201.1814v1]

- Search random loop of length l
- Add constraint (CP) to LP:

$$\sum_{i \in \text{loop}} x_i \geq \left\lceil \frac{l}{2} \right\rceil, \quad (*)$$

if loop has odd length and $(*)$ is not fulfilled yet.

Subgraphs:

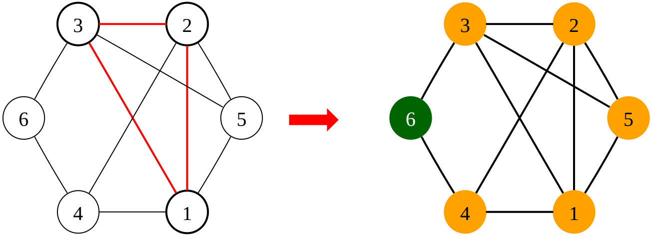
- Search random subgraph $G_S = (U, E_S)$ with $|U| \leq 10$
- Calculate minimum VC of size $X_C = |V_C(G_S)|$
- Add constraint (CP) to LP:

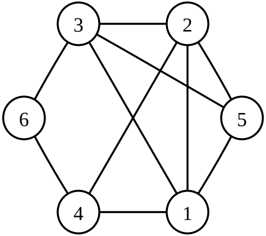
$$\sum_{i \in U} x_i \geq X_C, \quad (\star)$$

if $(\star)$ is not fulfilled yet.

Example for CP approach

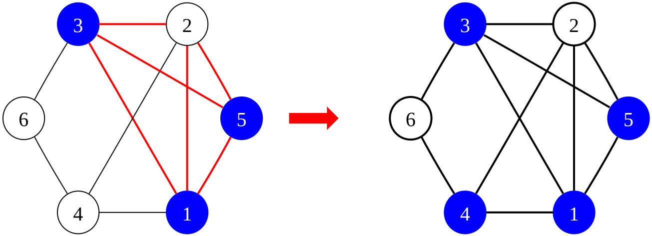
Loops:





$x_i = 2/3$ $x_i = 1/3$ $x_i = 1$

Subgraphs:



Node Heuristic (NH)

Aim: Get complete solution $\rightarrow$ all $x_i \in \{0, 1\}$

Algorithm:

- Set the **smallest** undecided variable $x_j \in]0, 1[$ to zero
- Add $x_j = 0$ to LP and solve it again

$\rightarrow$ Sets variables of adjacent nodes k to $x_k = 1$

$\rightarrow$ Repeated execution yields VC, but not necessarily of minimum size

General remarks

Used graph ensemble:

- Erdős-Rényi (ER) random graph ensemble: $\mathcal{G}(N, M)$ [ErdösMagTudAkMatKuIntKö60]
- All graphs with same N and M equiprobable

Important variables for graphs/VC:

- Connectivity (average number of neighbors): $c = 2\ {}^{M}\!/_{N}$
- Minimum relative cover size $x_c = {}^{X_C}\!/_{N}$

Details of simulations:

- Bland's first-index pivoting [BlandMathOperRes77]
- 10^3 realisations of random graphs
- Graph sizes up to $N = 570$

Phase transition in CP approach

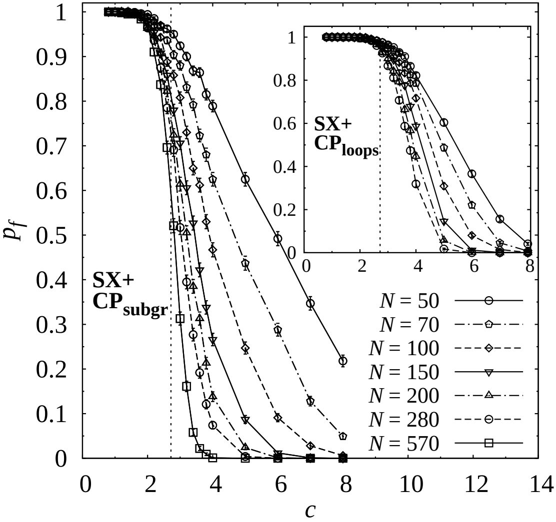


Figure: Fraction p_f of complete solutions for CP approach with subgraphs as a function of connectivity c and for CP approach with loops (inset). Vertical line denotes $c = e \approx 2.718$.

Phase diagram

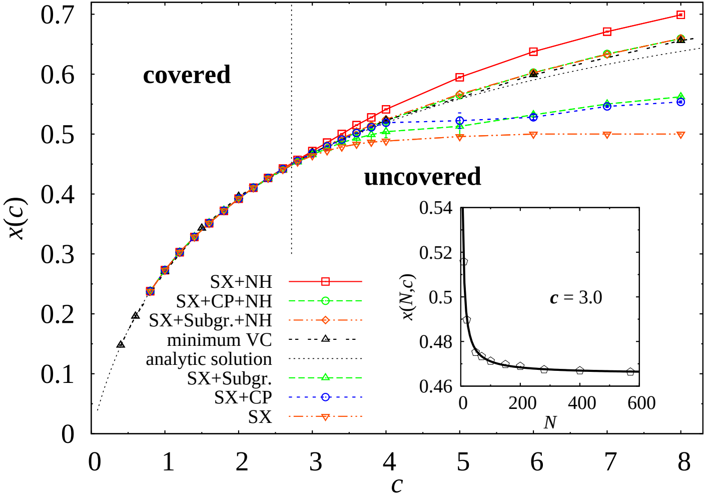


Figure: Phase diagram for the fraction of covered vertices x. Minimum VC found with exact branch-and-bound algorithm/analytics [HartmannPRL00]. Vertical line denotes $c = e \approx 2.718$, where RSB occurs. Inset: Finite-size scaling for CPs with subgraphs and $c = 3$

Summary/Conclusion

- Mapping of VC on ER random graphs on LP
- CP approach shows “easy-hard” transition close to $c = e$

  $\rightarrow$ Phase transition (PT) not only for configuration-space-based algorithms (e.g. branch-and-bound), but also for LP/CP approach (outside of feasible solutions)

  $\rightarrow$ Hardness of VC Problem is intrinsic property of problem

Thank you for your attention!

Announcements

Open access summary database:

www.papercore.org

Modern **C**omputational **S**cience Summerschool

August 20 – 31, 2012:

www.mcs.uni-oldenburg.de

DPG Physics School: *Efficient Algorithms in Computational Physics*, September 9 – 14, 2012:

www.pbh.de

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