The Floquet picture

for strongly driven quantum systems

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PART I:

Basics

- Consider (single particle/many-body) quantum system on $\mathcal{H}$ with time-periodic Hamiltonian

$$H(t) = H(t+T)$$

▷ **TASK:** Solve time-dependent Schrödinger equation

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle = H(t)|\psi(t)\rangle$$

▷ Introduce time-evolution operator

$$|\psi(t)\rangle = U(t,0)|\psi(0)\rangle$$

so that

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} U(t,0) = H(t)U(t,0) \qquad ; \qquad U(0,0) = \mathrm{id}$$

In any case,

$$U(t_1+t_2,0) = U(t_1+t_2,t_1)\, U(t_1,0)$$

▷ But since $H(t) = H(t+T)$, we have even more:

- **Assertion 1:**

  If $H(t) = H(t+T)$ *is periodic in time with period* T *, then* $U(t,0)$ *obeys the identity*

$$U(t+T,0) = U(t,0)U(T,0)$$

▷ **Proof:** Consider

$$V(t) := U(t+T,0)\,U^{-1}(T,0)$$

Then one has $V(0) = \mathrm{id} = U(0,0)$ and

$$\begin{aligned} \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}V(t) &= \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}U(t+T,0)\,U^{-1}(T,0) \\ &= H(t+T)\,U(t+T,0)\,U^{-1}(T,0) \\ &= H(t)\,V(t)\,. \end{aligned}$$

That's it. □

(For safety reasons: Let $\mathcal{H}$ be of finite dimension.)

- Consider one-cycle evolution operator

$$U(T,0) \equiv \exp(-\mathrm{i}GT/\hbar)$$

so that G is Hermitian.

- Define

$$P(t) := U(t,0)\exp(+\mathrm{i}Gt/\hbar)$$

Then

$$\begin{aligned} P(t+T) &= U(t+T,0)\exp\Big(+\mathrm{i}G(t+T)/\hbar\Big) \\ &= U(t,0)\Big(U(T,0)\exp(+\mathrm{i}GT/\hbar)\Big)\exp(+\mathrm{i}Gt/\hbar) \\ &= P(t) \end{aligned}$$

- **Assertion 2:**

Under suitable technical propositions, the time-evolution operator $U(t,0)$ of a T-periodically time-dependent quantum system has the form

$$U(t,0) = P(t)\exp(-\mathrm{i}Gt/\hbar)$$

where the unitary operator $P(t) = P(t+T)$ is T-periodic, and the operator G is Hermitian. □

▷ Write eigenvalues of $U(T,0) = \exp(-\mathrm{i}GT/\hbar)$ as $\{\mathrm{e}^{-\mathrm{i}\varepsilon_n T/\hbar}\}$:

$$U(T,0) = \sum_n |n\rangle\, \mathrm{e}^{-\mathrm{i}\varepsilon_n T/\hbar}\, \langle n|$$

implying

$$\mathrm{e}^{-\mathrm{i}Gt/\hbar}|n\rangle = \mathrm{e}^{-\mathrm{i}\varepsilon_n t/\hbar}|n\rangle$$

▷ Start from

$$
\begin{aligned}
|\psi(0)\rangle &= \sum_n |n\rangle\langle n|\psi(0)\rangle \\
&= \sum_n a_n |n\rangle
\end{aligned}
$$

and apply $U(t,0)$:

$$
\begin{aligned}
|\psi(t)\rangle &= U(t,0)|\psi(0)\rangle \\
&= \sum_n a_n \, P(t) \, \mathrm{e}^{-iGt/\hbar} \, |n\rangle \\
&= \sum_n a_n \, P(t) \, |n\rangle \, \mathrm{e}^{-\mathrm{i}\varepsilon_n t/\hbar} \\
&= \sum_n a_n |u_n(t)\rangle \, \mathrm{e}^{-\mathrm{i}\varepsilon_n t/\hbar}
\end{aligned}
$$

Here we have **defined** the Floquet functions $|u_n(t)\rangle \equiv P(t)|n\rangle$ which evidently are T -periodic: $|u_n(t)\rangle = |u_n(t+T)\rangle$

▷ **Definition:**

$$|\psi_n(t)\rangle = |u_n(t)\rangle \mathrm{e}^{-\mathrm{i}\varepsilon_n t/\hbar}$$

is called **Floquet state**

- **Assertion 3:**

Under suitable technical propositions, any solution $|\psi(t)\rangle$ to the time-dependent Schrödinger equation with a T -periodic Hamiltonian $H(t)$ can be expanded with respect to the Floquet states,

$$|\psi(t)\rangle = \sum_n a_n \, |u_n(t)\rangle \, \mathrm{e}^{-\mathrm{i}\varepsilon_n t/\hbar}$$

where the coefficients a_n do not *depend on time.* □

▷ **Definition:**
The quantities ε_n are called **quasienergies**
[Ya. B. Zel'dovich (1966); V. I. Ritus (1966)]

- **Interpretation:**

Consider

$$|\psi(t)\rangle = P(t)|\widetilde{\psi}(t)\rangle$$

Then, after some juggling,

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\widetilde{\psi}(t)\rangle \;=\; G|\widetilde{\psi}(t)\rangle$$

This appears to good to be true ???

- **Example 1**

The linearly driven harmonic oscillator:

$$H(x,t) = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial x^2} + \frac{1}{2}M\omega_0^2 x^2 + Fx\cos(\omega t)$$

▷ Strategy of solution (goes back to Husimi [1953]):

Let $\xi(t)$ be the T -periodic solution to the **classical** equation

$$M\ddot{\xi} = -M\omega_0^2\xi - F\cos(\omega t)$$

namely

$$\xi(t) = \frac{F}{M(\omega^2 - \omega_0^2)}\cos(\omega t)$$

Let $\chi_n(x)$ be an eigenfunction with $E_n = \hbar\omega(n + 1/2)$, and

$$L(t) = \frac{1}{2}M\dot{\xi}^2 - \frac{1}{2}M\omega_0^2\xi^2 - F\xi\cos(\omega t)$$

Then time-dependent wave functions are given by

$$\begin{aligned}\psi_n(x,t) &= \chi_n\big(x-\xi(t)\big)\,\mathrm{e}^{-\mathrm{i}E_n t/\hbar} \\ &\times \exp\left(\frac{\mathrm{i}}{\hbar}\left[M\dot{\xi}(t)\big(x-\xi(t)\big)+\int_0^t \mathrm{d}\tau\, L(\tau)\right]\right)\end{aligned}$$

▷ Extracting secular contributions, one finds Floquet functions

$$\begin{aligned}u_n(x,t) &= \chi_n\big(x-\xi(t)\big)\exp\left(\frac{\mathrm{i}}{\hbar}\left[M\dot{\xi}(t)\big(x-\xi(t)\big)\right.\right. \\ &\qquad \left.\left. +\int_0^t \mathrm{d}\tau\, L(\tau)-\frac{t}{T}\int_0^T \mathrm{d}\tau\, L(\tau)\right]\right)\end{aligned}$$

and quasienergies

$$\begin{aligned}\varepsilon_n &= E_n-\frac{1}{T}\int_0^T \mathrm{d}\tau\, L(\tau) \\ &= \hbar\omega_0(n+1/2)+\frac{F^2}{4M(\omega^2-\omega_0^2)}\end{aligned}$$

- All levels are shifted equally – this is an **exceptional** system!

- **Example 2**

The two-level system in a **circularly** polarized radiation field:

$$H_c(t) = \frac{\hbar\omega_0}{2}\sigma_z + \frac{\mu F}{2}\Big(\sigma_x \cos\omega t + \sigma_y \sin\omega t\Big)$$

▷ Transform to co-rotating frame:

$$P(t) = \exp\Big(\mathrm{i}\omega t(\mathbf{1} - \sigma_z)/2\Big)$$

This gives

$$P^\dagger(t)\left(H_c(t) - \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\right)P(t) = \frac{\hbar\omega}{2}\mathbf{1} + \frac{\hbar}{2}(\omega_0 - \omega)\sigma_z + \frac{\mu F}{2}\sigma_x - \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}$$

Therefore

$$G_c = \frac{\hbar\omega}{2}\mathbf{1} + \frac{\hbar}{2}(\omega_0 - \omega)\sigma_z + \frac{\mu F}{2}\sigma_x$$

Diagonalization yields quasienergies

$$\varepsilon_\pm = \frac{\hbar}{2}(\omega \pm \Omega)$$

with generalized Rabi frequency

$$\Omega = \sqrt{(\omega_0 - \omega)^2 + (\mu F/\hbar)^2}$$

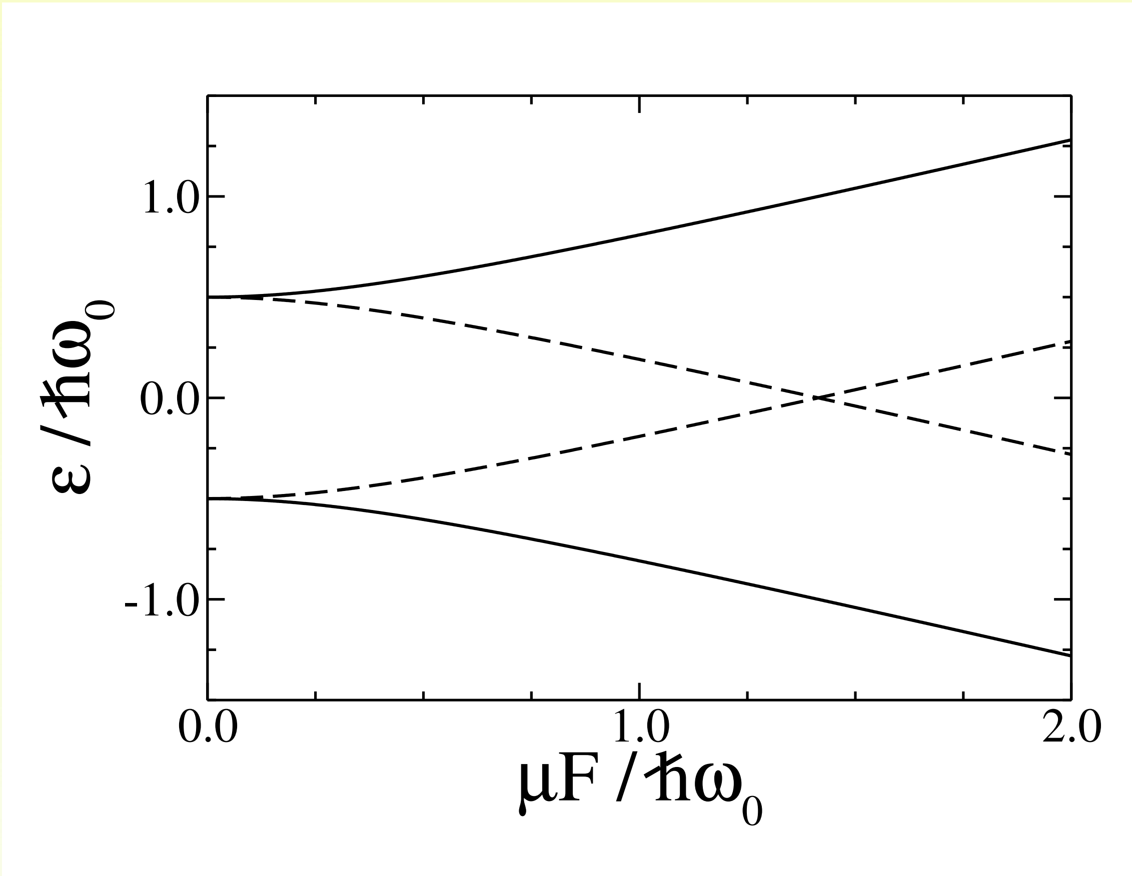
▷ **Observe:** For **red** detuning $(\omega < \omega_0)$, one has

$$\begin{aligned} \varepsilon_+ &\to +\hbar\omega_0/2 \\ \varepsilon_- &\to -\hbar\omega_0/2 + \hbar\omega \end{aligned}$$

whereas for **blue** detuning $(\omega > \omega_0)$

$$\begin{aligned} \varepsilon_+ &\to -\hbar\omega_0/2 + \hbar\omega \\ \varepsilon_- &\to +\hbar\omega_0/2 \end{aligned}$$

▷ Different **ac-Stark shifts**:



Full lines: Level repulsion for red detuning ($\omega/\omega_0 = 0.5$)

Dashed lines: Level crossing for blue detuning ($\omega/\omega_0 = 1.5$)

- **Example 3**

The two-level system in a **linearly** polarized radiation field:

$$
\begin{aligned}
H_l(t) &= \frac{\hbar\omega_0}{2}\sigma_z + \mu F \sigma_x \cos\omega t \\
&= \frac{\hbar\omega_0}{2}\sigma_z + \frac{\mu F}{2}\big(\sigma_x \cos\omega t + \sigma_y \sin\omega t\big) \\
&\quad + \frac{\mu F}{2}\big(\sigma_x \cos\omega t - \sigma_y \sin\omega t\big)
\end{aligned}
$$

▷ Transformation to rotating frame:

$$
P^\dagger(t)\left(H_l(t) - \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\right)P(t) = G_c - \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mu F}{2}\big(\sigma_x \cos 2\omega t - \sigma_y \sin 2\omega t\big)
$$

▷ **Rotating wave approximation (RWA)**:
Neglect high-frequency terms

- **Question:** Effect of the counter-rotating component?

- **Example 4**

Driven particle in a box (prototypical anharmonic oscillator):

$$H_0(x) = \frac{-\hbar^2}{2M}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$$

with

$$V(x) = \begin{cases} 0 & , \quad |x| < a \\ \infty & , \quad |x| \geq a \end{cases}$$

▷ Consider dipole-type driving:

$$H(x,t) = H_0(x) - F_0 x \cos(\omega t)$$

▷ Fully numerical approach:
- Truncate $\mathcal{H}$
- Compute $U(T,0)$
- Diagonalize
- Check "convergence"

▷ Use truncated basis of energy eigenstates:

Energy eigenvalues of H_0 :

$$E_n = \frac{\hbar^2 \pi^2}{8Ma^2}\, n^2 \quad ; \quad n = 1, 2, 3, \dots \; ,$$

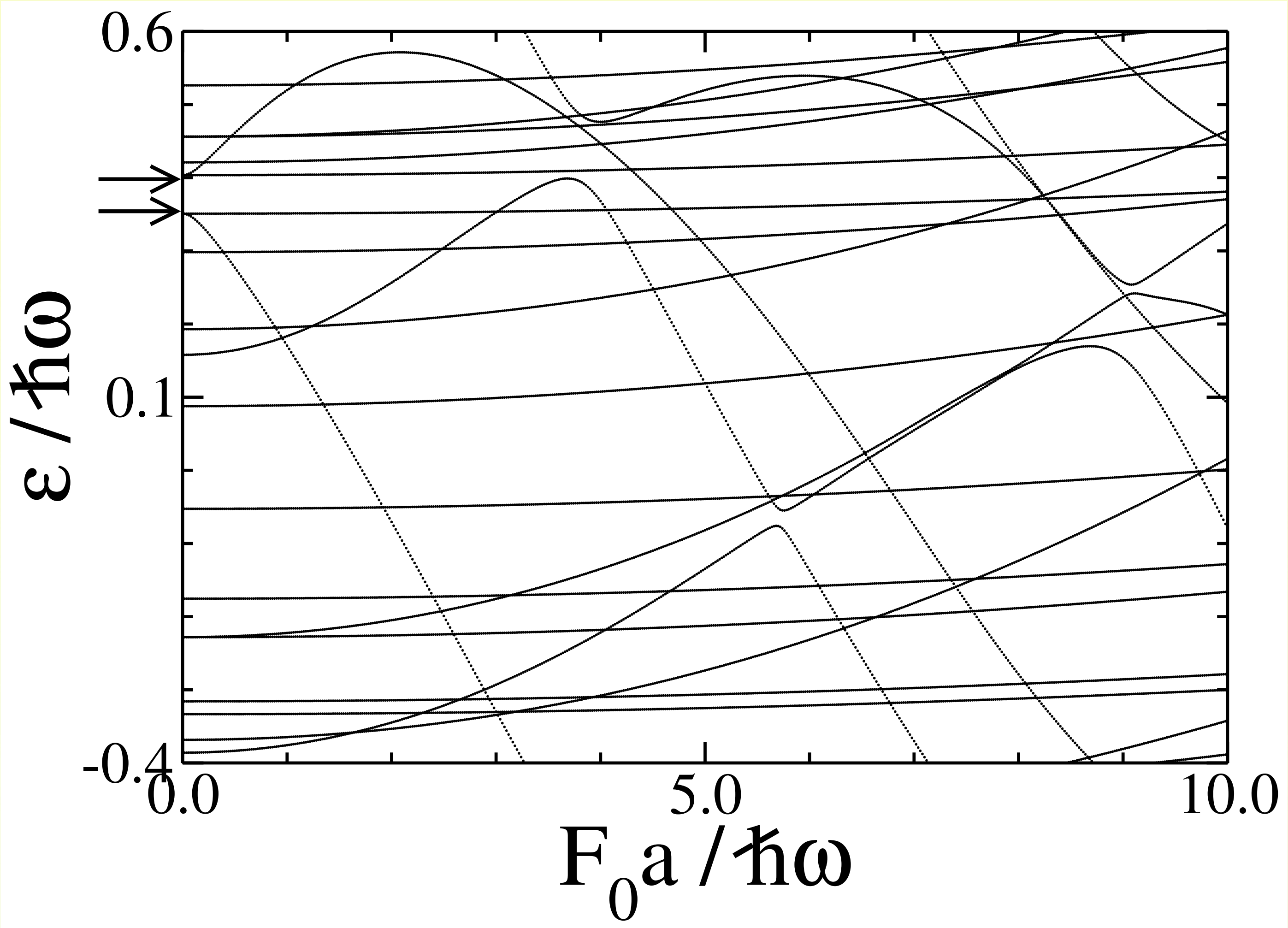
Dipole matrix elements:

$$\langle \varphi_m | x | \varphi_n \rangle = \begin{cases} -\dfrac{16a}{\pi^2} \dfrac{mn}{(m^2 - n^2)^2} & , \quad m + n \text{ odd} \\ 0 & , \quad m + n \text{ even} \; , \end{cases}$$

▷ **Example**:

$$\hbar\omega = 0.95\,(E_2 - E_1)$$

▷ Quasienergies vs. driving amplitude



- **Question:** How to interpret this figure?

PART II:

Extended Hilbert space and adiabatic principle

- **Numerical experiment**

▷ Consider **pulses:**

$$H(x,t) = H_0(x) - F_0(t)\, x \cos(\omega t)$$

with Gaussian envelope

$$F_0(t) = F_{\mathrm{max}} \exp\left(-\frac{t^2}{2\sigma^2}\right)$$

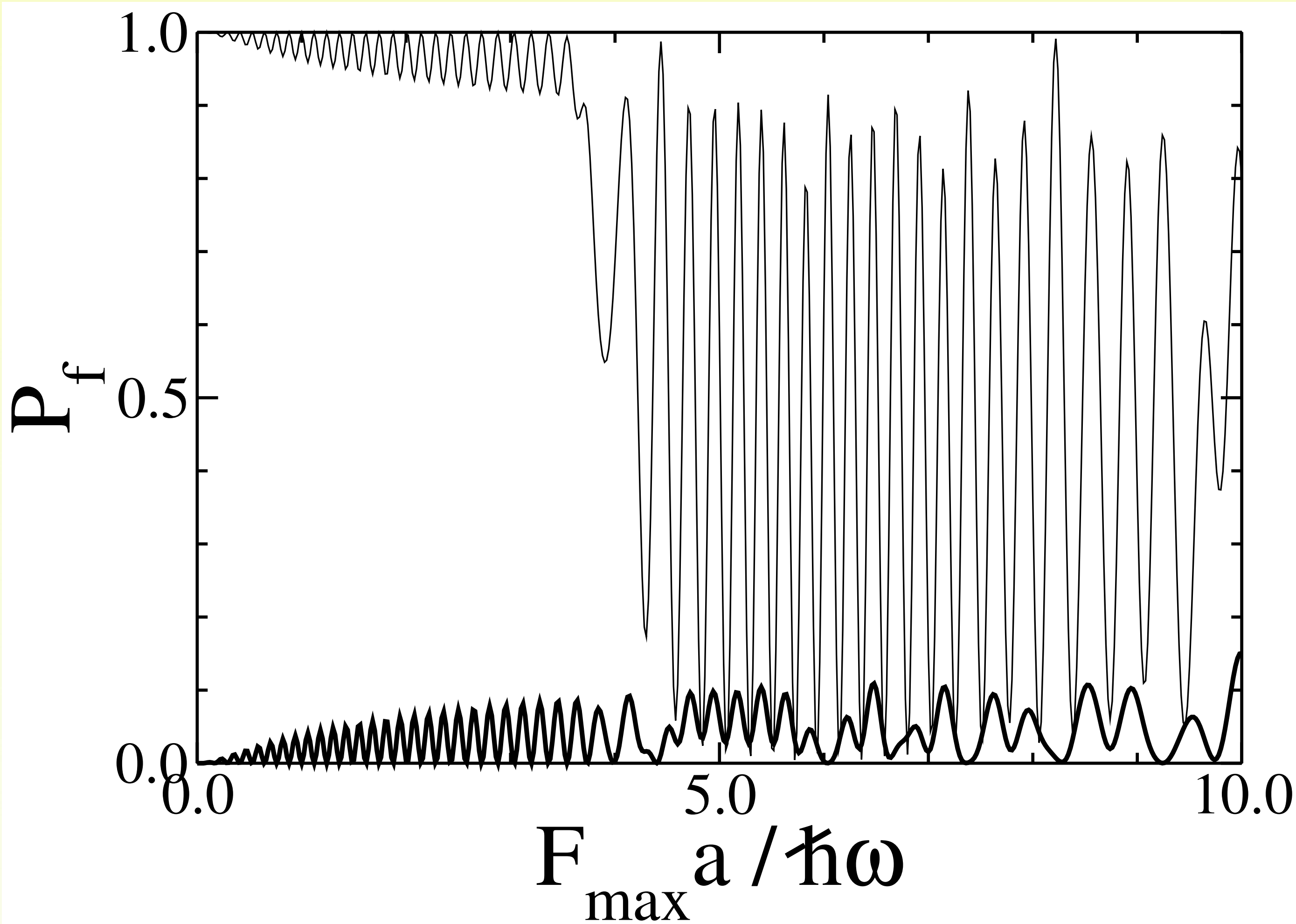
Initial state: $\psi(x,-\infty) = \varphi_1(x)$

Final state: $\psi(x,+\infty)$ (numerically)

▷ Compute probability for transition $1 \to n$:

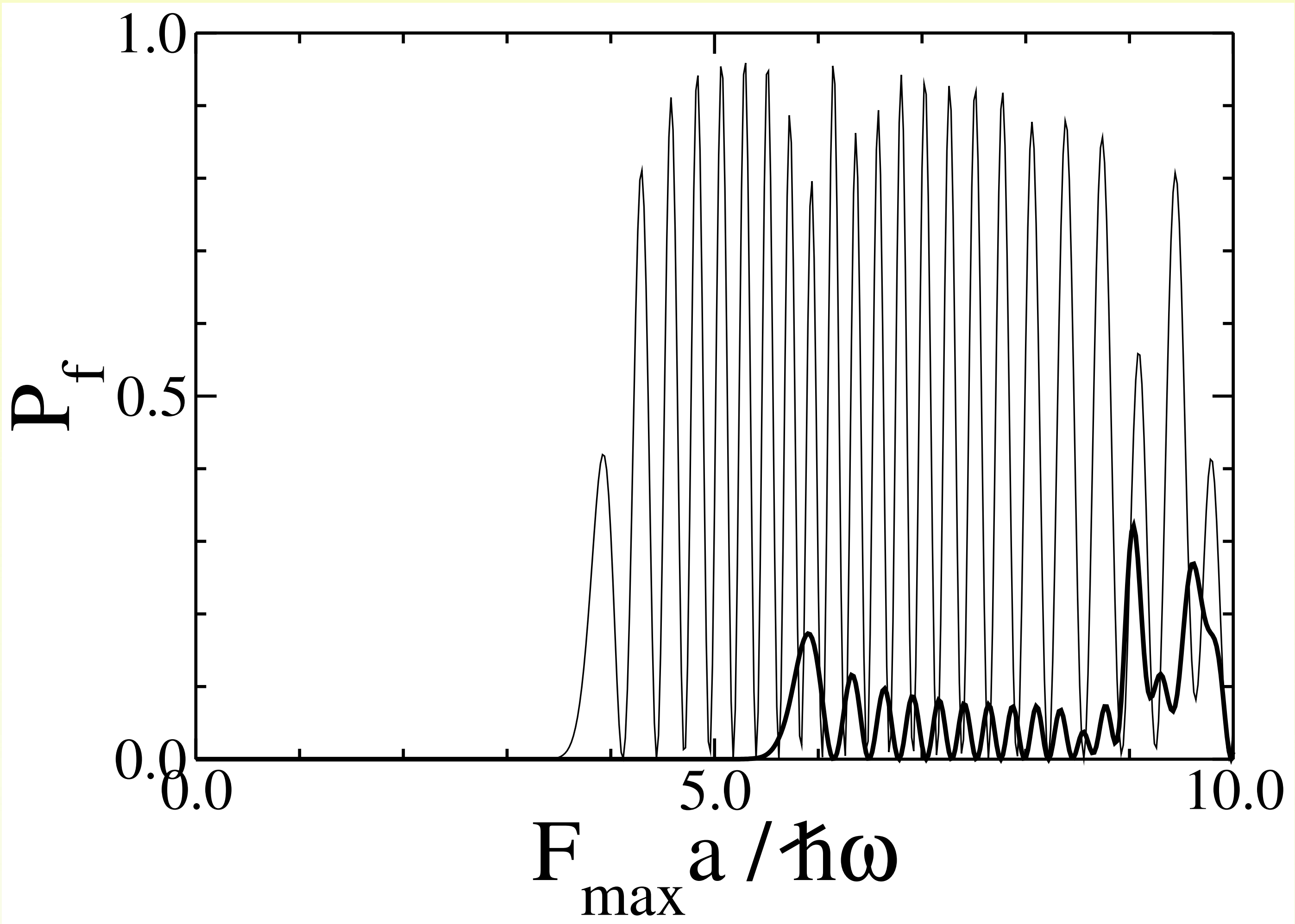
$$P_f(n) = \left| \langle \varphi_n | \psi(+\infty) \rangle \right|^2$$

▷ Final transition probabilities for $\sigma/T = 10$



Thin line: $1 \rightarrow 1$; heavy line: $1 \rightarrow 2$

▷ Final transition probabilities for $\sigma/T = 10$



Thin line: $1 \rightarrow 3$; heavy line: $1 \rightarrow 4$

- **Questions:**

▷ Why is the "dipole-allowed" transition $1 \to 2$ suppressed, and the "dipole-forbidden" transition $1 \to 3$ favored?

▷ What is the connection between
"avoided quasienergy crossings"
and
"multiphoton resonances"?

▷ The Hamiltonian is **not** periodic in time
— how can we apply Floquet theory?

???

- ▷ Let Hamiltonian depend on "slowly" changing parameters:

$$\boldsymbol{P}(t) = \Big(P_1(t), P_2(t), \ldots\Big)$$

such that

$$H^{\boldsymbol{P}}(t) = H^{\boldsymbol{P}}(t+T)$$

for each **fixed $\boldsymbol{P}$** .

- ▷ **Task:** Solve Schrödinger equation with "moving parameters"

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = H^{\boldsymbol{P}(t)}(t)|\psi(t)\rangle$$

- ▷ **Strategy:** Invoke **instantaneous Floquet states**

$$|\psi_n^{\boldsymbol{P}}(t)\rangle = |u_n^{\boldsymbol{P}}(t)\rangle \exp(-\mathrm{i}\varepsilon_n^{\boldsymbol{P}} t/\hbar)$$

▷ **Observe:** The Floquet states obey

$$
\begin{aligned}
& \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi_n^{\boldsymbol{P}}(t)\rangle \\
= \; & \left(\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|u_n^{\boldsymbol{P}}(t)\rangle + \varepsilon_n^{\boldsymbol{P}}|u_n^{\boldsymbol{P}}(t)\rangle\right)\exp(-\mathrm{i}\varepsilon_n^{\boldsymbol{P}}t/\hbar) \\
= \; & H^{\boldsymbol{P}}(t)|u_n^{\boldsymbol{P}}(t)\rangle\exp(-\mathrm{i}\varepsilon_n^{\boldsymbol{P}}t/\hbar)
\end{aligned}
$$

giving

$$
\left(H^{\boldsymbol{P}}(t) - \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\right)|u_n^{\boldsymbol{P}}(t)\rangle = \varepsilon_n^{\boldsymbol{P}}|u_n^{\boldsymbol{P}}(t)\rangle
$$

This is an **eigenvalue equation** in an **extended Hilbert space**, dubbed $L_2[0,T]\otimes\mathcal{H}$

▷ In this space, t is a **coordinate** ! [H. Sambe (1973)]

▷ Scalar product in $L_2[0,T] \otimes \mathcal{H}$:

$$\langle\langle u|v\rangle\rangle = \frac{1}{T}\int_0^T \mathrm{d}t\, \langle u(t)|v(t)\rangle$$

Observe:

$$p_t = \frac{\hbar}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}t}$$

is the momentum operator (!) conjugate to the t -coordinate

$$K^{\boldsymbol{P}} = H^{\boldsymbol{P}}(t) + p_t$$

is the **quasienergy operator**

▷ The eigenvalue equation

$$K^{\boldsymbol{P}}|u_n^{\boldsymbol{P}}(t)\rangle\rangle = \varepsilon_n^{\boldsymbol{P}}|u_n^{\boldsymbol{P}}(t)\rangle\rangle$$

adopts the role of the **stationary** Schrödinger equation!

• Classical analog:

Consider Hamiltonian $H_{\mathsf{Cl}}(p, x, t)$ in phase space $\{(p, x)\}$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial H_{\mathsf{Cl}}}{\partial p} \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H_{\mathsf{Cl}}}{\partial x}$$

▷ Introduce **extended phase space** $\{(p, p_t, x, t)\}$,

define "Kamiltonian" $K_{\mathsf{Cl}}(p, p_t, x, t) = H_{\mathsf{Cl}}(p, x, t) + p_t$

Need **new time variable** τ :

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\partial K_{\mathsf{Cl}}}{\partial p} = \frac{\partial H_{\mathsf{Cl}}}{\partial p} \qquad \frac{\mathrm{d}p}{\mathrm{d}\tau} = -\frac{\partial K_{\mathsf{Cl}}}{\partial x} = -\frac{\partial H_{\mathsf{Cl}}}{\partial x}$$

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\partial K_{\mathsf{Cl}}}{\partial p_t} = 1 \qquad \frac{\mathrm{d}p_t}{\mathrm{d}\tau} = -\frac{\partial K_{\mathsf{Cl}}}{\partial t}$$

▷ Recover old system by setting $\tau = t$!

- **Quantum system:**

Introduce "extended wave function" $|\Psi(\tau,t)\rangle\!\rangle$ such that

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}\tau}|\Psi(\tau,t)\rangle\!\rangle = K^{\boldsymbol{P}(\tau)}|\Psi(\tau,t)\rangle\!\rangle$$

and

$$|\psi(t)\rangle = |\Psi(\tau,t)\rangle\!\rangle\Big|_{\tau=t}$$

▷ Then we find the proper Schrödinger equation:

$$\begin{aligned}
\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle &= \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}\tau}|\Psi(\tau,t)\rangle\!\rangle\Big|_{\tau=t} + \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Psi(\tau,t)\rangle\!\rangle\Big|_{\tau=t} \\
&= \left(H^{\boldsymbol{P}(\tau)}(t) - \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\right)|\Psi(\tau,t)\rangle\!\rangle\Big|_{\tau=t} + \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Psi(\tau,t)\rangle\!\rangle\Big|_{\tau=t} \\
&= H^{\boldsymbol{P}(t)}(t)|\psi(t)\rangle
\end{aligned}$$

▷ **Observe:** $K^{\boldsymbol{P}(\tau)}$ remains periodic in time t for **any** $\boldsymbol{P}(\tau)$!

- **Adiabatic principle:**

▷ Go to extended space $L_2[0,T] \otimes \mathcal{H}$: Assume

$$|\Psi(\tau = 0, t)\rangle\!\rangle = |u_n^{\boldsymbol{P}(\tau=0)}(t)\rangle\!\rangle$$

Then (under appropriate conditions)

$$|\Psi(\tau, t)\rangle\!\rangle = \exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^\tau \mathrm{d}\tau'\, \varepsilon_n^{\boldsymbol{P}(\tau')}\right) \mathrm{e}^{\mathrm{i}\gamma_n(\tau)} |u_n^{\boldsymbol{P}(\tau)}(t)\rangle\!\rangle$$

with

$$\dot{\gamma}_n(\tau) = -\mathrm{Im}\, \langle\!\langle u_n^{\boldsymbol{P}(\tau)} | \nabla_{\boldsymbol{P}} u_n^{\boldsymbol{P}(\tau)} \rangle\!\rangle \cdot \dot{\boldsymbol{P}}(\tau)$$

Return to actual Hilbert space $\mathcal{H}$:

$$|\psi(t)\rangle = \exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^t \mathrm{d}t'\, \varepsilon_n^{\boldsymbol{P}(t')}\right) \mathrm{e}^{\mathrm{i}\gamma_n(t)} |u_n^{\boldsymbol{P}(t)}(t)\rangle$$

- **Remark:** There is a **Berry phase** for closed contours

$$\gamma_n(\mathcal{C}) = -\mathrm{Im} \oint_{\mathcal{C}} \langle\langle u_n^{\boldsymbol{P}} | \nabla_{\boldsymbol{P}} u_n^{\boldsymbol{P}} \rangle\rangle \cdot \mathrm{d}\boldsymbol{P}$$

Parallel transport in $L_2[0,T] \otimes \mathcal{H}$ is given by

$$\langle\langle u_n^{\boldsymbol{P}} | \nabla_{\boldsymbol{P}} u_n^{\boldsymbol{P}} \rangle\rangle = 0$$

- **Remark:** Assume

$$K^{\boldsymbol{P}} | u_n^{\boldsymbol{P}}(t) \rangle\rangle = \varepsilon_n^{\boldsymbol{P}} | u_n^{\boldsymbol{P}}(t)$$

Then, for $\omega = 2\pi/T$ and any integer m :

$$K^{\boldsymbol{P}} | u_n^{\boldsymbol{P}}(t) \mathrm{e}^{\mathrm{i}m\omega t} \rangle\rangle = (\varepsilon_n^{\boldsymbol{P}} + m\hbar\omega) | u_n^{\boldsymbol{P}}(t) \mathrm{e}^{\mathrm{i}m\omega t} \rangle\rangle$$

Different solutions in $L_2[0,T] \otimes \mathcal{H}$ give the same state in $\mathcal{H}$:

$$| u_n^{\boldsymbol{P}}(t) \mathrm{e}^{\mathrm{i}m\omega t} \rangle \exp(-\mathrm{i}[\varepsilon_n^{\boldsymbol{P}} + m\hbar\omega] t/\hbar) = | u_n^{\boldsymbol{P}}(t) \rangle \exp(-\mathrm{i}\varepsilon_n^{\boldsymbol{P}} t/\hbar)$$

- Keep in mind:

  A quasienergy is a class of equivalent representatives,

  $$\{\varepsilon_n + m\hbar\omega \,|\, m = 0, \pm 1, \pm 2, \ldots\}$$

  Each **Brillouin zone** of width $\hbar\omega$ contains one quasienergy representative of each Floquet state

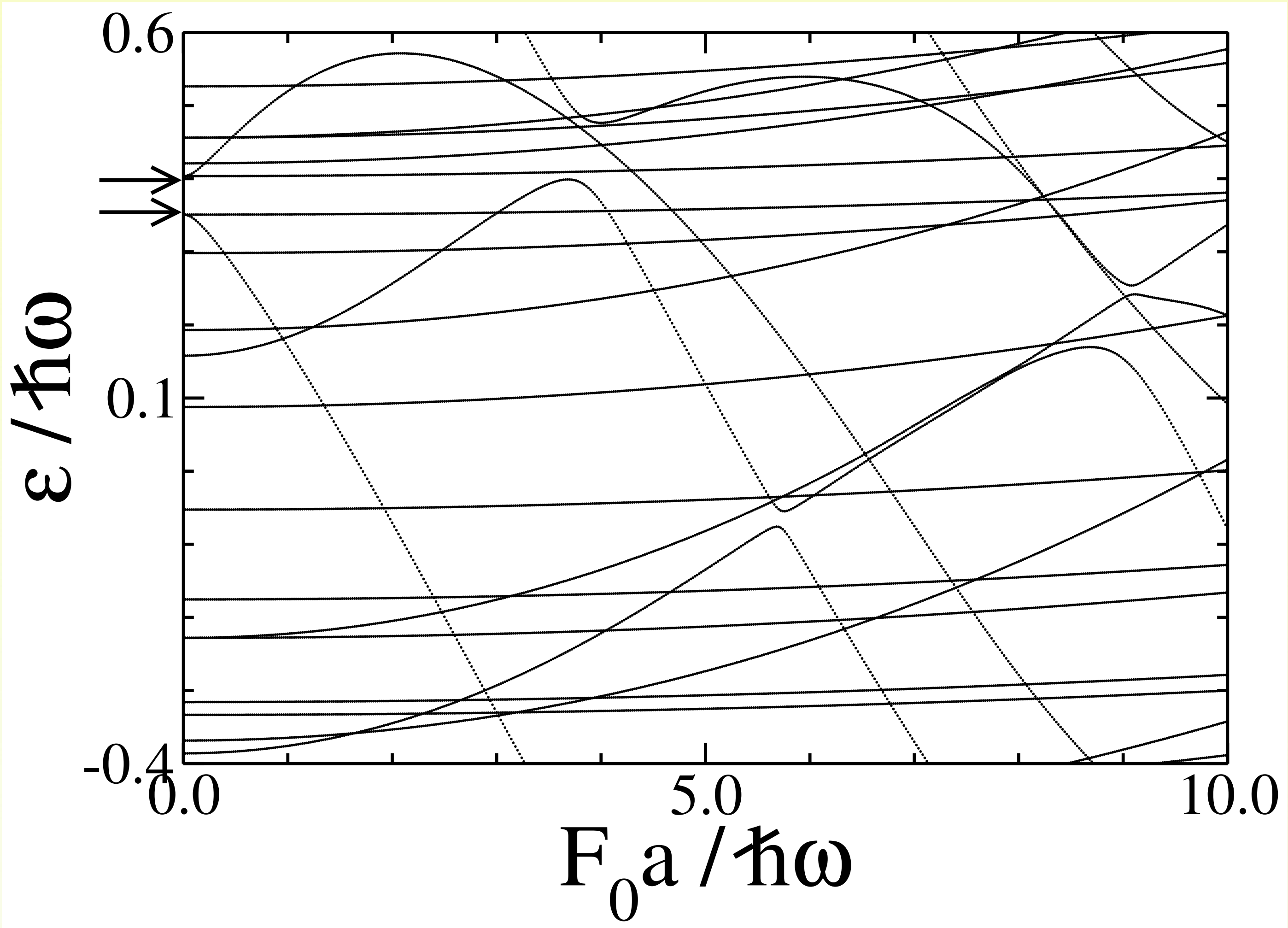
- **Remark**:

  From the rigorous mathematical viewpoint, the quasi-energy eigenvalue problem is extremely delicate even for such "simple" systems as the driven particle in a box: Is the spectrum pure point?

  This is the problem of **quantum stability**

  [J. S. Howland (1989), (1992)]

- Back to “???”:



▷ Adiabatic-diabatic motion on “quasienergy surfaces”

- Resolution of "???"

▷ *Amplitude rises:* The initial state is adiabatically shifted into the "connected" Floquet state

▷ At **avoided quasienergy crossings**, Landau-Zener-type transitions to the anticrossing state occur

▷ All components then again move adiabatically, each acquiring their own dynamical phase

▷ *Amplitude decreases:* At the second traversal of the anti-crossing, the components **interfere**

▷ Interference pattern determines final transition probabilities

- **New "?"**

What decides whether there is an anticrossing or a crossing?

▷ **von Neumann-Wigner noncrossing rule** (1929)

Eigenvalues of a Hermitian operator which belong to the same symmetry class generically do not cross.

▷ Here: K is invariant under generalized parity

$$P : \begin{cases} x & \to & -x \\ t & \to & t + T/2 \end{cases}$$

▷ Label Floquet functions such that $|u_n^{\boldsymbol{P}}(t)\rangle\rangle$ "connects" to energy eigenstate $|\varphi_n\rangle$ of H_0 :

$|u_n^{\boldsymbol{P}}(t)\mathrm{e}^{\mathrm{i}m\omega t}\rangle\rangle$ has generalized parity $(-1)^{n+m+1}$

- Assign label (n,m) to $|u_n^{\boldsymbol{P}}(t)\mathrm{e}^{\mathrm{i}m\omega t}\rangle\rangle$:

  Anticrossing of $(1,1)$ and $(3,-3)$ at $F_0 a/(\hbar\omega) \approx 4.0$ corresponds to "4-photon resonance"

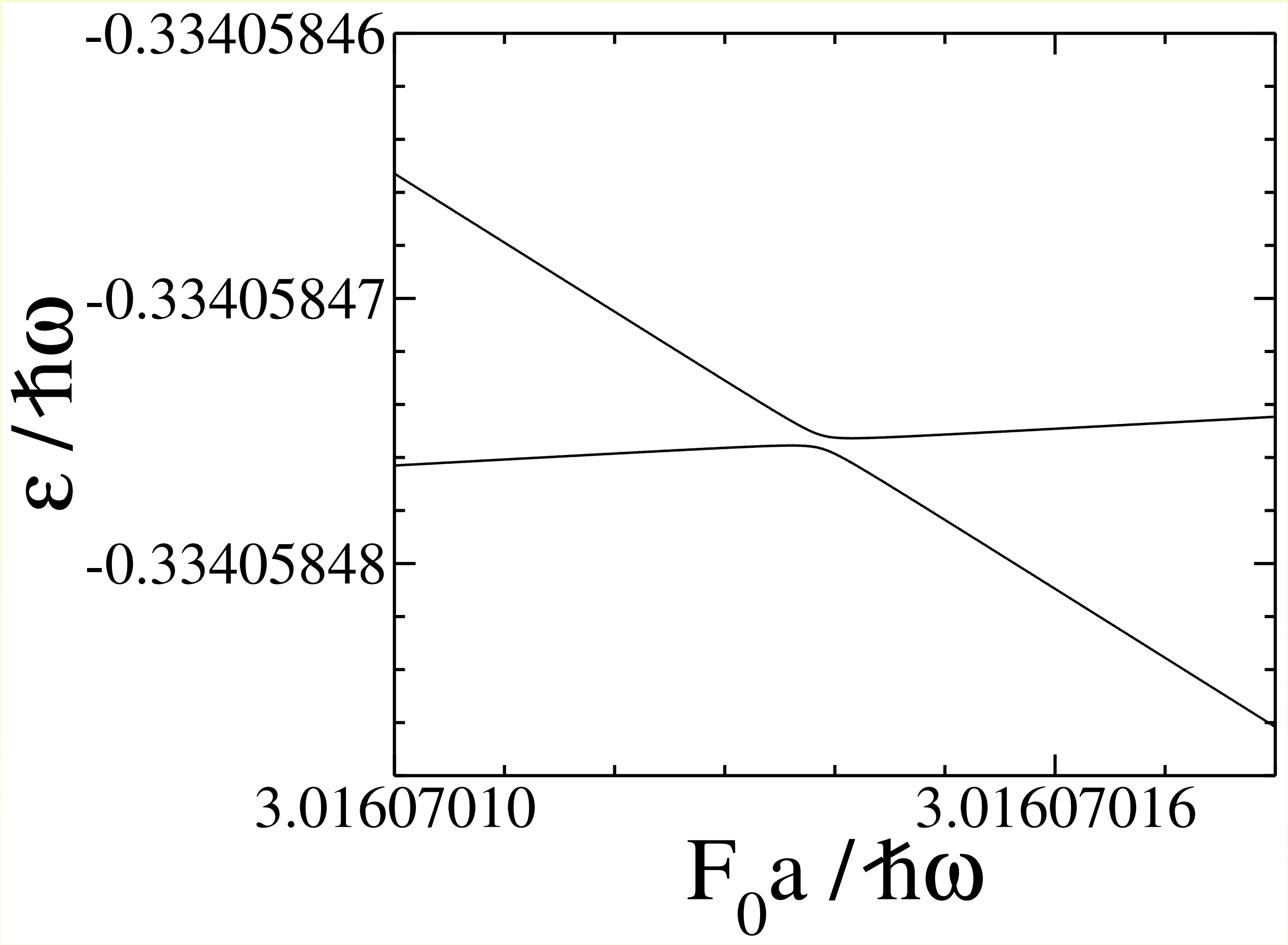
  Anticrossing of $(1,1)$ and $(4,-6)$ at $F_0 a/(\hbar\omega) \approx 5.7$ corresponds to "7-photon resonance"

- **Keep in mind:**

  Selection rules for **strongly** *driven systems are determined by symmetries of K in $L_2[0,T] \otimes \mathcal{H}$,* **not** *by those of H in $\mathcal{H}$!*

- **Major problem:**
  What about all the other apparent crossings?

▷ Anticrossing among $(1, 0)$ and $(6, -13)$:



- There are no smooth quasienergy surfaces!

- **Keep in mind:**

Ignoring (possibly infinitely many) "small" anticrossings corresponds to **coarse graining** *— may be justified by the time scales of the respective experiment*

Effectively adiabatic motion on coarse-grained quasienergy surfaces actually is highly diabatic motion on "rough" surfaces — there is **no adiabatic limit**

(This is a physicist's view on the "quantum stability problem")

- **Remark**

The famous "area theorem" is an **application** *of the adiabatic principle for Floquet states:*

Two-level system, RWA: For $\omega = \omega_0$, one has

$$\varepsilon_\pm^F = \pm\frac{\mu F}{2} \qquad \mathrm{mod}\ \hbar\omega$$

▷ For a resonant pulse with envelope $F(t)$, this gives

$$\begin{aligned} P_{-\to+} &= \sin^2\left(\frac{1}{2\hbar}\int_0^{T_p} \mathrm{d}t\, (\varepsilon_+^{F(t)} - \varepsilon_-^{F(t)})\right) \\ &= \sin^2\left(\frac{\mu}{2\hbar}\int_0^{T_p} \mathrm{d}t\, F(t)\right) \end{aligned}$$

▷ "π -pulse" (yielding $P_{-\to+} = 1$) for

$$\frac{\mu}{\hbar}\int_0^{T_p} \mathrm{d}t\, F(t) = \pi$$

- **Remark**

Beyond RWA: Effect of “counterrotating” terms?

▷ Recall transformation to rotating frame:

$$P^\dagger(t)\left(H_l(t) - \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\right)P(t)$$
$$= \quad G_c - \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mu F}{2}\Big(\sigma_x\cos 2\omega t - \sigma_y \sin 2\omega t\Big)$$

▷ Perform ordinary Rayleigh-Schrödinger perturbation theory in extended Hilbert space:

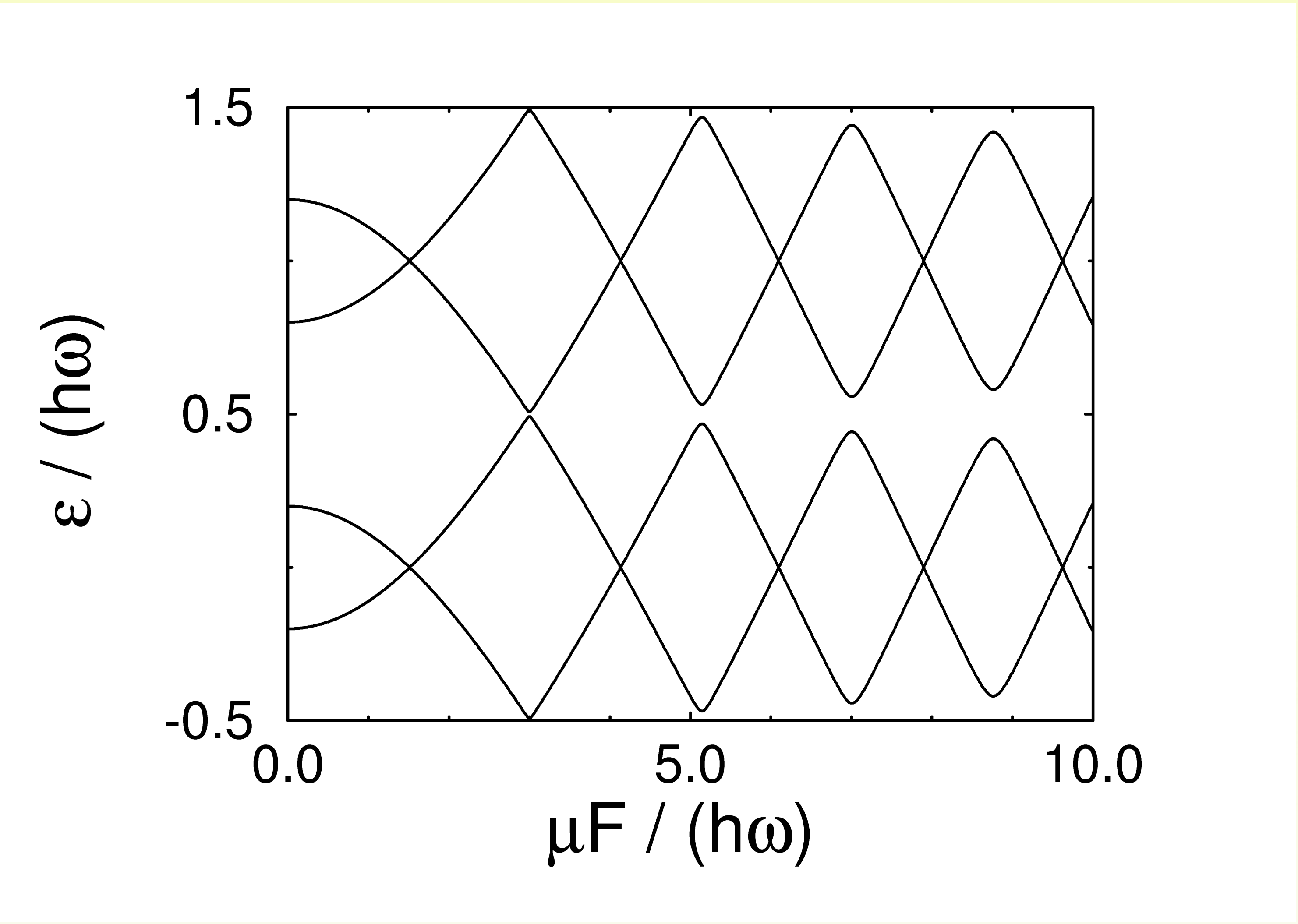
$$E_n^{(1)} = \langle n|H_{\mathrm{pert}}|n\rangle$$

translates into

$$\varepsilon_n^{(1)} = \langle\!\langle u_n(t)|H_{\mathrm{pert}}(t)|u_n(t)\rangle\!\rangle$$

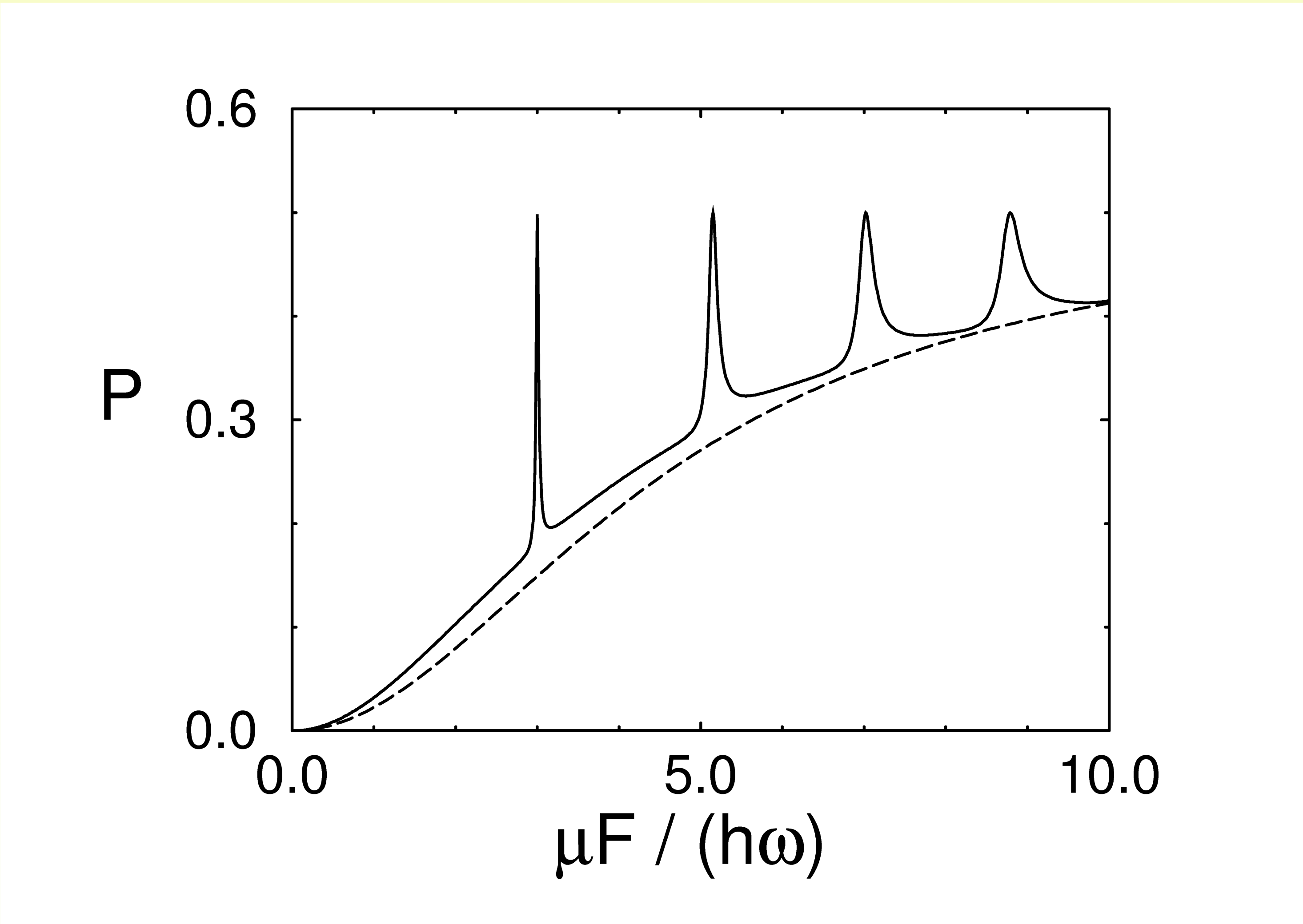
etc.

- Linearly driven TLS



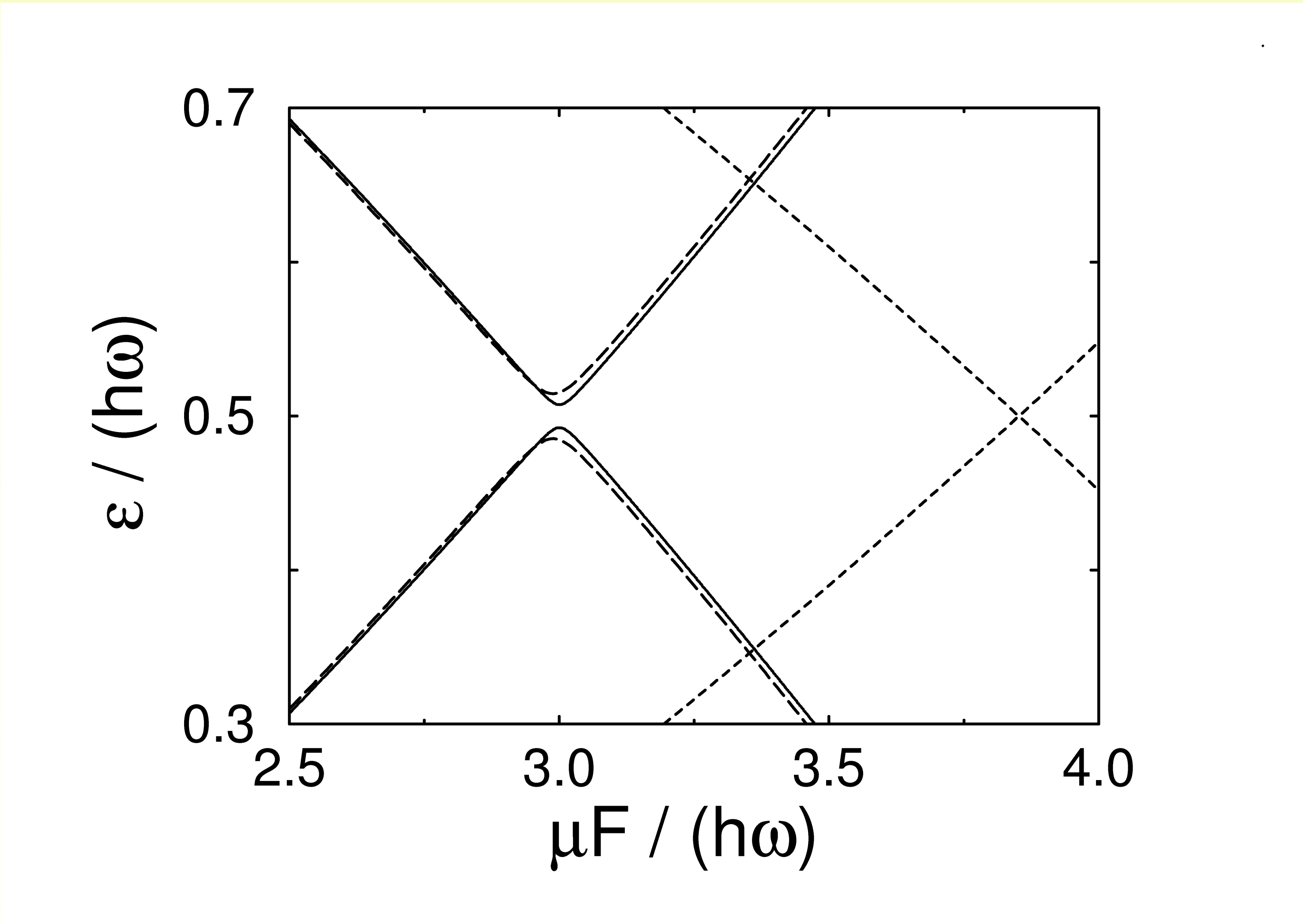
Exact quasienergies for $\omega_0/\omega = 5.6$

- Long-time averaged transition probabilities



▷ RWA (dashed) misses multiphoton resonances!

- Bloch-Siegert shift and avoided crossings



▷ Full line: exact; short dashes: RWA; long dashes: deg. RSPT

PART III:

Floquet engineering with optical lattices

- Consider particle in **spatially** periodic potential

$$V(x) = V(x+a)$$

acted on by **temporally** periodic force

$$F(t) = F(t+T)$$

▷ Assume dipole-type coupling:

$$\widetilde{H}(x,t) = -\frac{\hbar^2}{2M}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) - xF(t)$$

▷ **TASK:** Solve time-dependent Schrödinger equation

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\psi}(x,t) = \widetilde{H}(x,t)\widetilde{\psi}(x,t)$$

- Perform unitary transformation:

$$\widetilde{\psi}(x,t) = \exp\left(\frac{\mathrm{i}}{\hbar} x \int_0^t \mathrm{d}\tau\, F(\tau)\right) \psi(x,t)$$

This gives $\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t}\psi(x,t) = H(x,t)\psi(x,t)$ with new Hamiltonian

$$H(x,t) = \frac{1}{2M}\Big(p - A(t)\Big)^2 + V(x)$$

where

$$A(t) = -\int_0^t \mathrm{d}\tau\, F(\tau)$$

- Assume forcing without dc component:

$$\frac{1}{T}\int_0^T \mathrm{d}t\, F(t) = 0$$

Then

$$H(x,t) = H(x+a,t) = H(x,t+T)$$

- Hence, we have **spatio-temporal Bloch waves:**

$$\psi_{n,k}(x,t) = \exp\left[\mathrm{i}kx - \mathrm{i}\varepsilon_n(k)t/\hbar\right] u_{n,k}(x,t)$$

with doubly periodic Floquet functions

$$u_{n,k}(x,t) = u_{n,k}(x+a,t) = u_{n,k}(x,t+T)$$

- Build wave packet in n -th **quasienergy band**:

$$\psi_n(x,t) = \sqrt{\frac{a}{2\pi}}\int \mathrm{d}k\, g_n(k)\ \exp[\mathrm{i}kx - \mathrm{i}\varepsilon_n(k)t/\hbar]\, u_{n,k}(x,t)$$

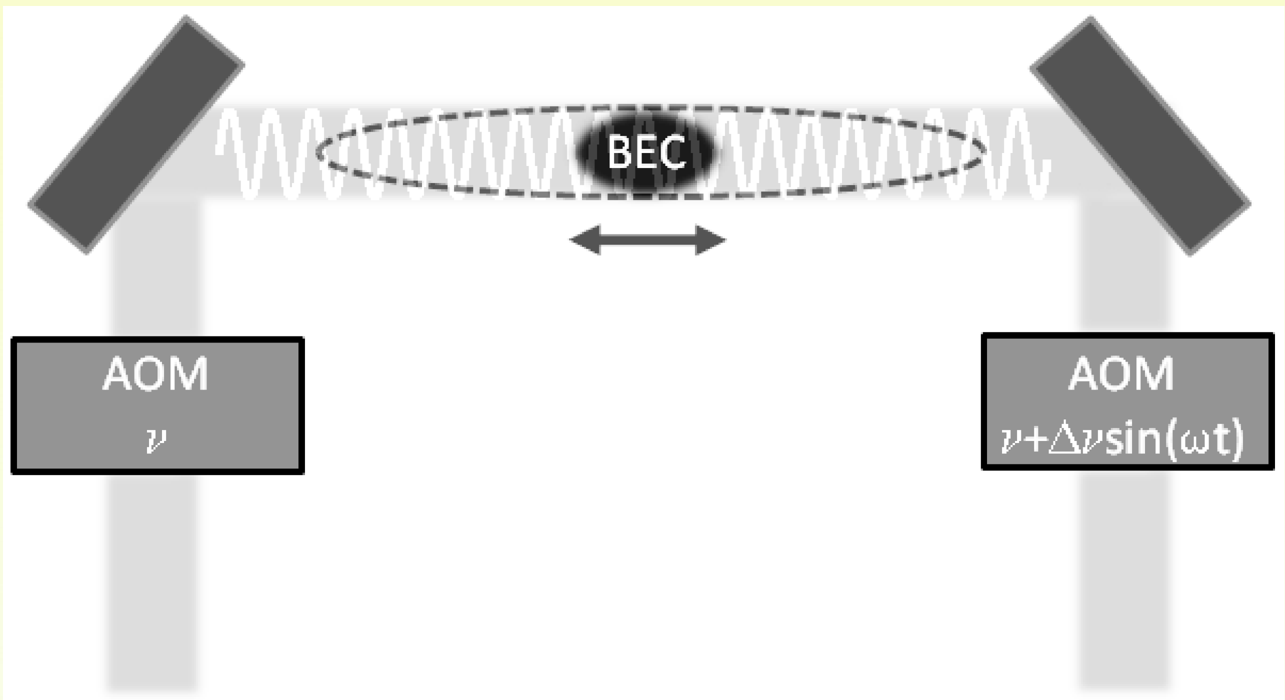
- Quasienergy eigenvalue problem in extended Hilbert space:

$$\left[\frac{1}{2M}\left(p + \hbar k + \int_0^t \mathrm{d}\tau\, F(\tau)\right)^2 + V(x) - \mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\right] u_{n,k}(x,t) = \varepsilon_n(k) u_{n,k}(x,t)$$

(Very similar to "particle in the box", but with additional parameter k !)

- Experimental realization:

Ultracold atoms in shaken optical lattices



▷ 1d optical lattice potential: $V(x) = \frac{V_0}{2}\cos(2k_\mathrm{L}x)$

▷ Characteristic energy scale: $E_\mathrm{R} = \frac{\hbar^2 k_\mathrm{L}^2}{2M}$

▷ Laboratory frame:

$$H^{\mathrm{lab}}(x,t) = \frac{p^2}{2M} + \frac{V_0}{2}\cos\Big(2k_{\mathrm{L}}[x - \Delta L\cos(\omega t)]\Big)$$

▷ This is unitarily equivalent (and isospectral) to

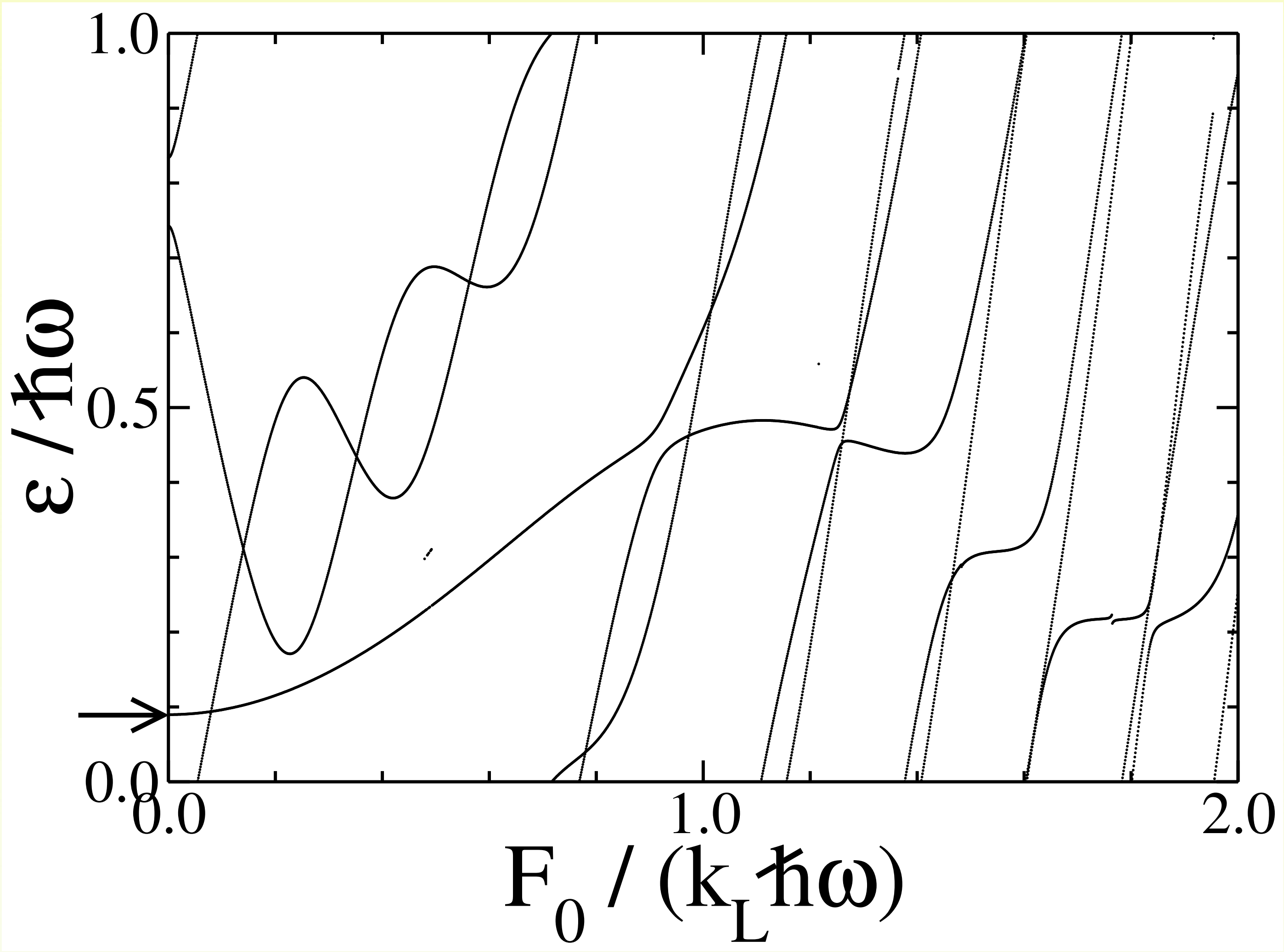
$$H(x,t) = \frac{1}{2M}\Big(p + M\Delta L\omega\sin(\omega t)\Big)^2 + \frac{V_0}{2}\cos(2k_{\mathrm{L}}x) - \frac{M}{4}(\Delta L\omega)^2$$

Thus, $F(t) = F_0\cos(\omega t)$ with $F_0 = M\Delta L\omega^2$

- **Keep in mind:**

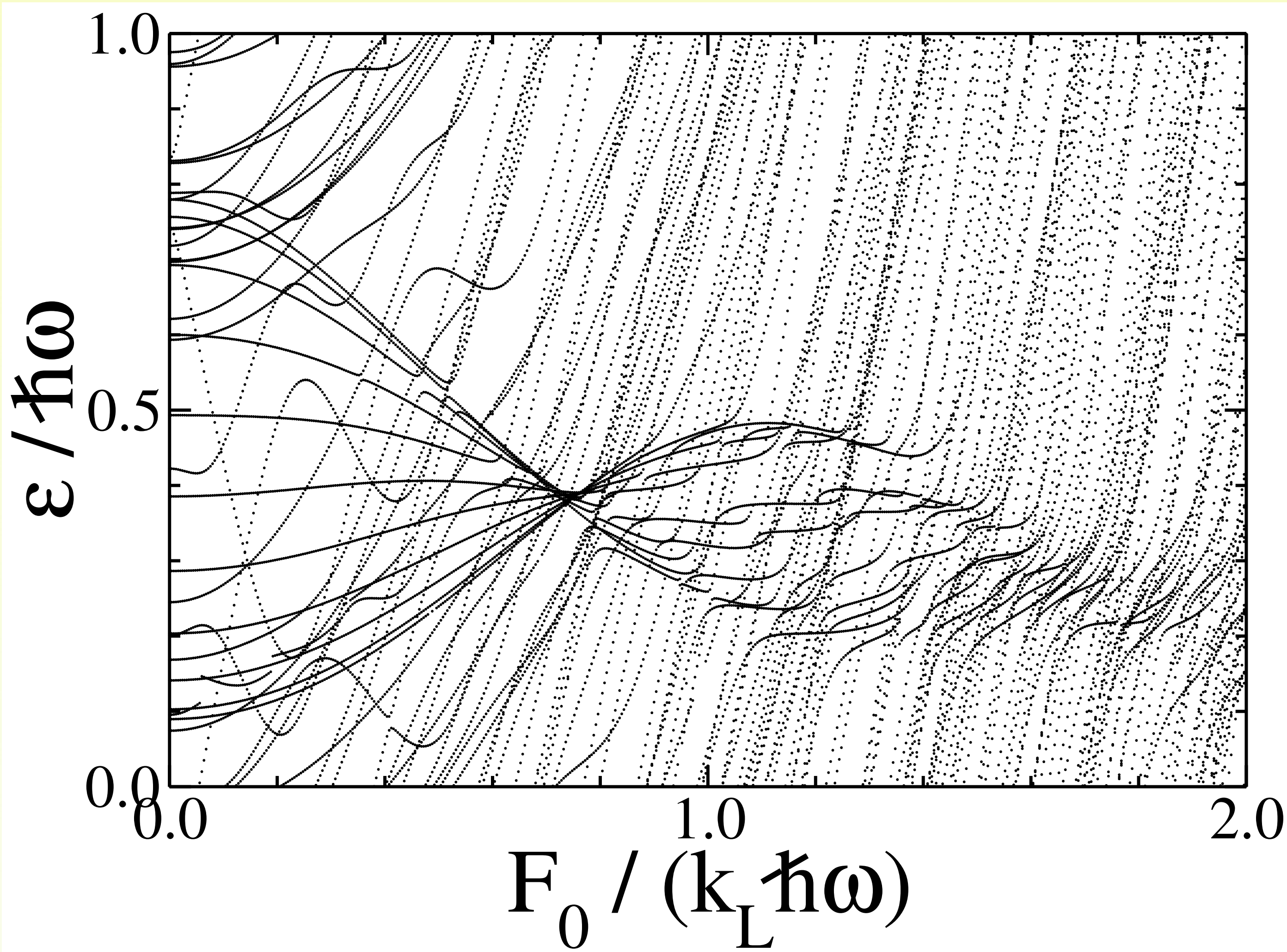
A time-periodically driven optical lattice is like a "spatio-temporal crystal" with quasienergy-quasimomentum dispersion relations which can be manipulated at will!

▷ ac-Stark shift of band edge $k/k_L = 0$



Lattice depth: $V_0/E_\mathrm{R} = 4.0$, driving frequency: $\hbar\omega/E_\mathrm{R} = 0.5$

▷ Destruction of quasienergy band ($k/k_L = 0.0,\ 0.1,\ \ldots,\ 1.0$)



Lattice depth: $V_0/E_{\rm R} = 4.0$, driving frequency: $\hbar\omega/E_{\rm R} = 0.5$

- Tight-binding approximation (deep lattices):

$$E_0(k) = E_c - \frac{W_0}{2}\cos(ka)$$

- Single-band approximation (neglect interband transitions)

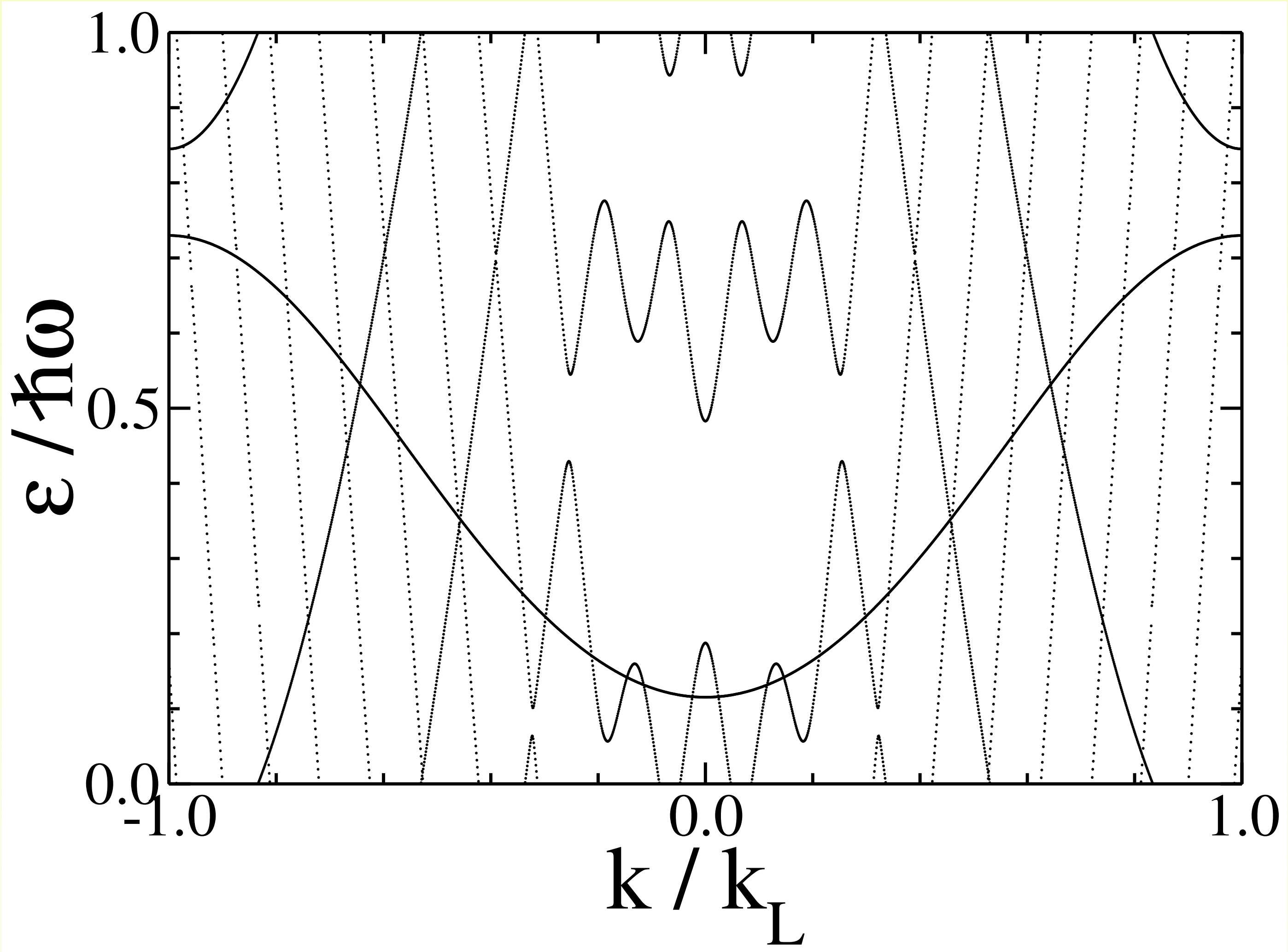
$$\varepsilon_0(k) = E_c - \frac{W_0}{2} J_0\left(\frac{F_0 a}{\hbar\omega}\right)\cos(ka) \quad \mathrm{mod}\ \hbar\omega$$

(*Driving effectuates renormalization of band width*)

- **Keep in mind:**

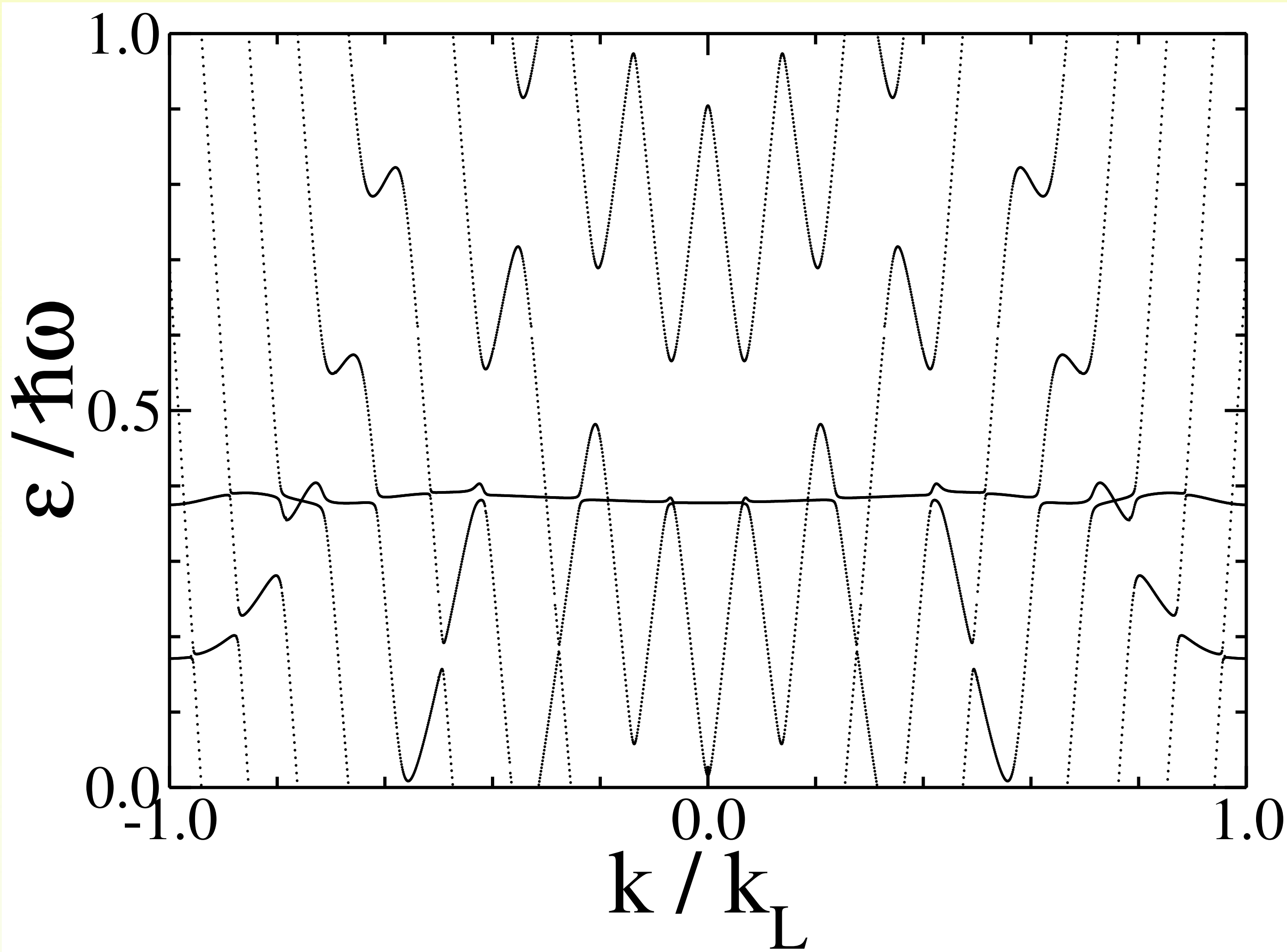
The "quasienergy band collapse" at the zeros of J_0 corresponds to **dynamic localization**, *due to "prohibited dephasing" in a flat band*

▷ Quasienergy dispersion relation for $F_0/(k_L \hbar\omega) = 0.20$



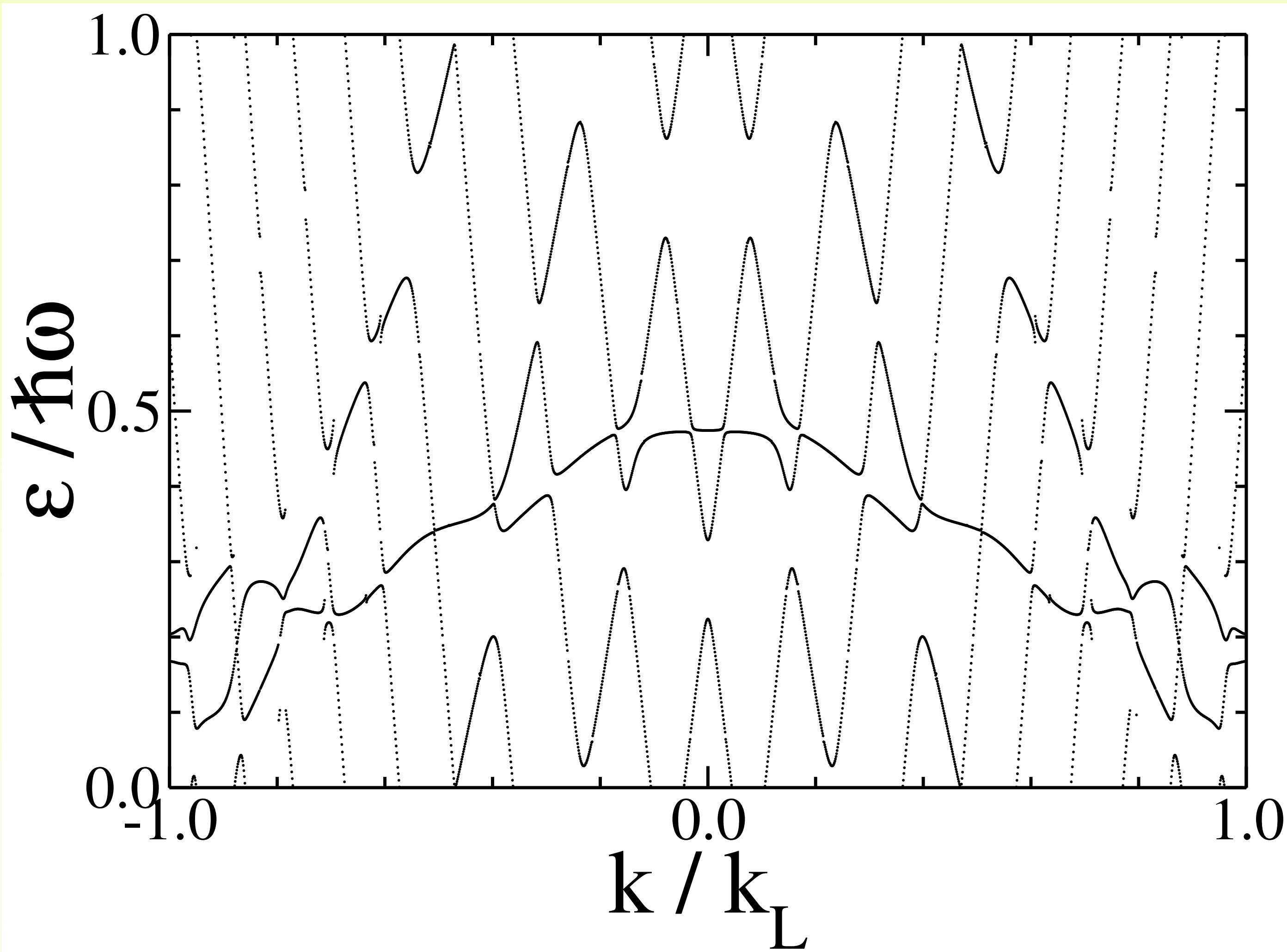
Lattice depth: $V_0/E_R = 4.0$, driving frequency: $\hbar\omega/E_R = 0.5$

▷ Quasienergy dispersion relation for $F_0/(k_L \hbar\omega) = 0.74$



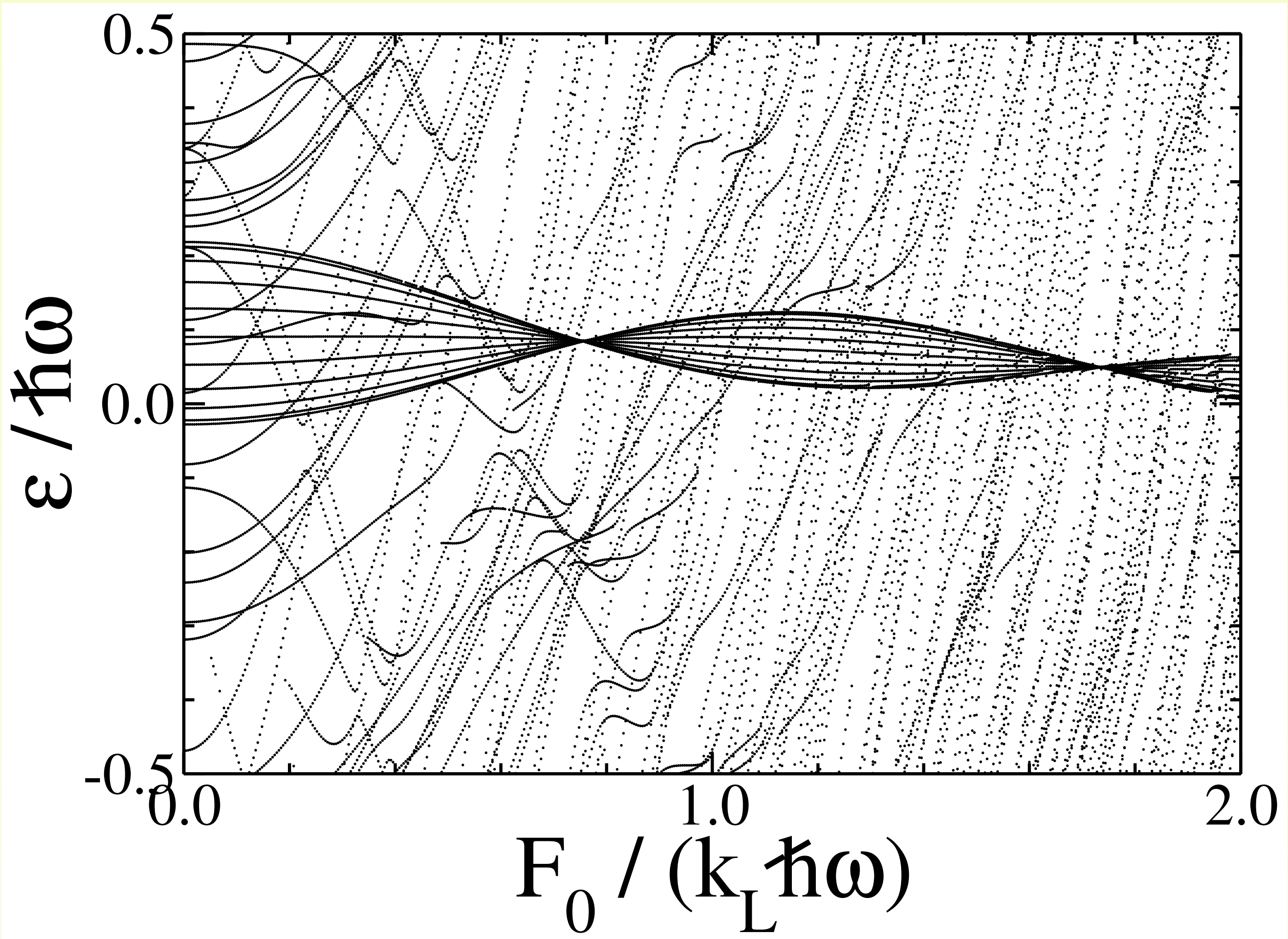
Lattice depth: $V_0/E_R = 4.0$, driving frequency: $\hbar\omega/E_R = 0.5$

▷ Quasienergy dispersion relation for $F_0/(k_\mathrm{L}\hbar\omega) = 1.21$



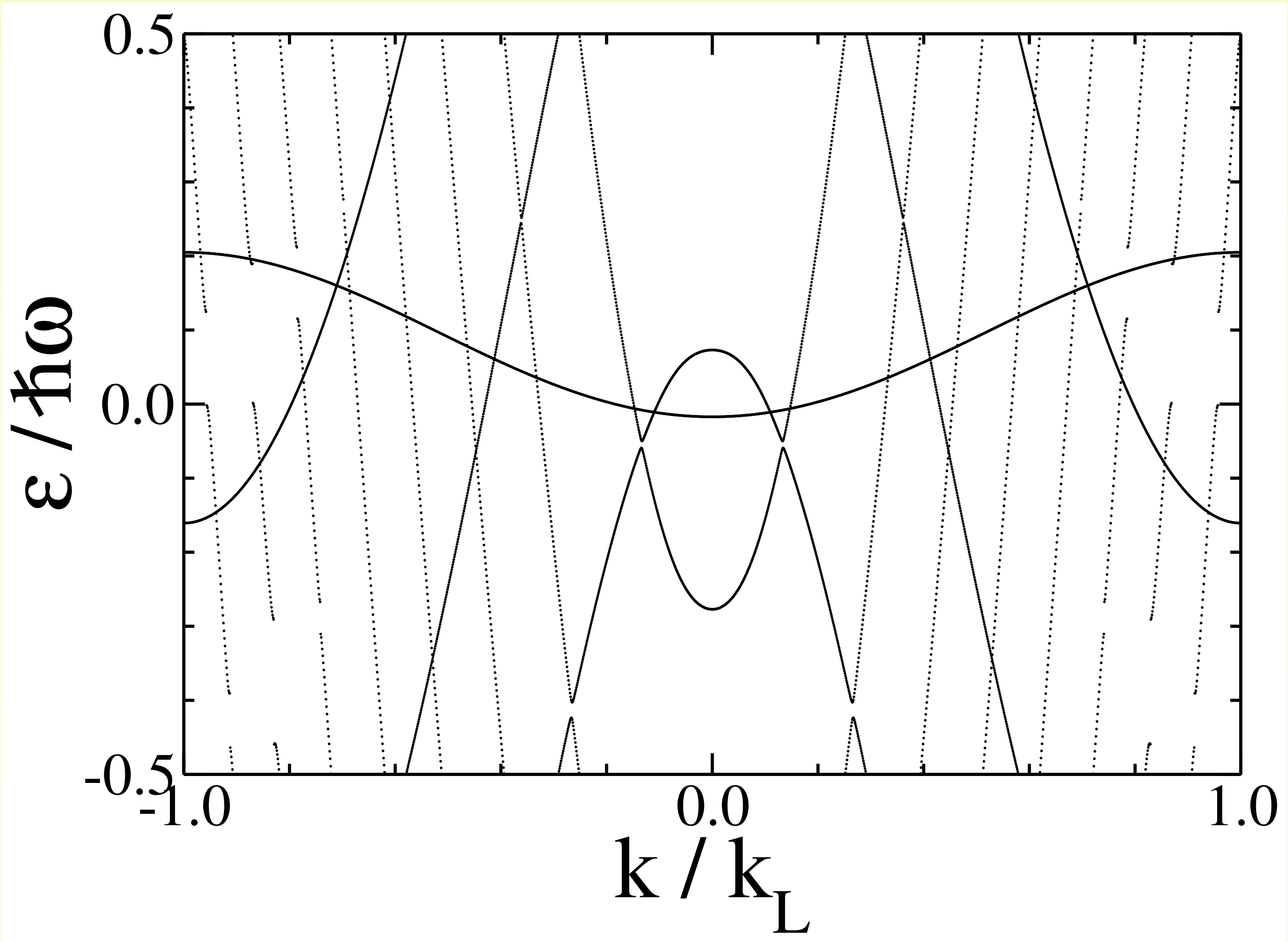
Lattice depth: $V_0/E_\mathrm{R} = 4.0$, driving frequency: $\hbar\omega/E_\mathrm{R} = 0.5$

▷ Preservation of quasienergy band ($k/k_L = 0.0, 0.1, \ldots, 1.0$)



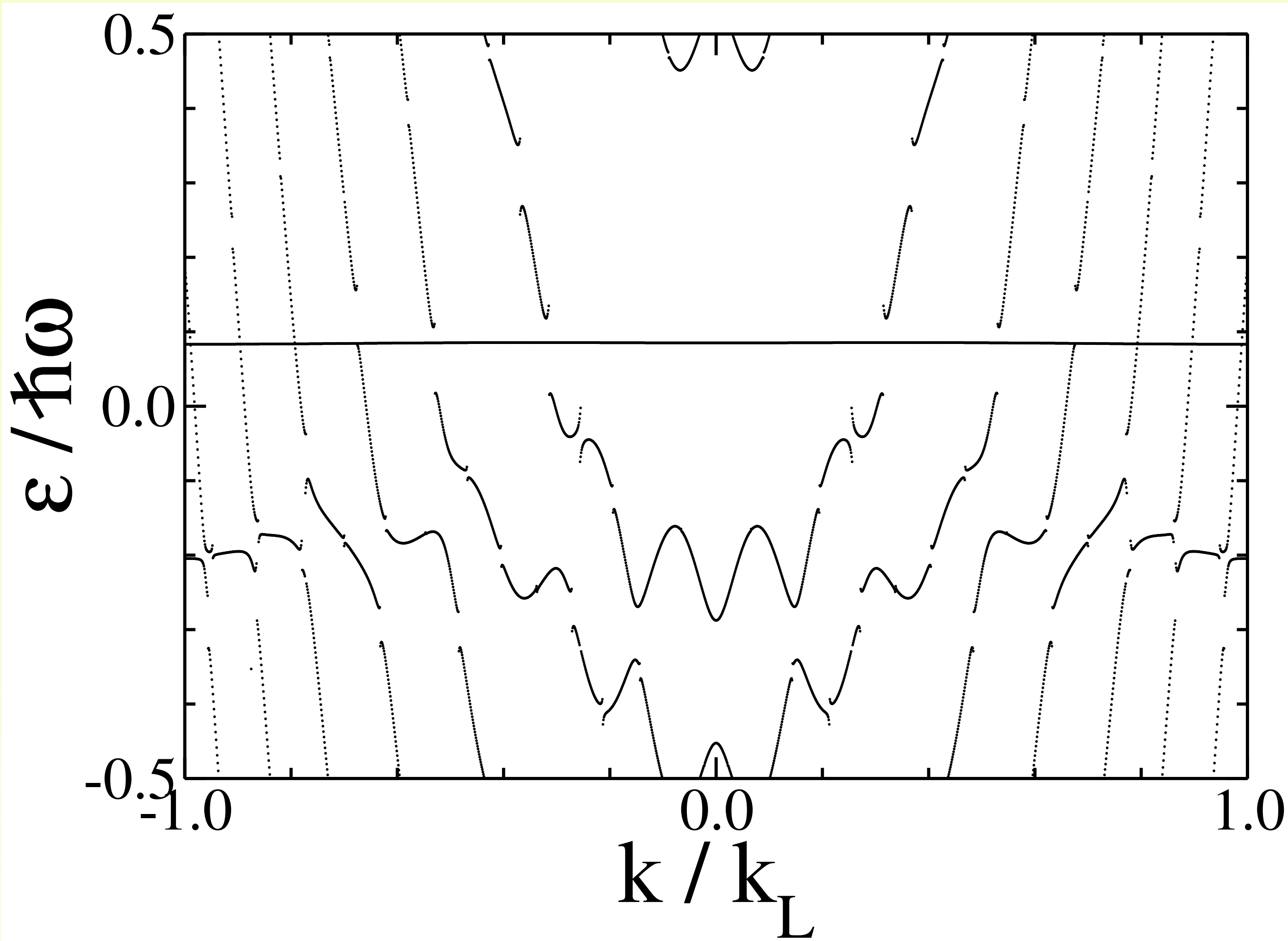
Lattice depth: $V_0/E_\mathrm{R} = 8.0$, driving frequency: $\hbar\omega/E_\mathrm{R} = 0.5$

▷ Quasienergy dispersion relation for $F_0/(k_L \hbar\omega) = 0.20$



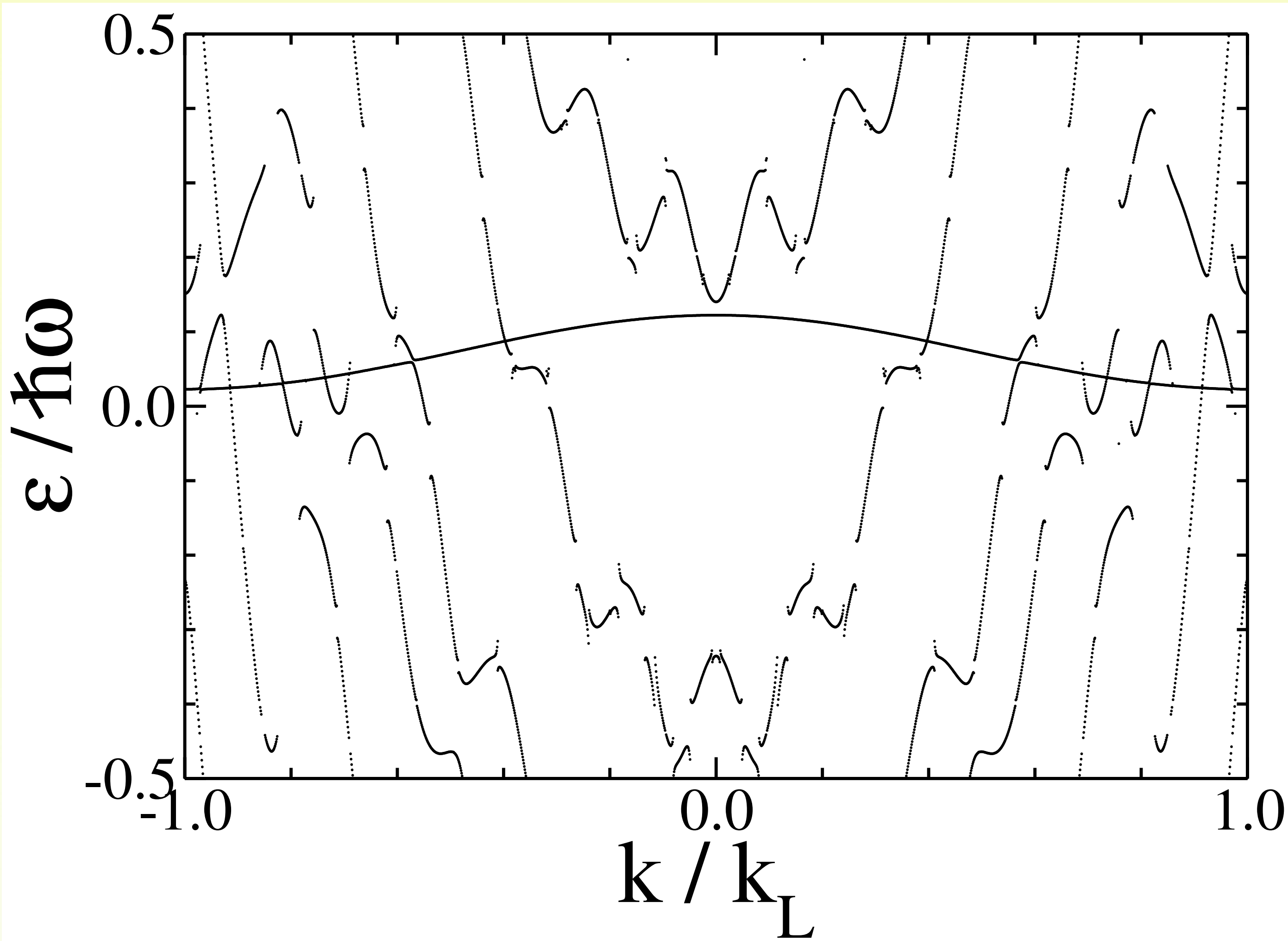
Lattice depth: $V_0/E_R = 8.0$, driving frequency: $\hbar\omega/E_R = 0.5$

▷ Quasienergy dispersion relation for $F_0/(k_L \hbar\omega) = 0.76$



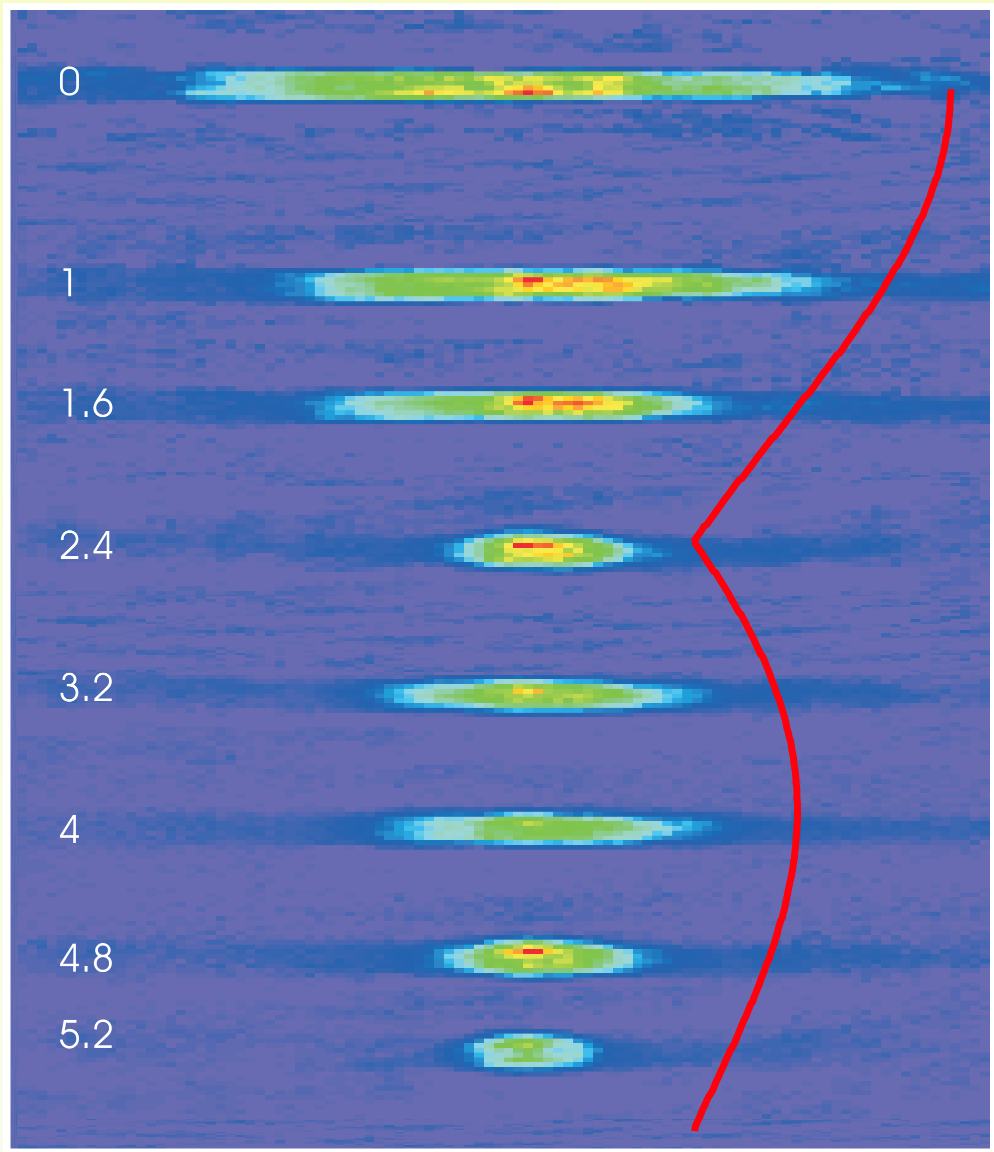
Lattice depth: $V_0/E_R = 8.0$, driving frequency: $\hbar\omega/E_R = 0.5$

▷ Quasienergy dispersion relation for $F_0/(k_L \hbar\omega) = 1.21$

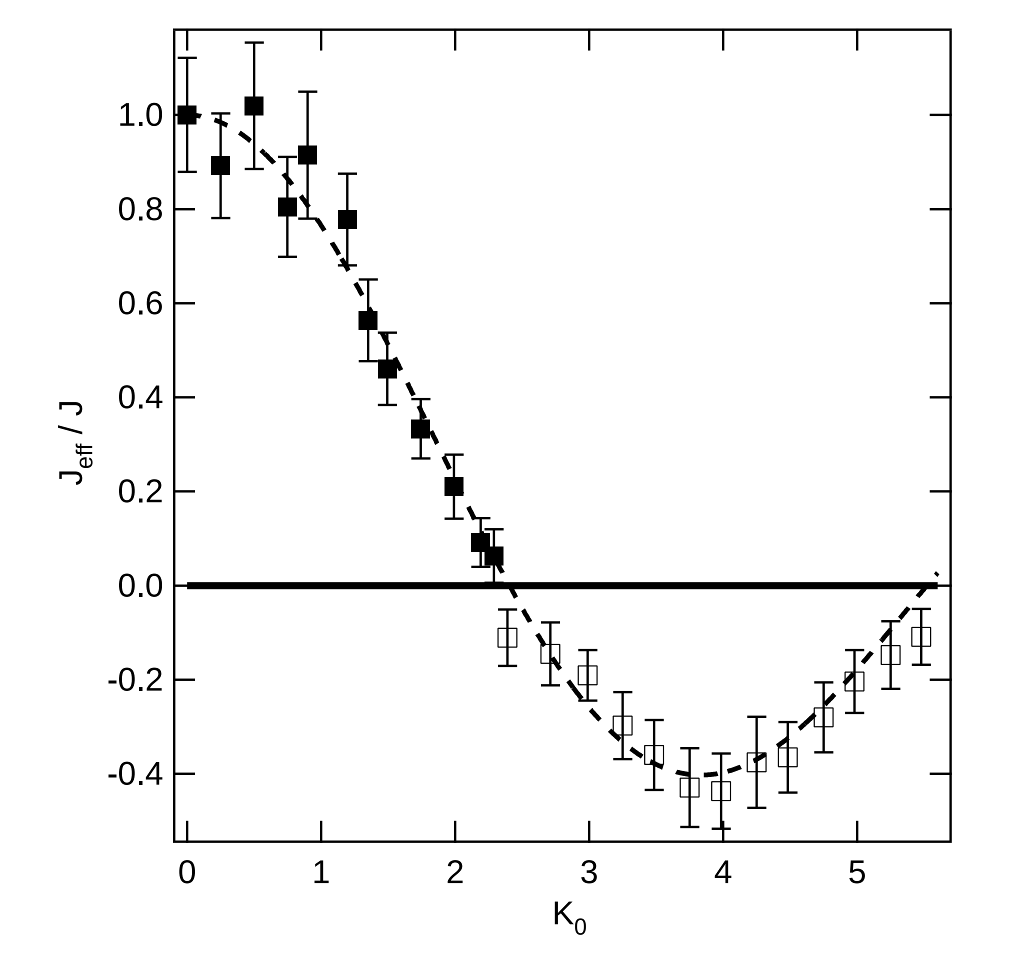


Lattice depth: $V_0/E_R = 8.0$, driving frequency: $\hbar\omega/E_R = 0.5$

- In-situ measurements of expansion rate

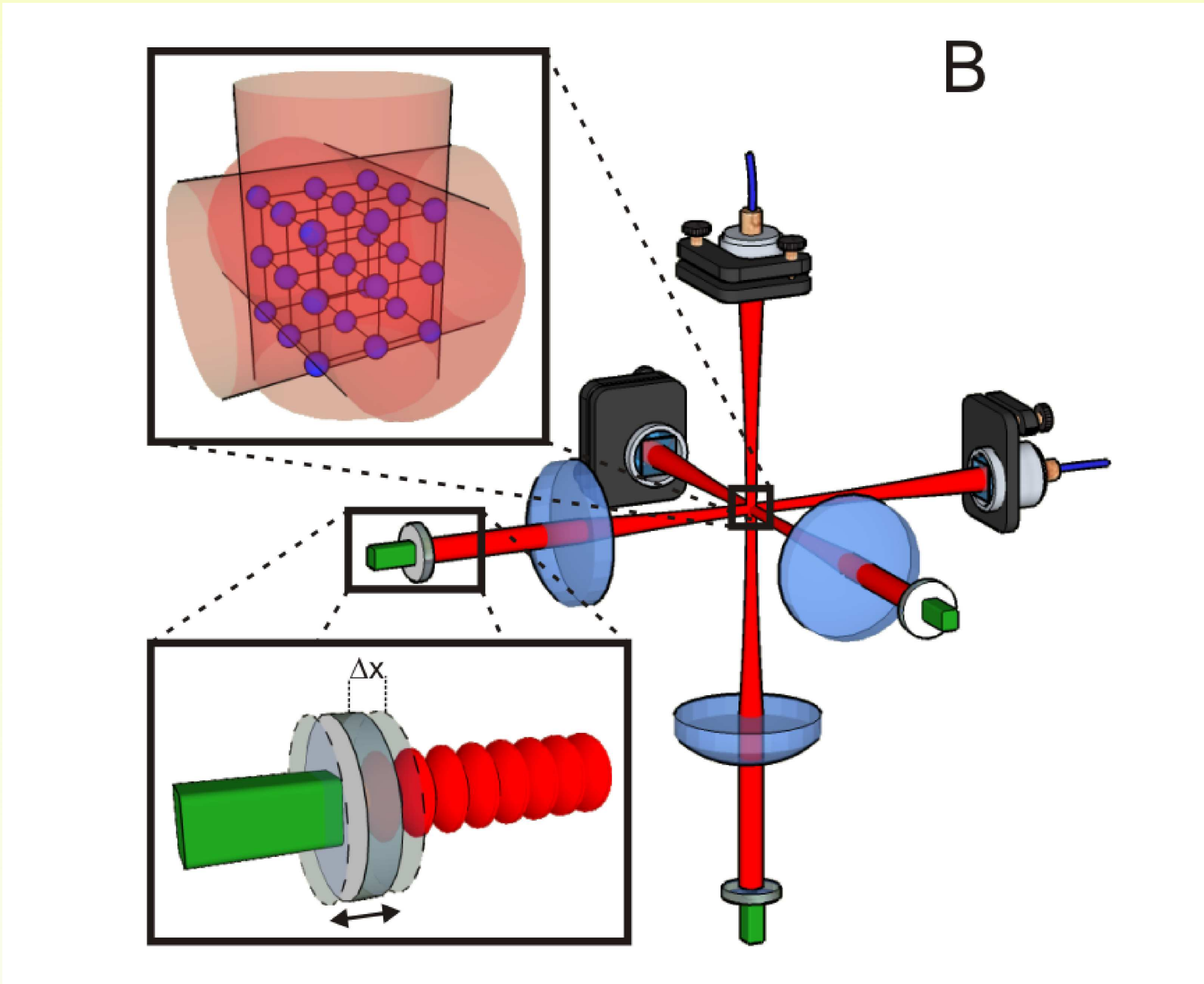


- Experimental data (Pisa group; ^{87}Rb at $\lambda = 842\ nm$)



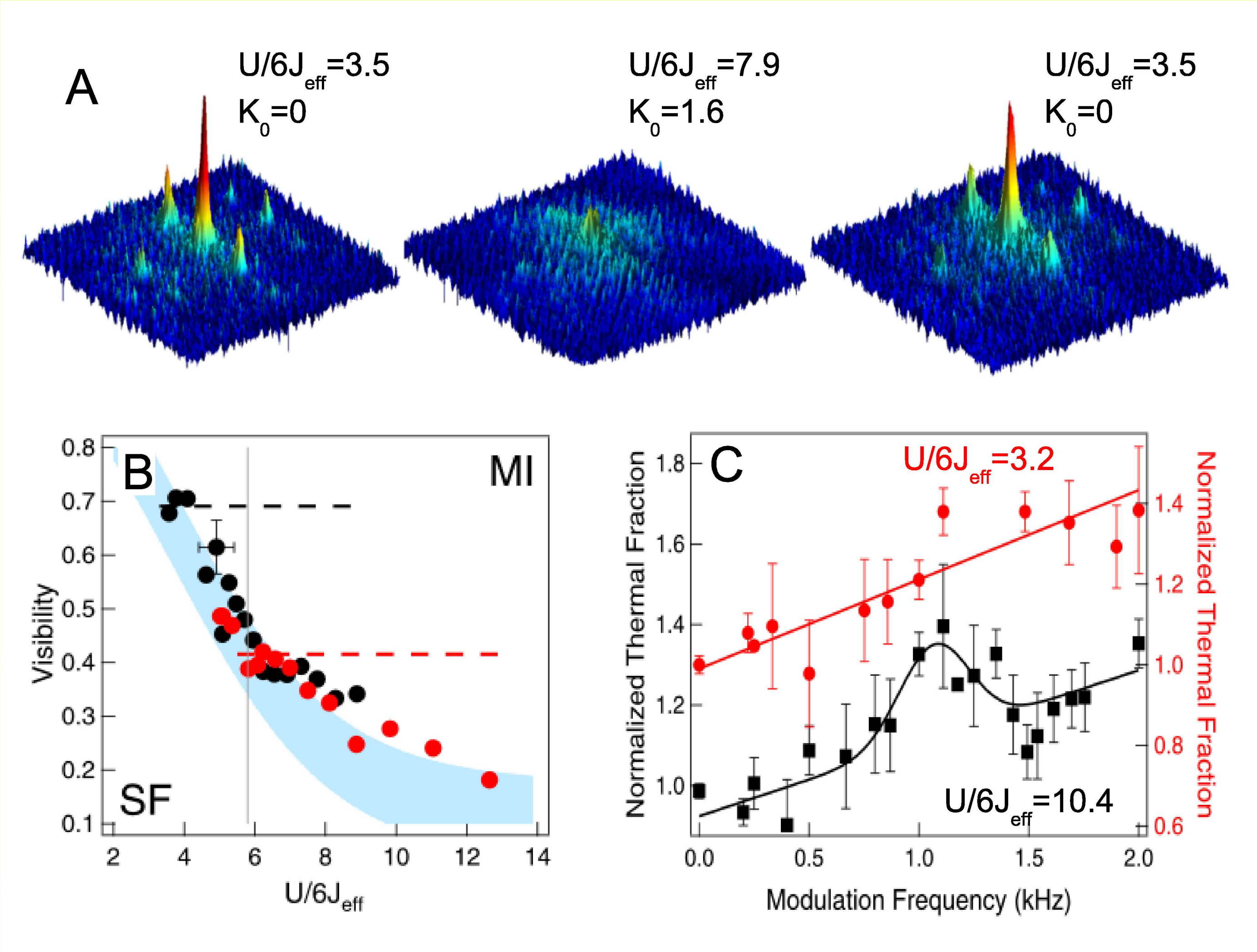
$V_0/E_\mathrm{R} = 6.0$, $\omega/(2\pi) = 4.0$ kHz , expansion time 150 ms

- **Shaken $3d$ optical lattices: The Pisa setup**



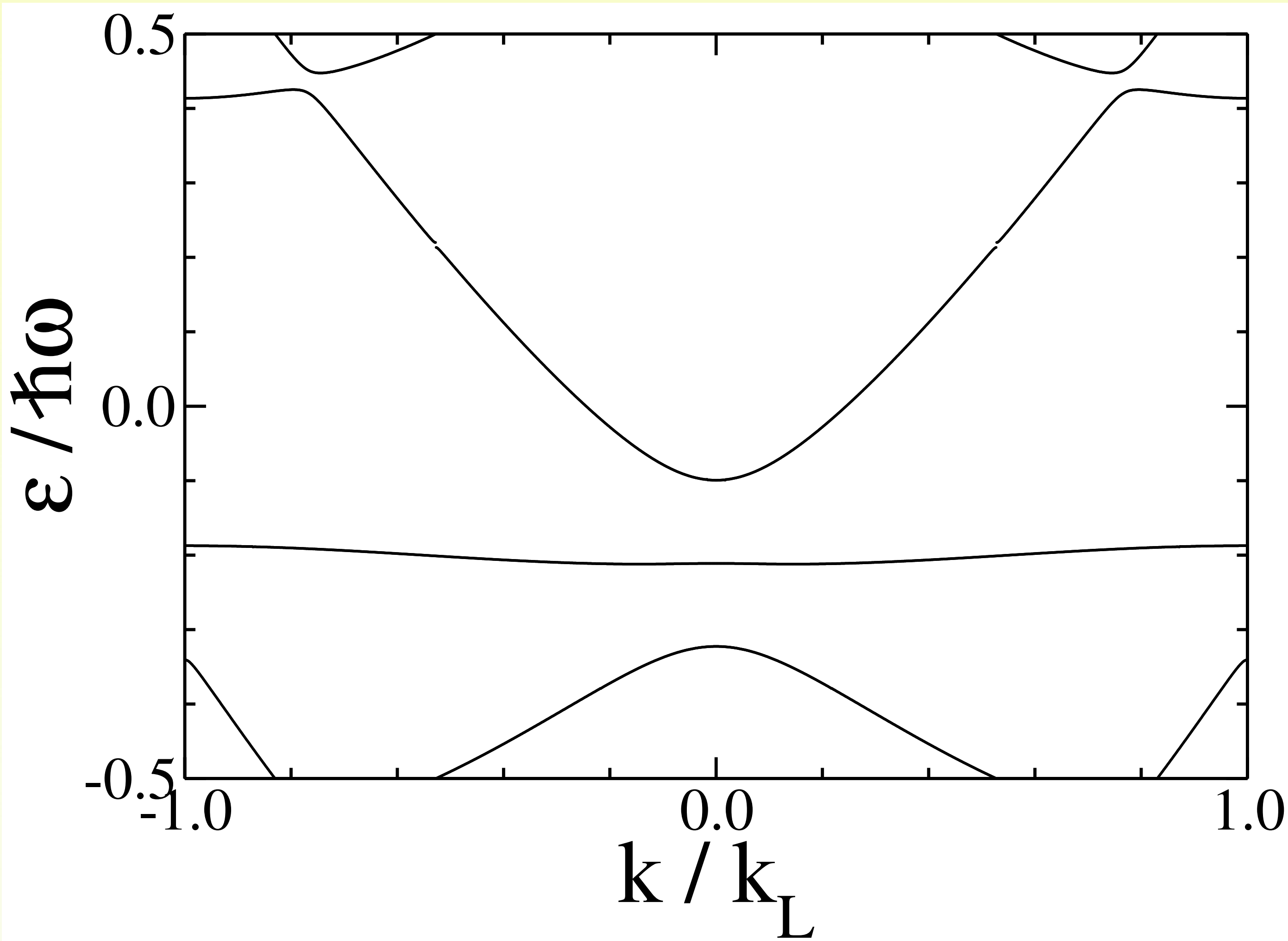
Zenesini et al., PRL **102**, 100403 (2009)

- **SF-MI transition (and back) induced by ac forcing**



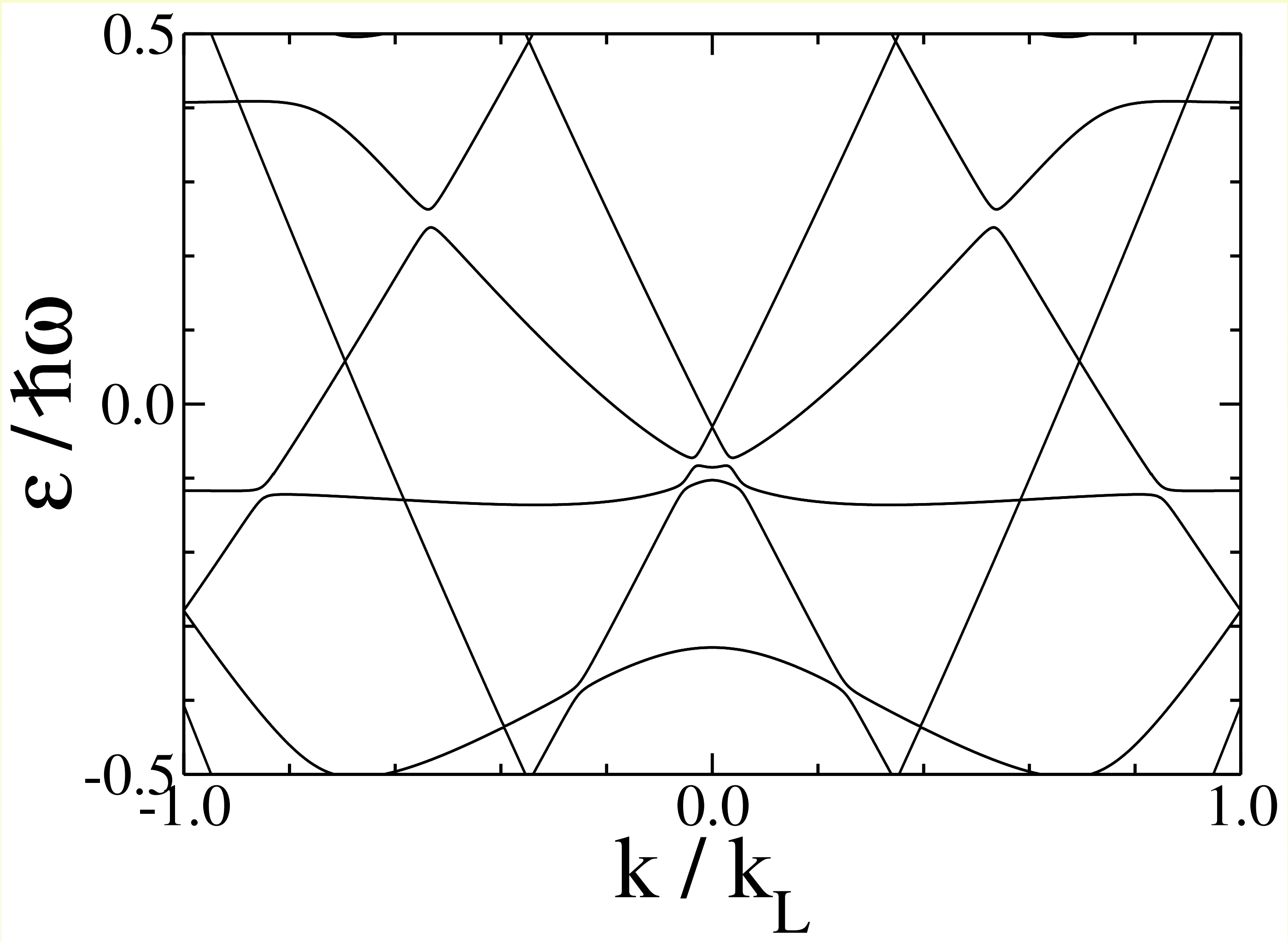
Zenesini et al., PRL **102**, 100403 (2009)

▷ Quasienergy dispersion relation for $F_0/(k_\mathrm{L}\hbar\omega) = 0.17$



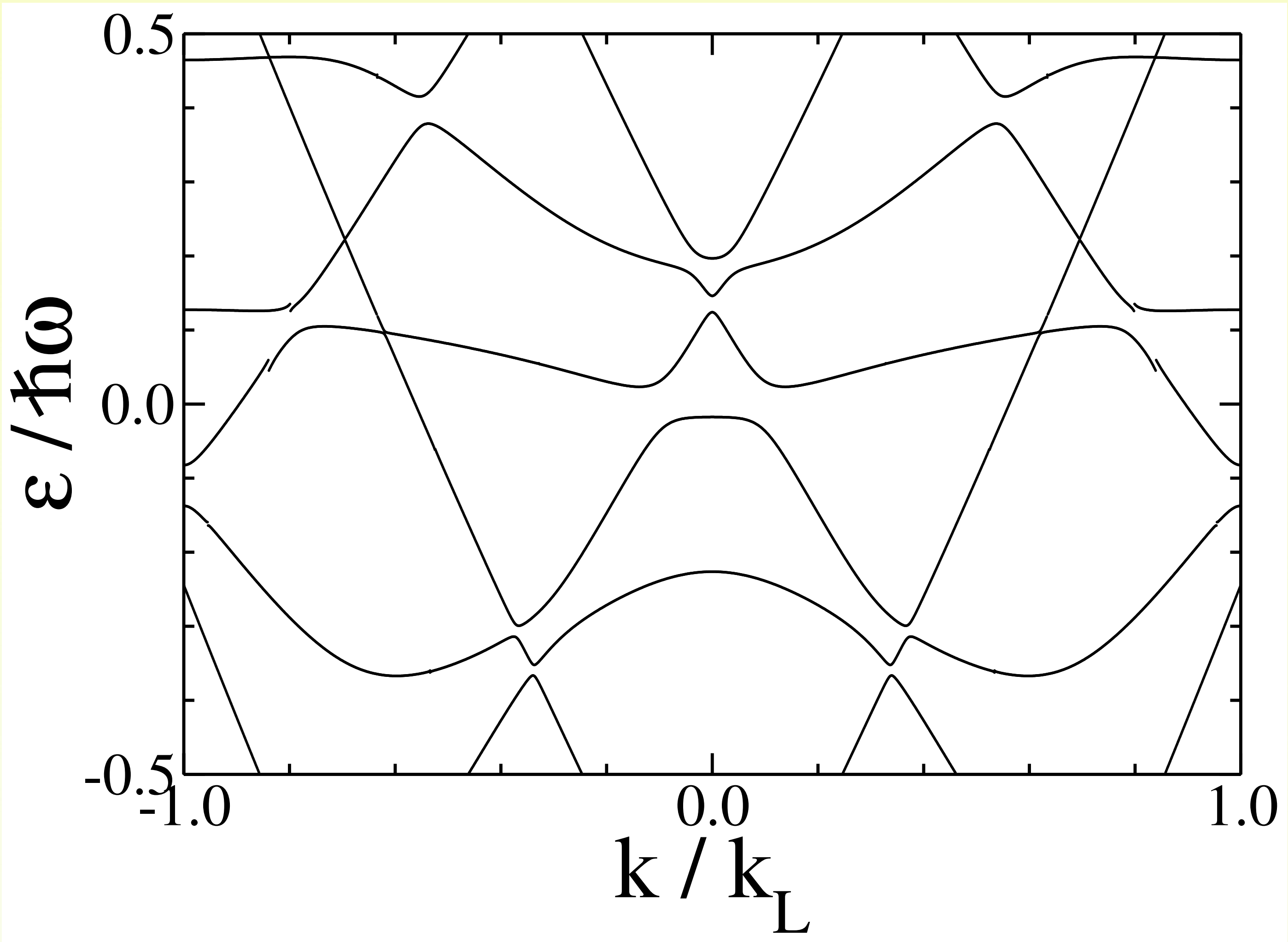
Depth: $V_0/E_\mathrm{R} = 7.0$, frequency: $\hbar\omega/E_\mathrm{R} = 5.51$

▷ Quasienergy dispersion relation for $F_0/(k_\mathrm{L}\hbar\omega) = 0.69$



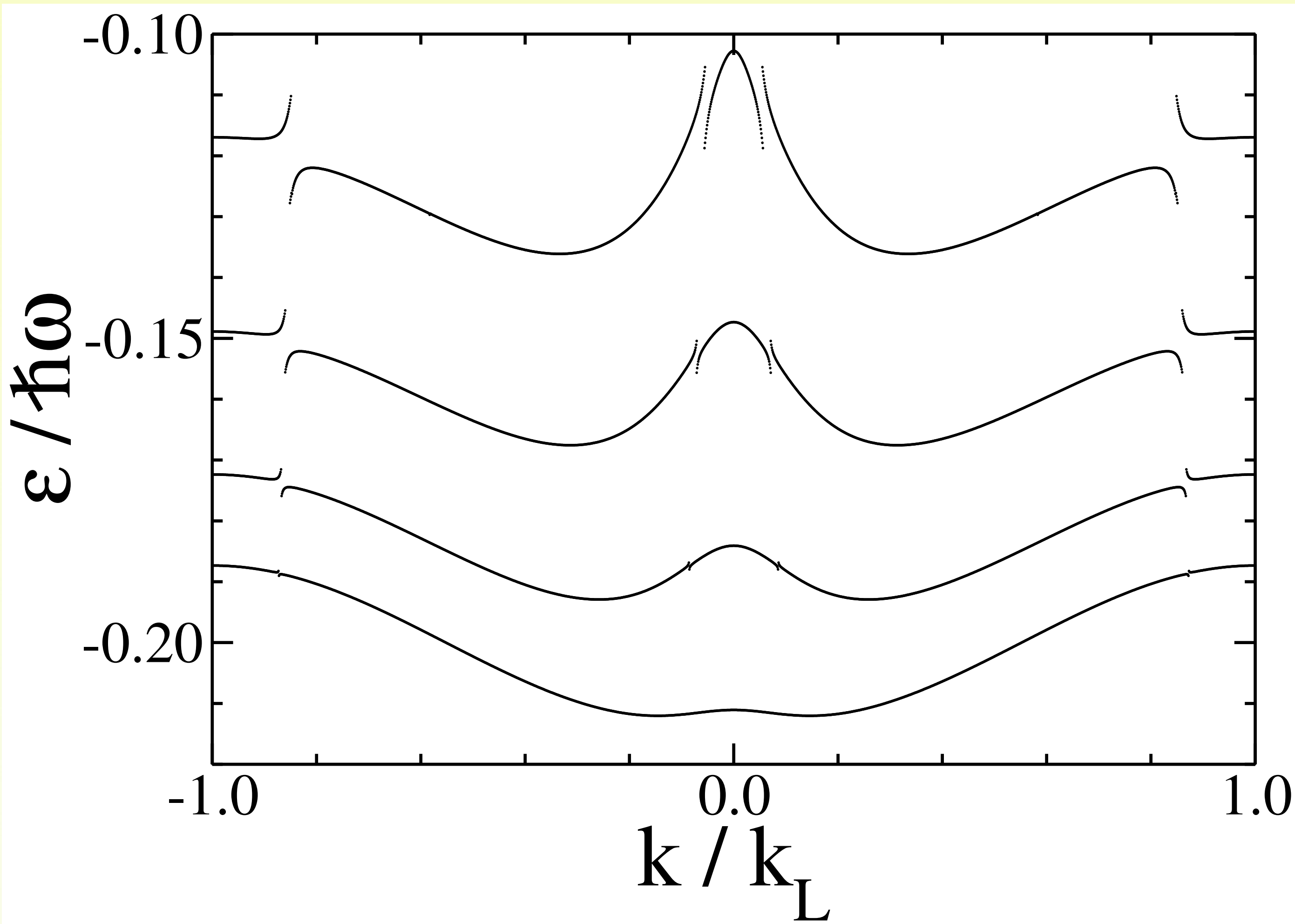
Depth: $V_0/E_\mathrm{R} = 7.0$, frequency: $\hbar\omega/E_\mathrm{R} = 5.51$

▷ Quasienergy dispersion relation for $F_0/(k_\mathrm{L}\hbar\omega) = 1.50$



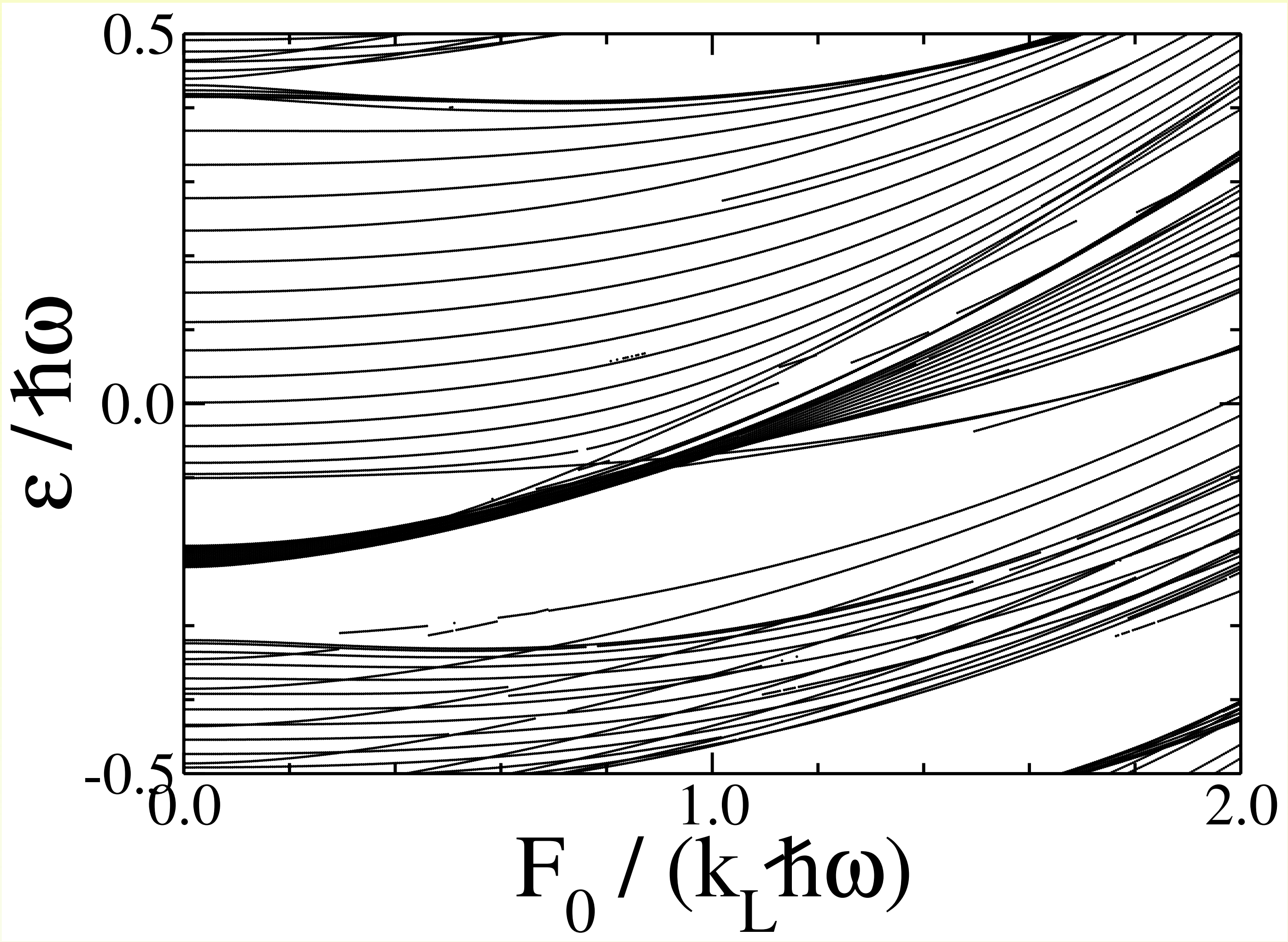
Depth: $V_0/E_\mathrm{R} = 7.0$, frequency: $\hbar\omega/E_\mathrm{R} = 5.51$

▷ “Lowest” band for $F_0/(k_\mathrm{L}\hbar\omega) = 0.17,\ 0.35,\ 0.52,\ 0.69$



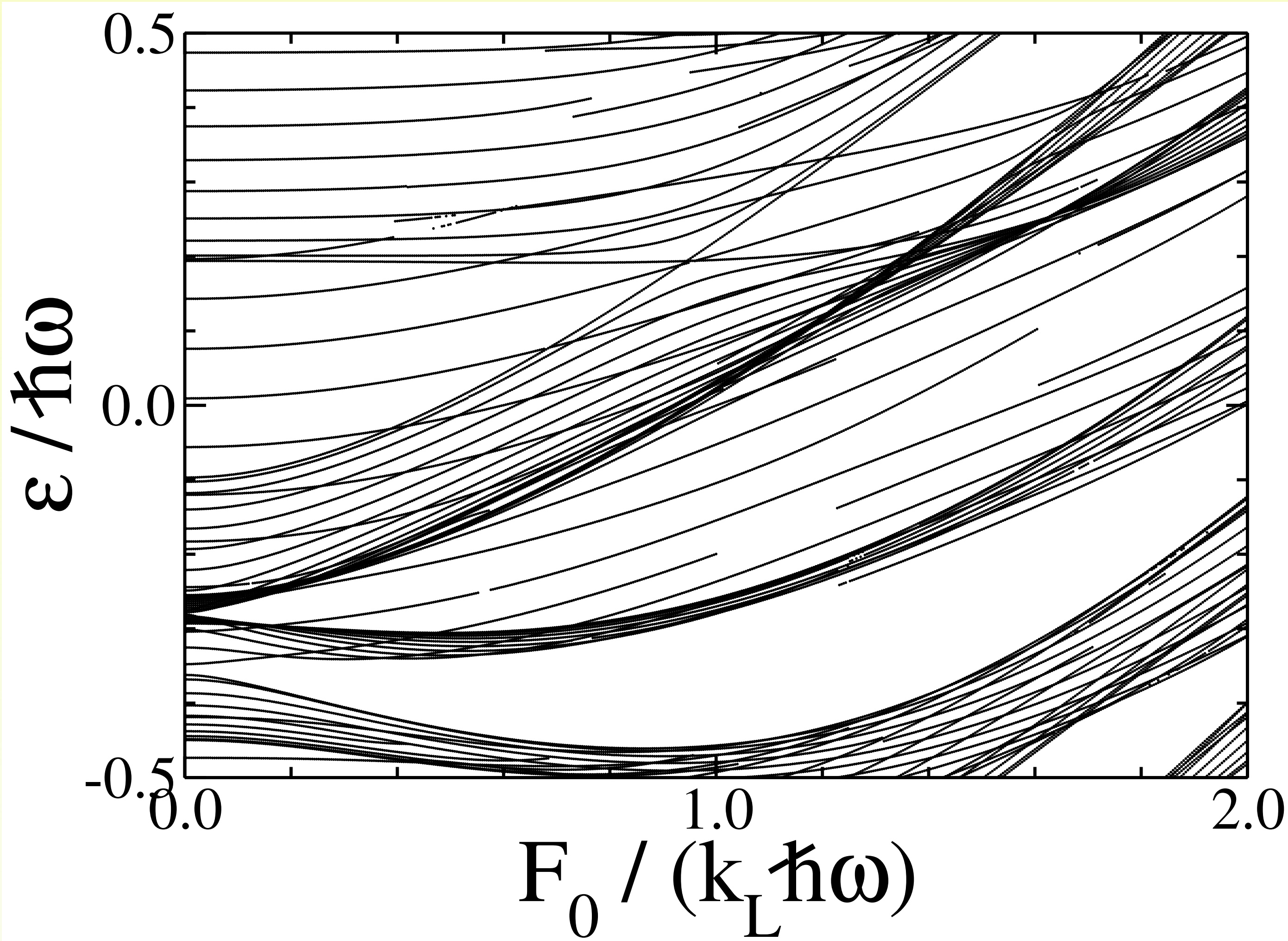
▷ Compare: C. V. Parker, L.-C. Ha, and C. Chin, Nat. Phys. **9**, 769 (2013)

▷ Morphology of quasienergy bands: Dirac points!



Depth: $V_0/E_\mathrm{R} = 7.0$, frequency: $\hbar\omega/E_\mathrm{R} = 5.51$

▷ Morphology of quasienergy bands: Dirac points!



Depth: $V_0/E_R = 7.0$, frequency: $\hbar\omega/E_R = 4.15$

PART IV:

The driven Josephson junction

- **Model:** N Bose particles occupying two sites

$$H_0 = -\frac{\hbar\Omega}{2}\left(a_1 a_2^\dagger + a_1^\dagger a_2\right) + \hbar\kappa\left(a_1^\dagger a_1^\dagger a_1 a_1 + a_2^\dagger a_2^\dagger a_2 a_2\right)$$

with

$$\left[a_j, a_k\right] = 0\,, \qquad \left[a_j^\dagger, a_k^\dagger\right] = 0\,, \qquad \left[a_j, a_k^\dagger\right] = \delta_{jk}$$

Convenient: $\dim \mathcal{H} = N + 1$!

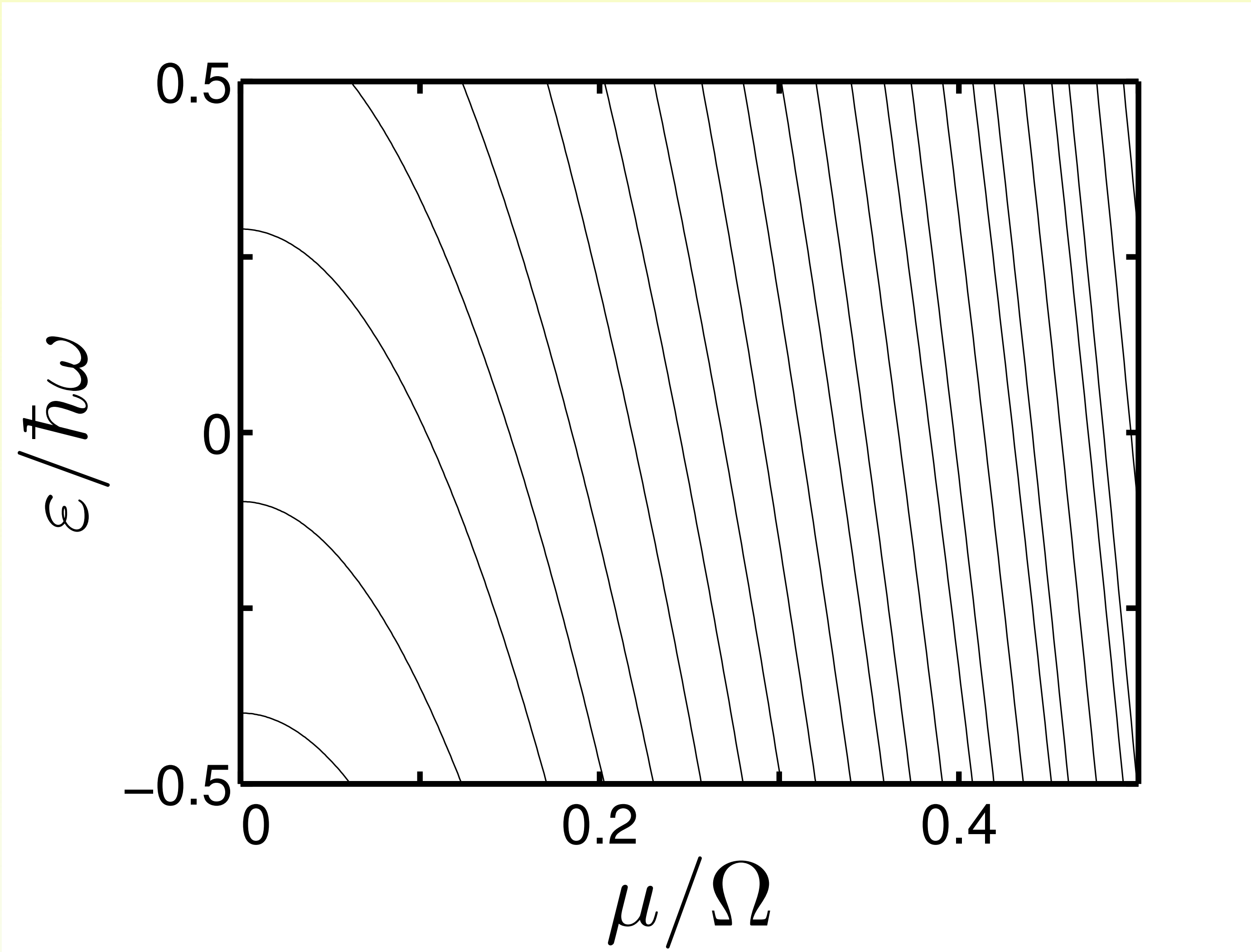
▷ Add site-diagonal forcing:

$$H_1(t) = \hbar\mu(t)\sin(\omega t)\left(a_1^\dagger a_1 - a_2^\dagger a_2\right)$$

▷ Instantaneous quasienergy operators:

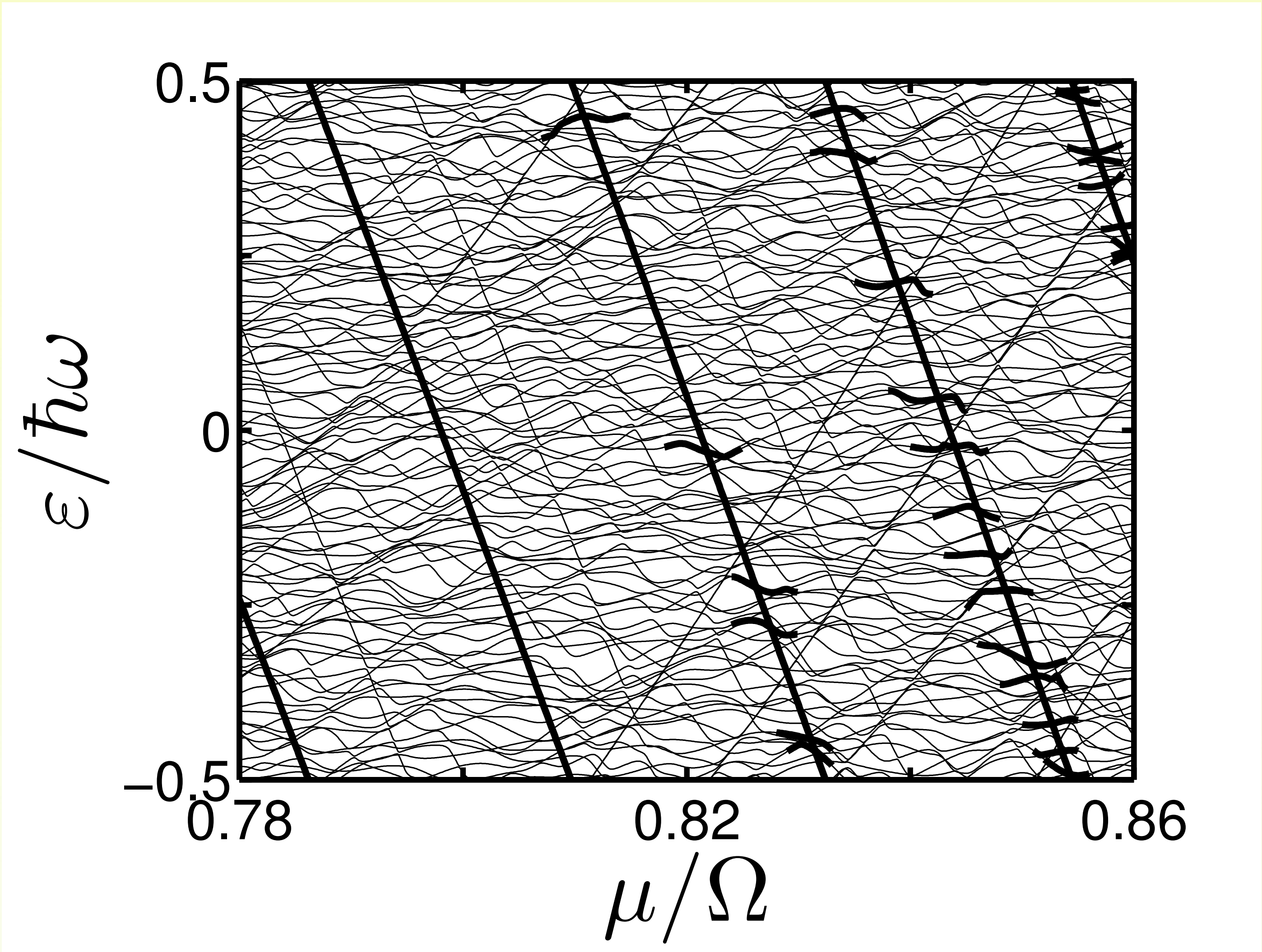
$$K^\mu = H_0 + \hbar\mu\sin(\omega t)\left(a_1^\dagger a_1 - a_2^\dagger a_2\right) + \frac{\hbar}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}t}$$

▷ Quasienergies of “lowest” three Floquet states



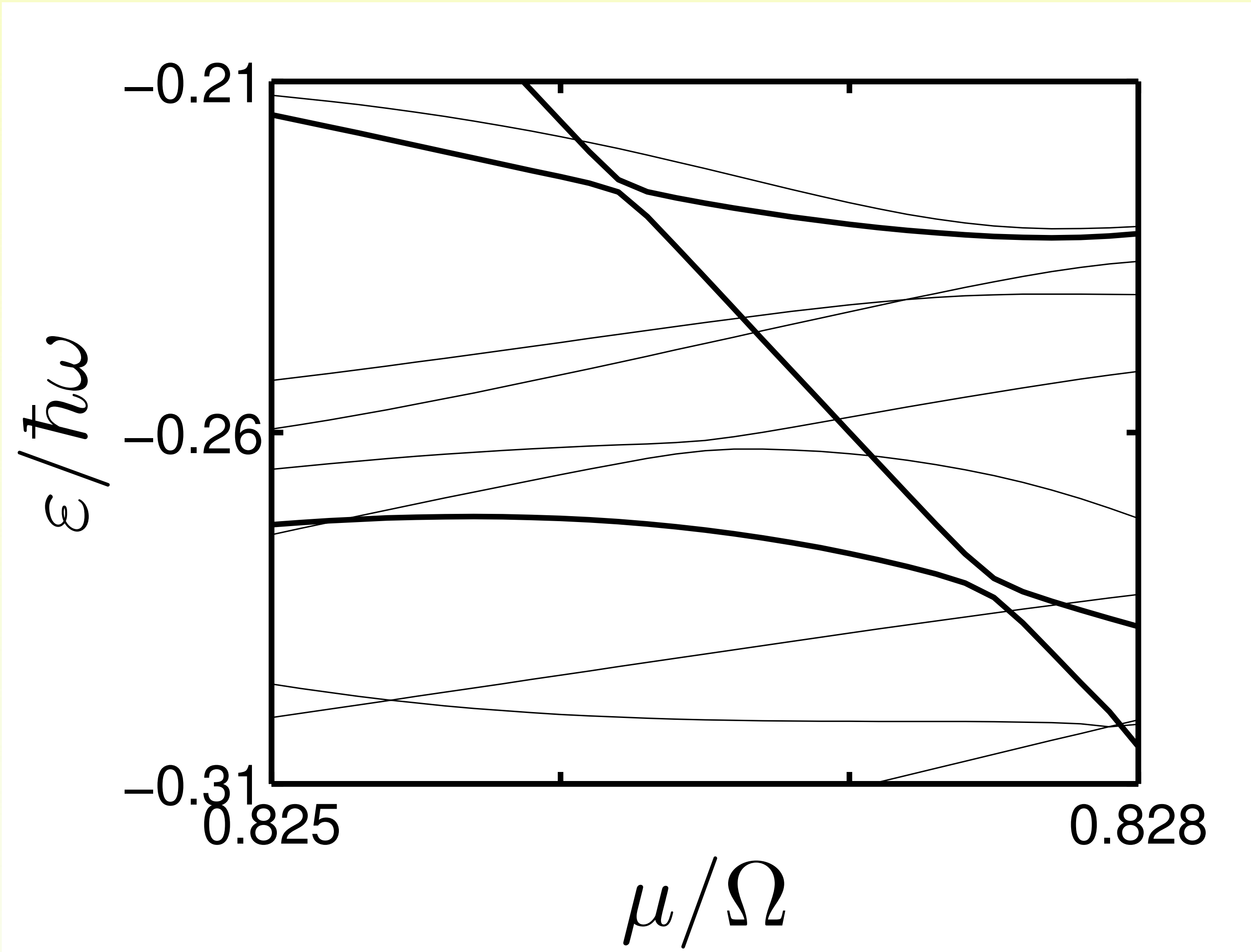
$N = 100$, $N\kappa/\Omega = 0.95$, $\omega/\Omega = 1.0$

▷ Quasienergies of all Floquet states



$N = 100$, $N\kappa/\Omega = 0.95$, $\omega/\Omega = 1.0$

▷ Details: roughness beyond coarse graining



$N = 100$, $N\kappa/\Omega = 0.95$, $\omega/\Omega = 1.0$

- Are there **Floquet condensates** ?

▷ Compute one-particle reduced density matrices

$$\varrho_n = \begin{pmatrix} \langle a_1^\dagger a_1 \rangle_n & \langle a_1^\dagger a_2 \rangle_n \\ \langle a_2^\dagger a_1 \rangle_n & \langle a_2^\dagger a_2 \rangle_n \end{pmatrix}$$

(cf. Penrose-Onsager criterion)

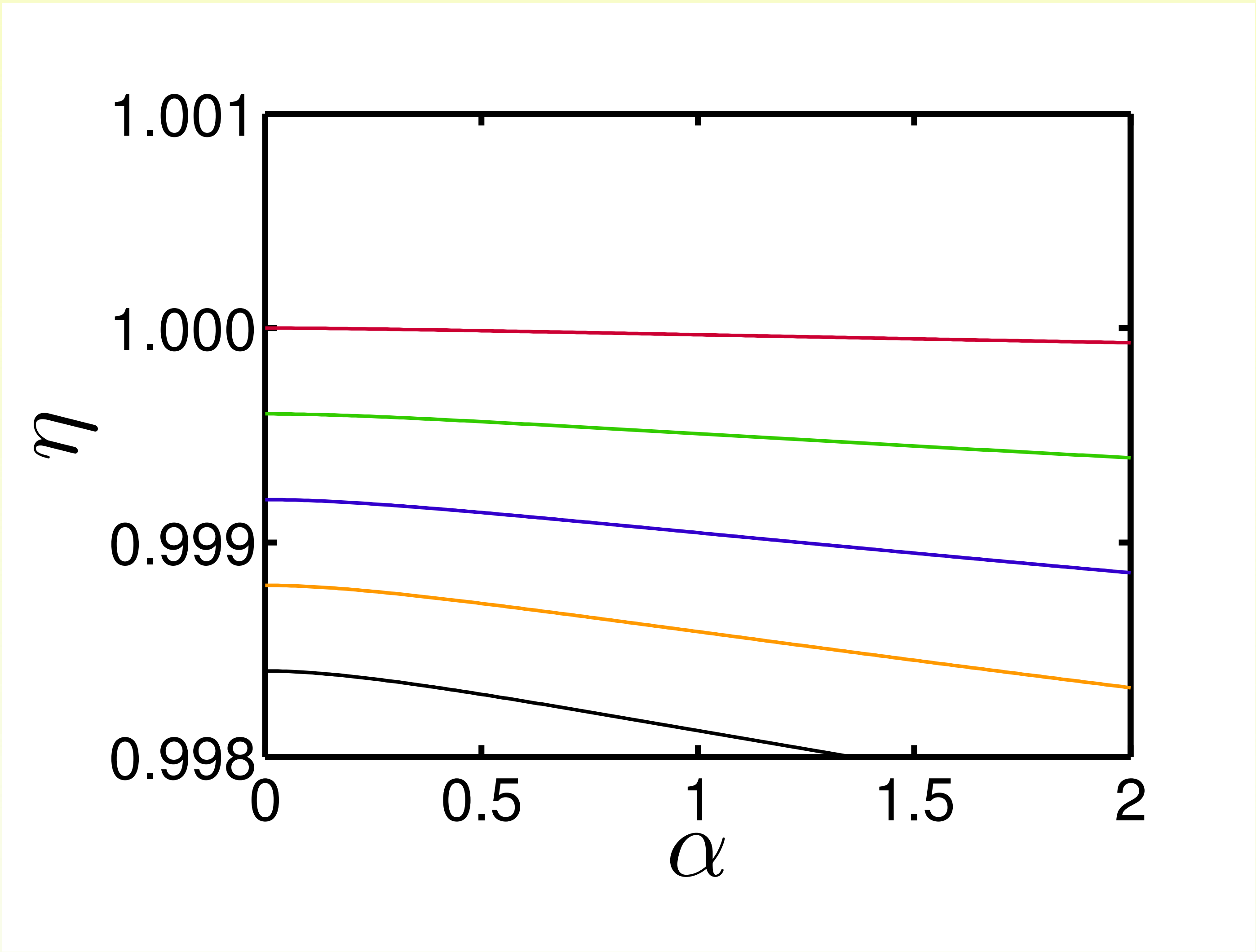
▷ "Degree of coherence" (simplicity)

$$\eta_n = 2N^{-2} \,\mathrm{tr}\, \varrho_n^2 - 1$$

$\eta_n = 1$ for N -fold occupied single particle states

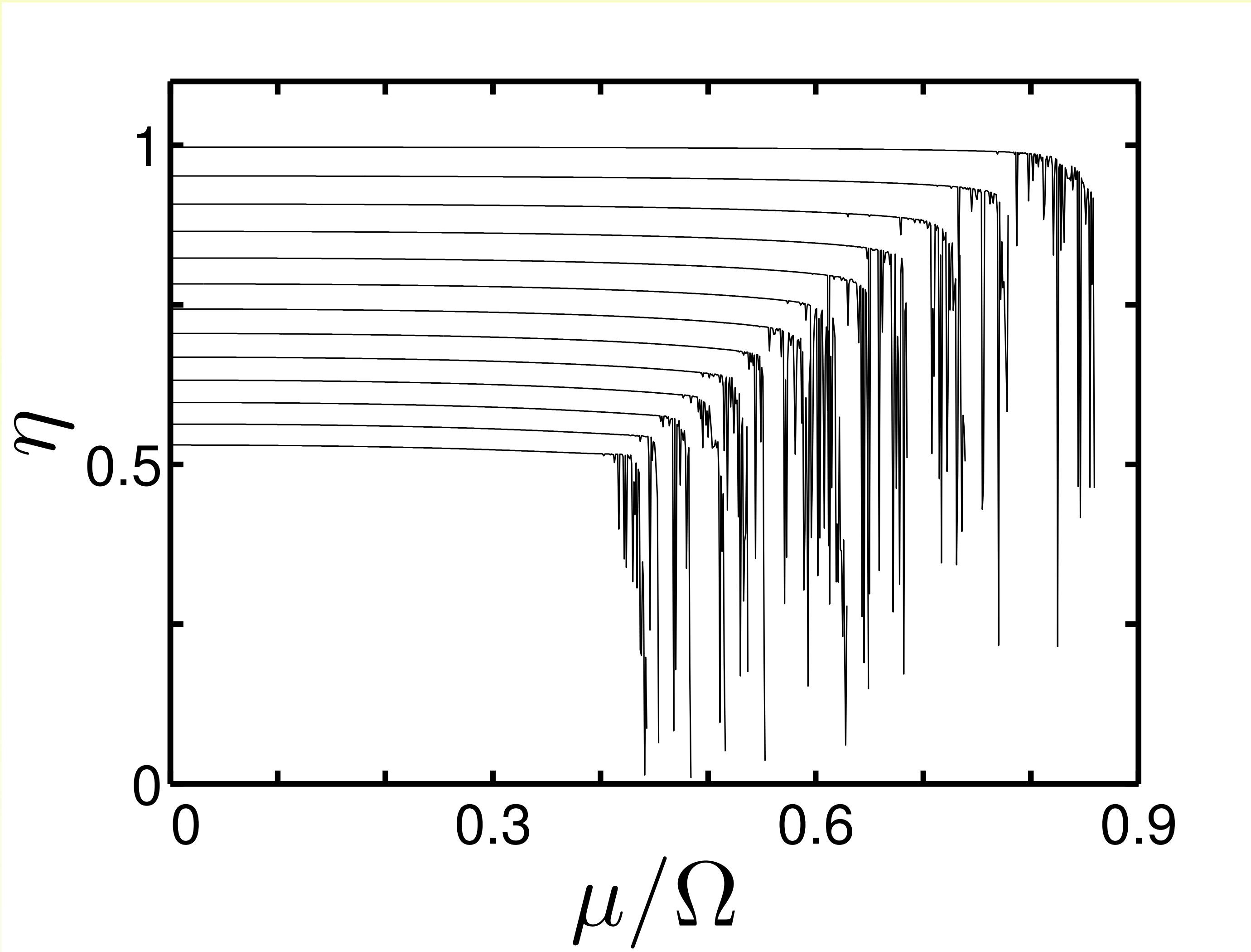
$\eta_n = 0$ for maximally fractionalized states

▷ Degree of coherence for lowest energy eigenstates



$N = 10.000$, $\alpha = N\kappa/\Omega$

▷ Degree of coherence for "lowest" Floquet states



$N = 100$, $N\kappa/\Omega = 0.95$, $\omega/\Omega = 1.0$

• Adiabatic preparation of Floquet condensates

▷ Gaussian turn-on of driving:

$$\mu(t) = \begin{cases} \mu_{\mathrm{max}} \exp\left(-\frac{t^2}{2\sigma^2}\right) & , \ t \leq 0 \\ \mu_{\mathrm{max}} & , \ t > 0 \end{cases}$$

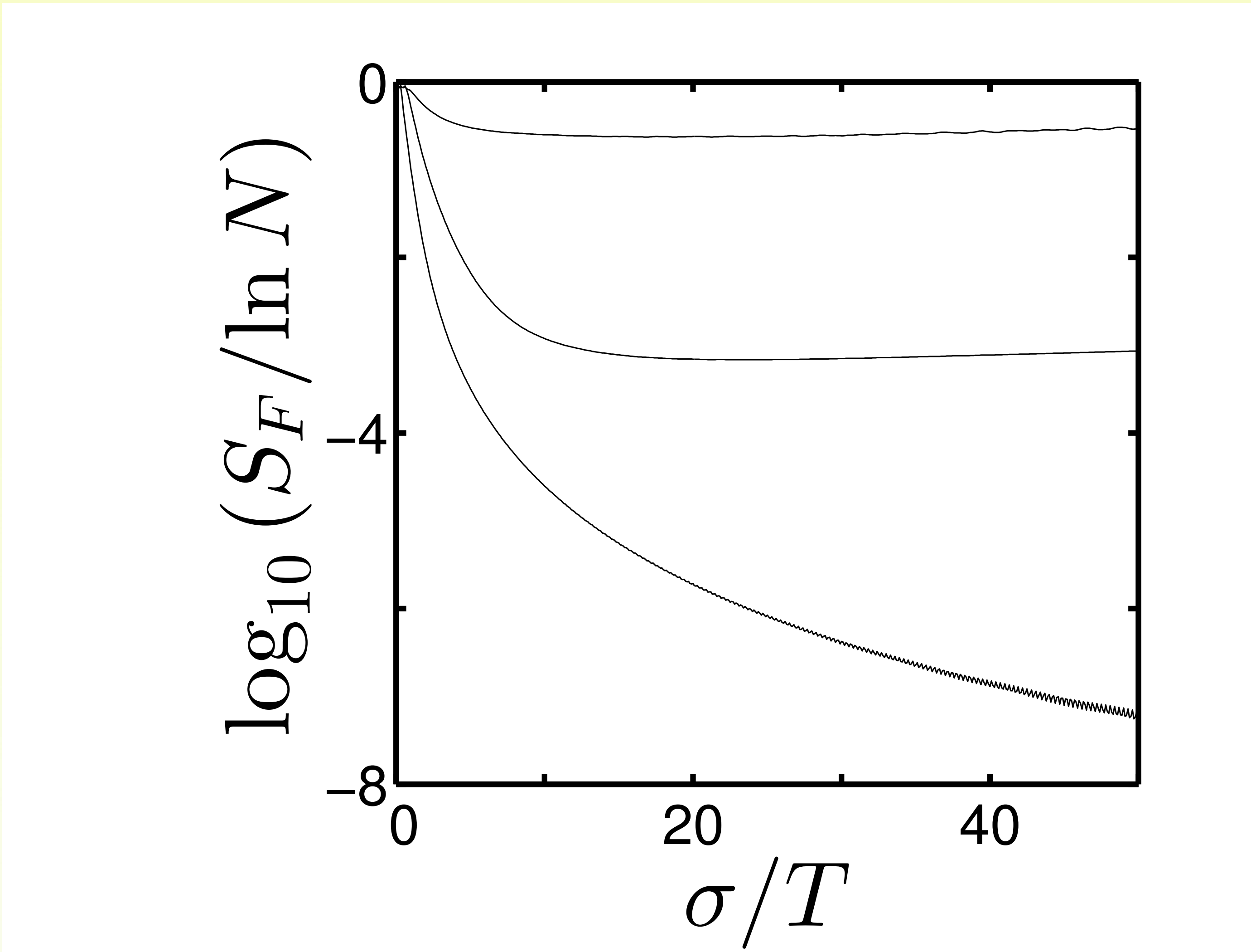
▷ Decompose w.r.t. instantaneous Floquet states:

$$|\psi(t)\rangle = \sum_n a_n \, |u_n^\mu(t)\rangle \ \exp(-\mathrm{i}\varepsilon_n^\mu t/\hbar)$$

▷ Compute Floquet entropy:

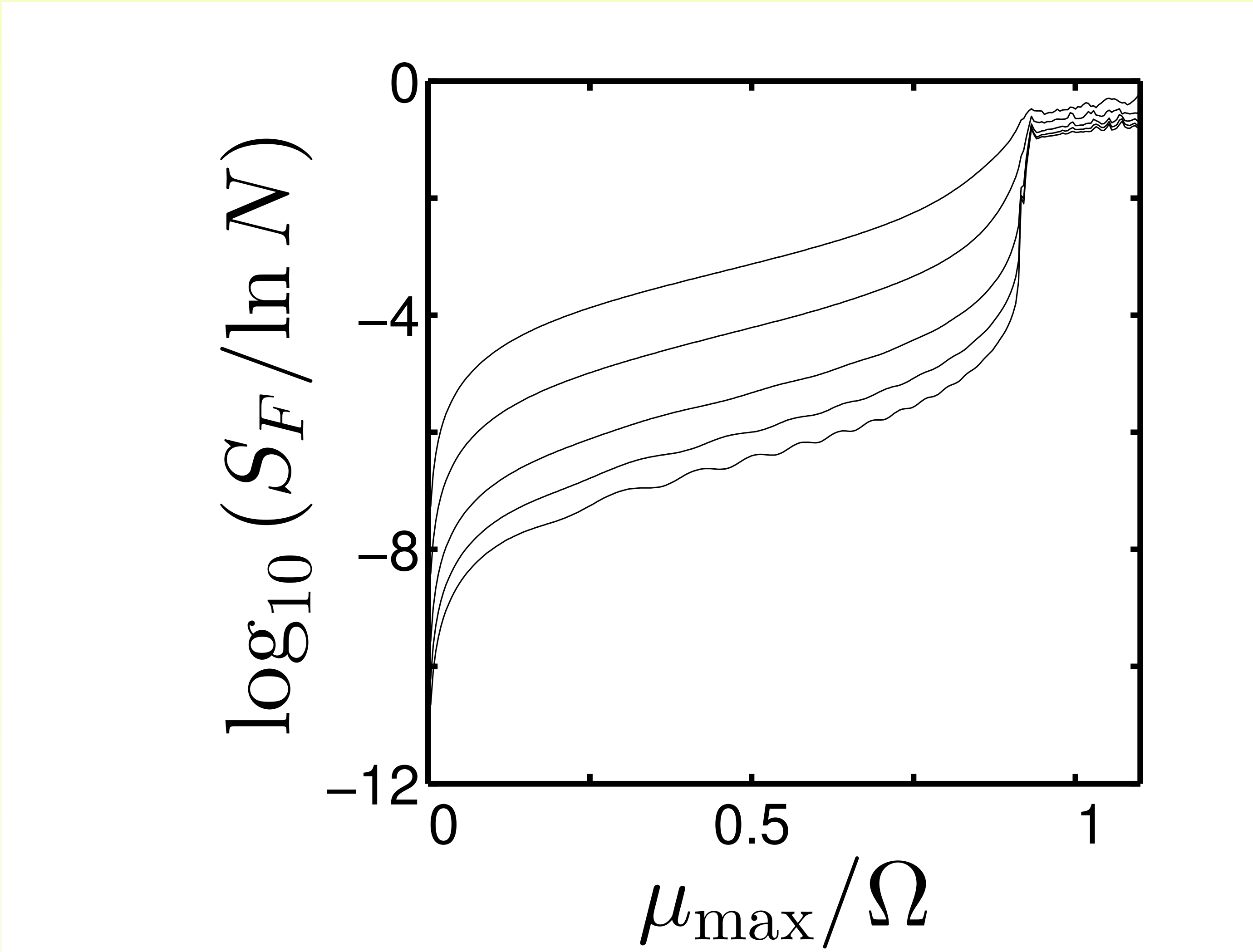
$$S_F(t) = -\sum_n |a_n(t)|^2 \ \ln |a_n(t)|^2$$

▷ Floquet entropy after turn-on: $\mu_{\mathrm{max}}/\Omega = 0.6,\ 0.8,\ 0.9$



$N = 100$, $N\kappa/\Omega = 0.95$, $\omega/\Omega = 1.0$

▷ Floquet entropy after turn-on: $\sigma/T = 5,\ 10,\ 20,\ 30,\ 40$



$N = 1000$, $N\kappa/\Omega = 0.95$, $\omega/\Omega = 1.0$

• Entropy production within a pulsed BEC

▷ Full Gaussian pulses:

$$\mu(t) = \mu_{\mathsf{max}} \exp\left(-\frac{t^2}{2\sigma^2}\right) \quad , \quad -\infty < t < +\infty$$

▷ Monitor population imbalance:

$$\langle J_z \rangle / N = \frac{1}{2N} \langle \psi(t) | a_1^\dagger a_1 - a_2^\dagger a_2 | \psi(t) \rangle$$

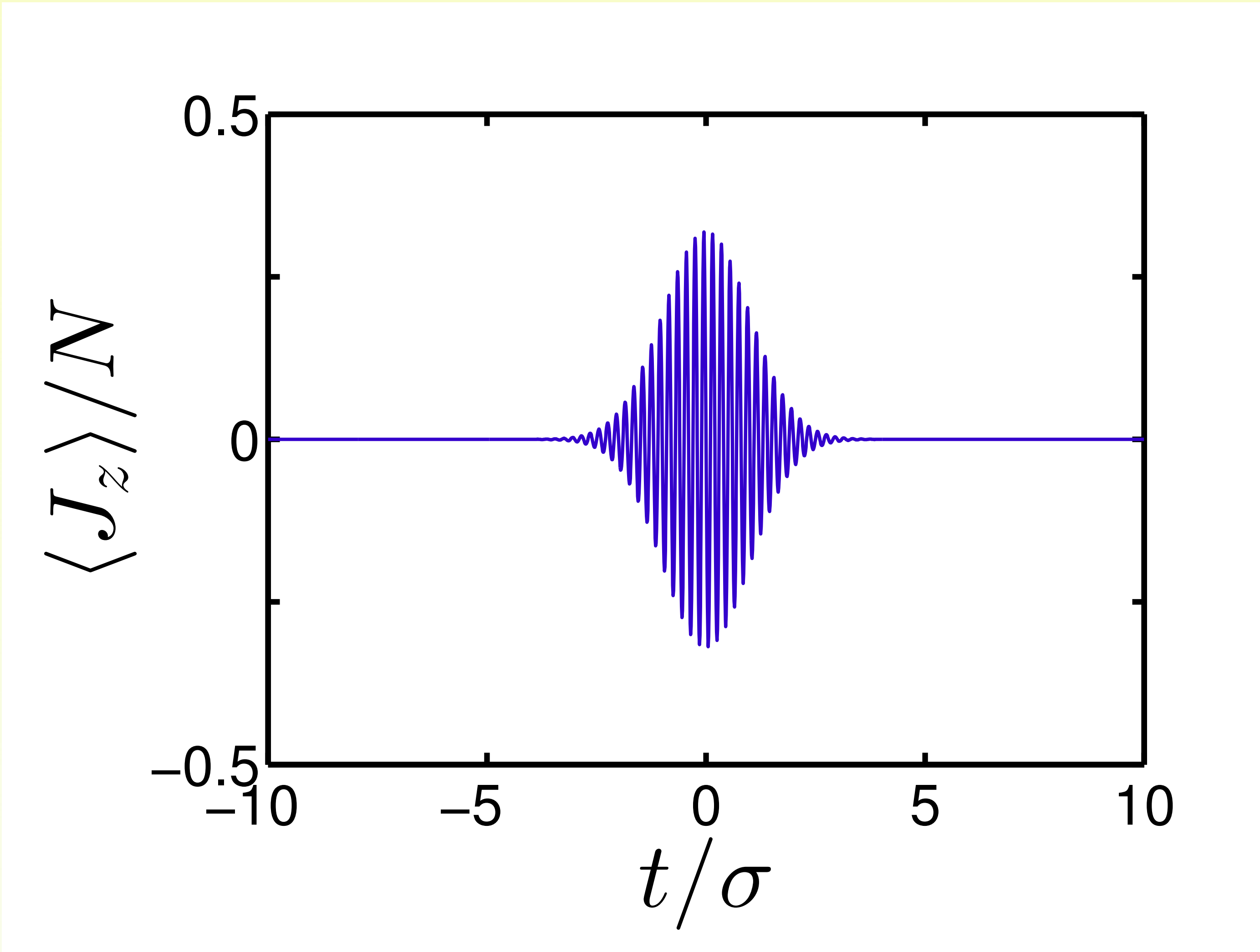
▷ Determine final occupation probabilities:

$$p_n = \left| \langle n | \psi(t_f) \rangle \right|^2$$

▷ Compute von Neumann entropy generated by pulse:

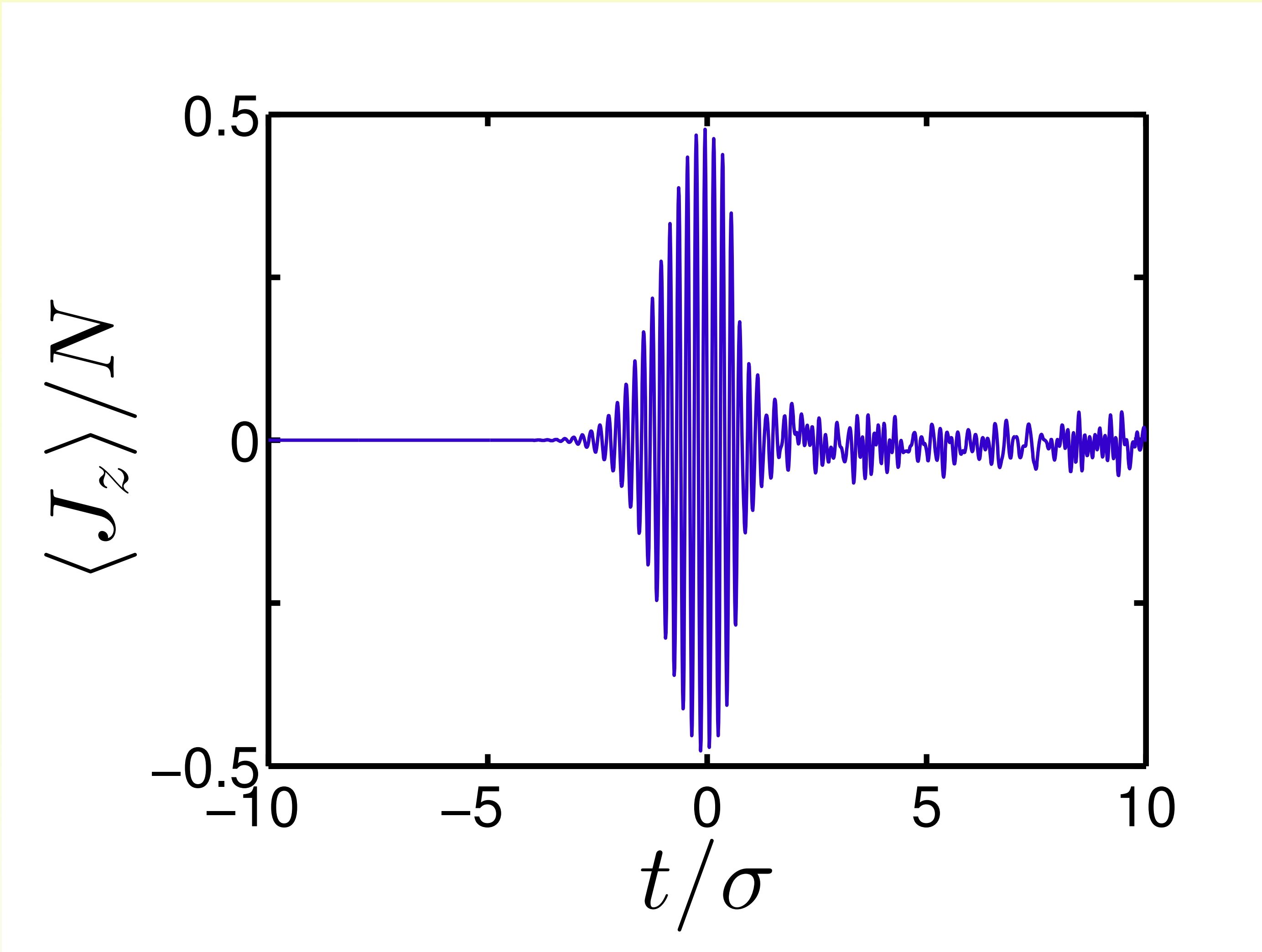
$$S = -\sum_n p_n \ \ln p_n$$

▷ Adiabatic response for moderate maximum amplitude



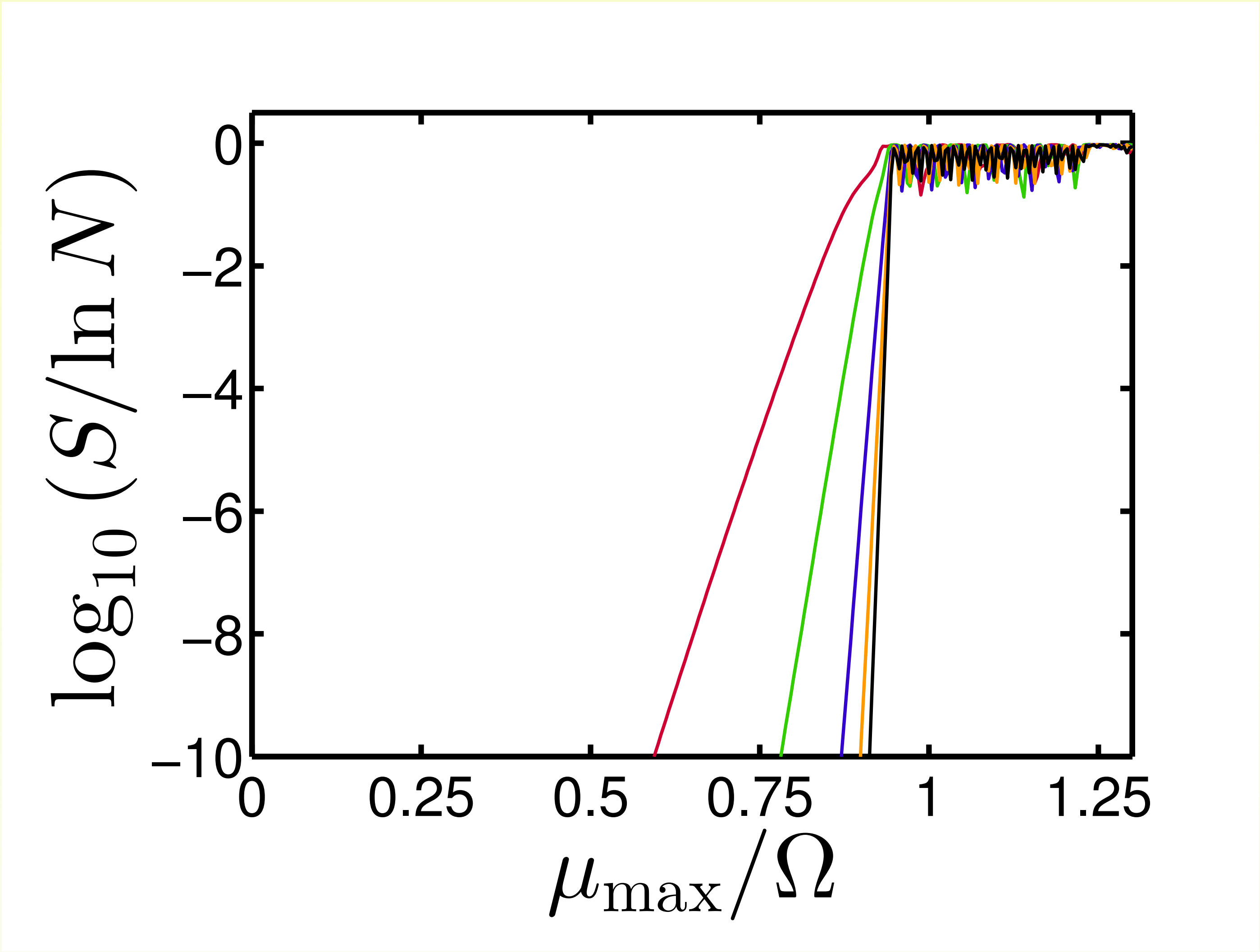
$N = 100$, $N\kappa/\Omega = 0.95$, $\mu_{\mathrm{max}}/\Omega = 0.60$, $\sigma/T = 5.0$

▷ Loss of adiabaticity for strong driving



$N = 100$, $N\kappa/\Omega = 0.95$, $\mu_{\max}/\Omega = 0.90$, $\sigma/T = 5.0$

▷ Sharp "chaos border": $\sigma/T = 5,\ 10,\ 20,\ 30,\ 40$



$N = 10.000$, $N\kappa/\Omega = 0.95$, $\omega/\Omega = 1.0$

• Universal behavior for quantum resonances

▷ Assume $E'_r \equiv E_{r+1} - E_r \approx \hbar\omega$

▷ Ansatz:

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}\eta t/\hbar} \sum_n b_n \, |n\rangle \exp\left[-\frac{\mathrm{i}}{\hbar}\Big(E_r + (n-r)\hbar\omega\Big)t\right]$$

This gives

$$\eta b_n = \Big(E_n - E_r - (n-r)\hbar\omega\Big) b_n + 2\hbar\mu\cos(\omega t) \sum_m \mathrm{e}^{\mathrm{i}(n-m)\omega t} \langle n|J_z|m\rangle b_m$$

with

$$J_z = \left(a_1^\dagger a_1 - a_2^\dagger a_2\right)/2$$

▷ RWA-type “resonance” approximation:

$$\eta b_n = \frac{1}{2}(n-r)^2 E''_r b_n + \hbar\mu \langle r|J_z|r-1\rangle \Big(b_{n+1} + b_{n-1}\Big)$$

▷ Fourier representation . . .

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\ f(\theta)\mathrm{e}^{-\mathrm{i}(n-r)\theta}$$

. . . yields

$$\eta f(\theta) = -\frac{1}{2}E_r'' f''(\theta) + 2\hbar\mu\langle r|J_z|r-1\rangle \cos\theta\ f(\theta)$$

▷ This is a **Mathieu equation** ("pendulum approximation"):

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \alpha - 2q\cos(2z)\right)\chi(z) = 0$$

with

$$\alpha = \frac{8\eta}{E_r''}\,,$$

$$q = \frac{4}{E_r''/(\hbar\omega)}\frac{2\mu}{\omega}\langle r|J_z|r-1\rangle$$

▷ π -periodic solutions $\chi(z)$ require **characteristic values**

$$\alpha_k(q) = \begin{cases} a_k(q) & \text{for} \quad k = 0, 2, 4, \ldots \\ b_{k+1}(q) & \text{for} \quad k = 1, 3, 5, \ldots \ , \end{cases}$$

▷ Approximation for near-resonant Floquet states:

$$|\psi_k(t)\rangle = \exp\left(-\frac{\mathrm{i}}{8\hbar} E_r'' \alpha_k t\right) \sum_\ell f_{\ell,k} |r + \ell\rangle \exp\left[-\frac{\mathrm{i}}{\hbar}\Big(E_r + \ell\hbar\omega\Big)t\right]$$

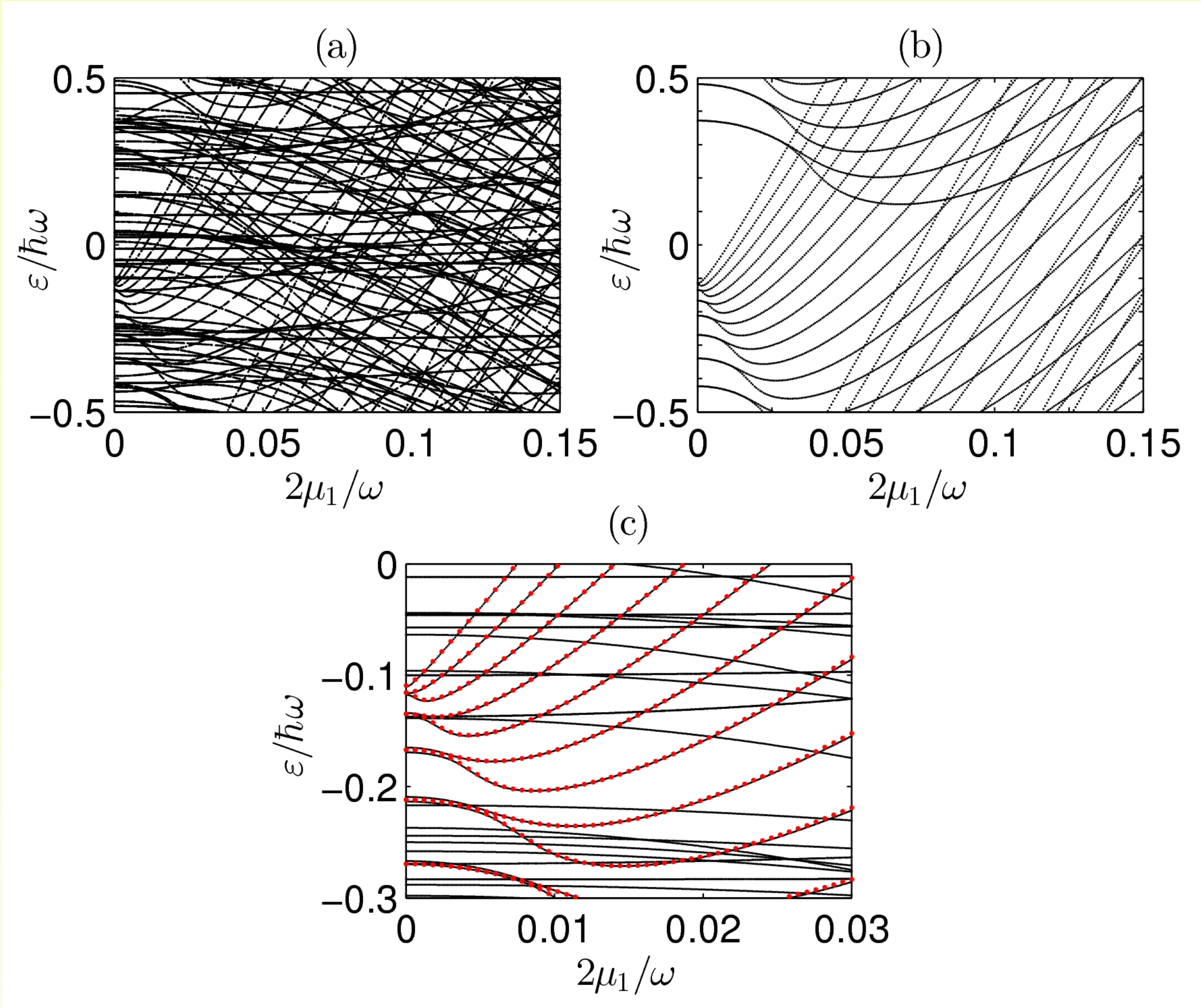
with quasienergies

$$\varepsilon_k = E_r + \frac{1}{8} E_r'' \alpha_k(q) \quad \text{mod } \hbar\omega$$

- **Observe:** New quantum number k

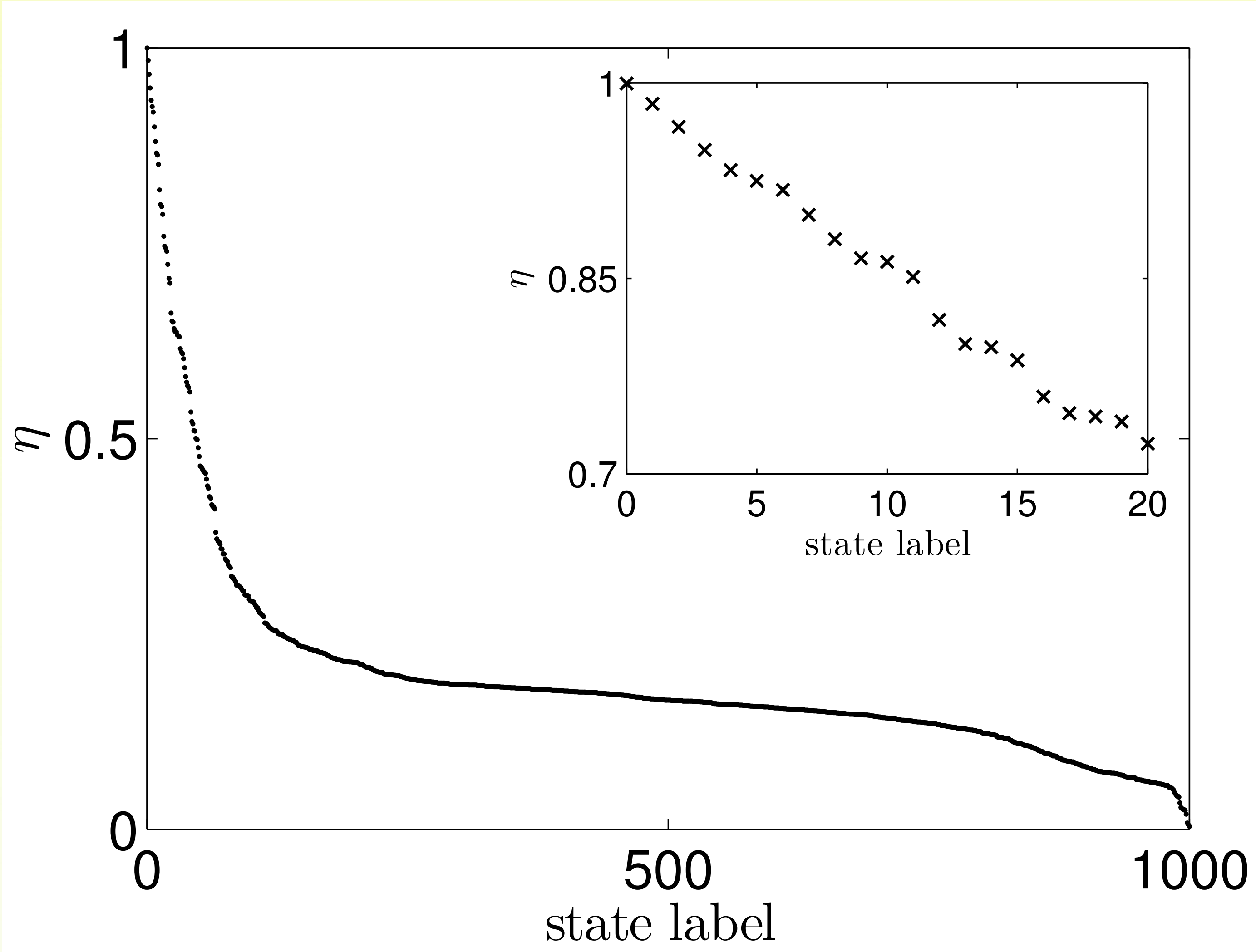
Resonances effectuate a nonperturbative reorganization of the quasienergy spectrum!

▷ Comparison: Exact and approximate quasienergies



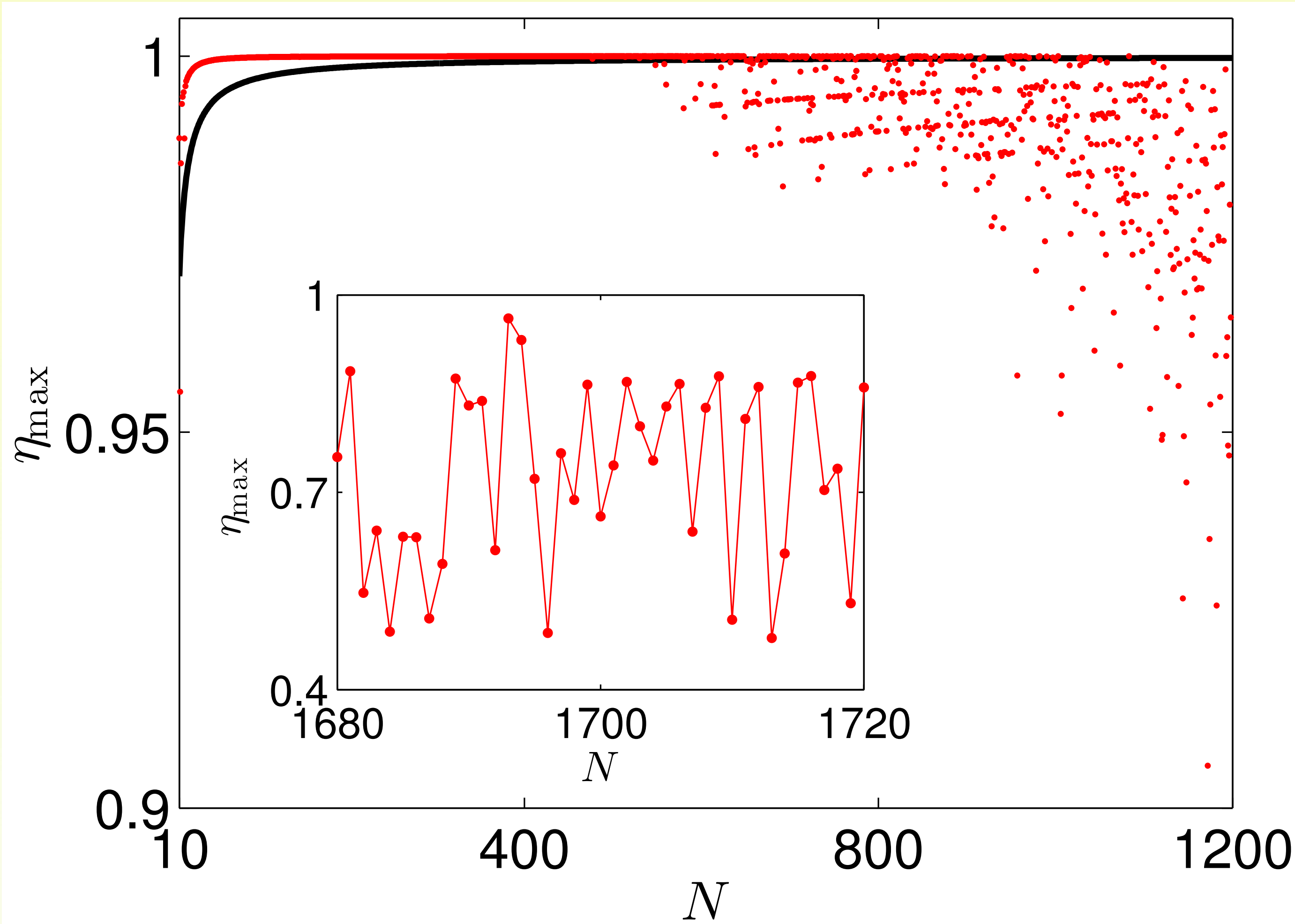
$N = 100$, $N\kappa/\Omega = 1.9$, $\omega/\Omega = 1.6$ $(r = 34)$

▷ Degree of coherence for all Floquet states at given μ



$N = 1000$, $N\kappa/\Omega = 0.95$, $\omega/\Omega = 1.62$, $2\mu/\omega = 0.3$

▷ Maximum degree of coherence for increasing N



$$N\kappa/\Omega = 0.95\ ,\ \omega/\Omega = 1.62\ ,\ 2\mu/\omega = 0.3$$

- **Remarks:**

  Condensate-carrying Floquet state (i.e., ground state of quantum pendulum) usually **not** *connected to ground state of undriven system*

  Only "mesoscopic" Floquet condensates possible — this is just another manifestation of the "quantum stability problem"

  Lots *of further issues to explore — e.g., connection between "quantum chaos" and destruction of macroscopic wave function*

Thank you!

Main sources of these lectures:

- M.H.
  Floquet engineering with quasienergy bands of periodically driven optical lattices (Tutorial)
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  Adiabatic preparation of Floquet condensates
  J. Mod. Opt. **63**, 1768 (2016)
  (Special issue: *20 years of Bose-Einstein Condensates*)

- B. Gertjerenken, M.H.
  Trojan quasiparticles
  New J. Phys. **16**, 093009 (2014)

- B. Gertjerenken, M.H.
  N-coherence vs. t-coherence: An alternative route to the Gross-Pitaevskii equation
  Annals of Physics **362**, 482 (2015)