# Physical Properties of squashed Kaluza-Klein Black Holes

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**2009/7/16 MG12**

# 1. Introduction

# Higher Dimensions

## v.s. Four-Dimensions

Dimensional reduction:

Compactification of extra-dimensions

Brane universe admits large extra-dimensions

- Geometry of higher dim. Black holes.

# Kaluza-Klein Black Hole

## BH with compact extra dimensions

P.D.Dobiash&D.Maison, G.W.Gibbons&D.L.Wiltshire, R.C.Myers…..
B.Kol, T.Harmark, N.Obers, T.Wiseman, H.Kudoh….

5-dim. Kaluza-Klein black hole

Near horizon: ~ 5-dim. BH

$$ds^2 \sim -\left(1-\frac{r_g^2}{r^2}\right)dt^2+\frac{dr^2}{1-\frac{r_g^2}{r^2}}+r^2 d\Omega_{S^3}^2$$

Far region: ~ 4-dim. BH x $S^1$

$$ds^2 \sim -\left(1-\frac{r_g}{r}\right)dt^2+\frac{dr^2}{1-\frac{r_g}{r}}+r^2 d\Omega_{S^2}^2+r_0^2 d\phi^2$$

# 2. Cohomogeneity-One Black Holes

# 4-dim. Cohomogeneity-One Black Holes

Black hole spacetime foliated by
homogeneous subspace

$$ds^2 = -\left(1 - \frac{r_g}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_g}{r}} + r^2 d\Omega_{S^2}^2$$

$\Sigma$ : $r$=const surface

$$ds_\Sigma^2 = -a_0^2 dt^2 + b_0^2 d\Omega_{S^2}^2$$

# Cohomogeneity-One (C-1) Spacetime

$$ds^2 = g(r)dr^2 + ds_\Sigma^2 \qquad D = 5$$

$$\begin{aligned} ds_\Sigma^2 &= \gamma_{ab}(r)\sigma^a\sigma^b \\ &= -f(r)\left(dt + A_i(r)\sigma^i\right)^2 + \gamma_{ij}(r)\sigma^i\sigma^j \end{aligned}$$

$\sigma^i$ : invariant one form

$\Sigma$ : homogeneous spacetime

Tangherlini (1963), Myers-Perry [$J_1$=$J_2$](1987), BMPV(1997)

Cvetic-Lu-Pope (2005), Gimon-Hashimoto(2003), ……….

# C-1 Black Holes in 5-dim.

$M$ , $Q$

( 5-dim Reissner-Nordström )

**Asymptotic Flat Black Holes**

# C-1 Black Holes in 5-dim.

$M\ ,\ Q$

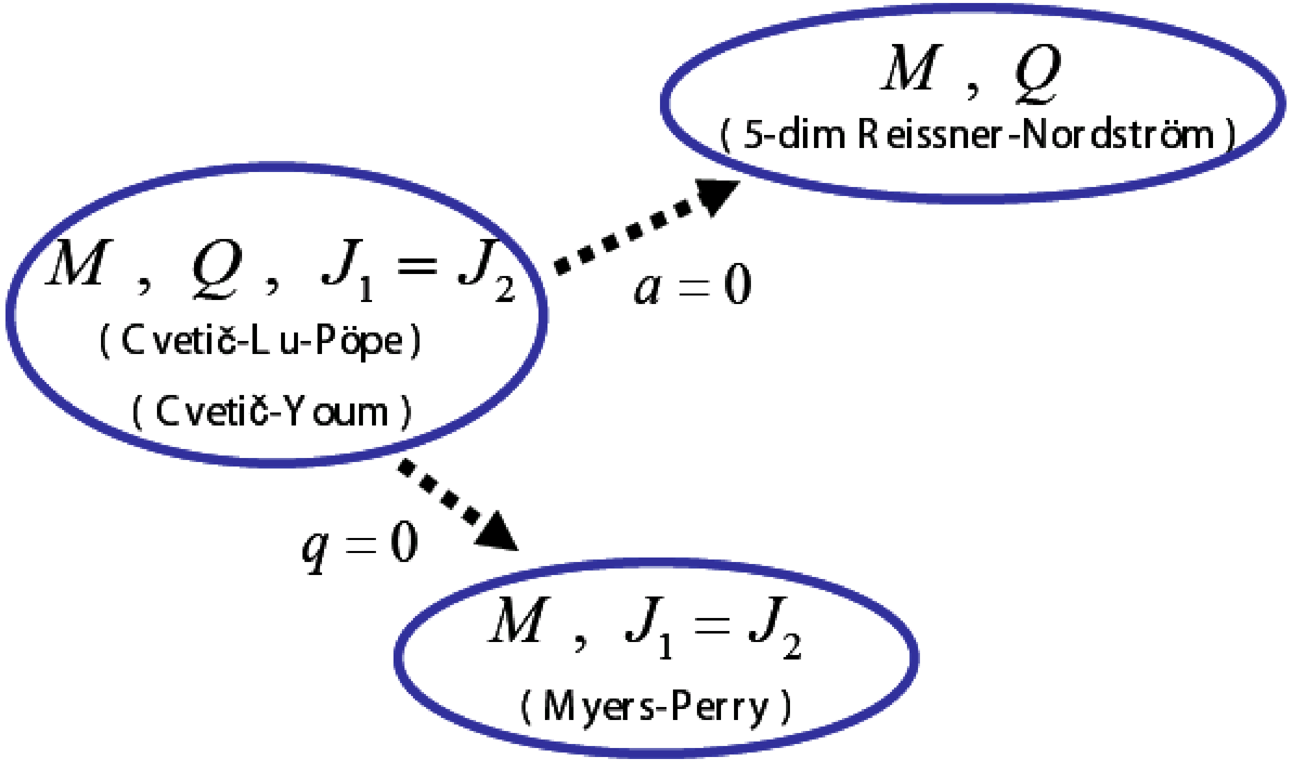
( 5-dim Reissner-Nordström )

$M\ ,\ J_1 = J_2$

( Myers-Perry )

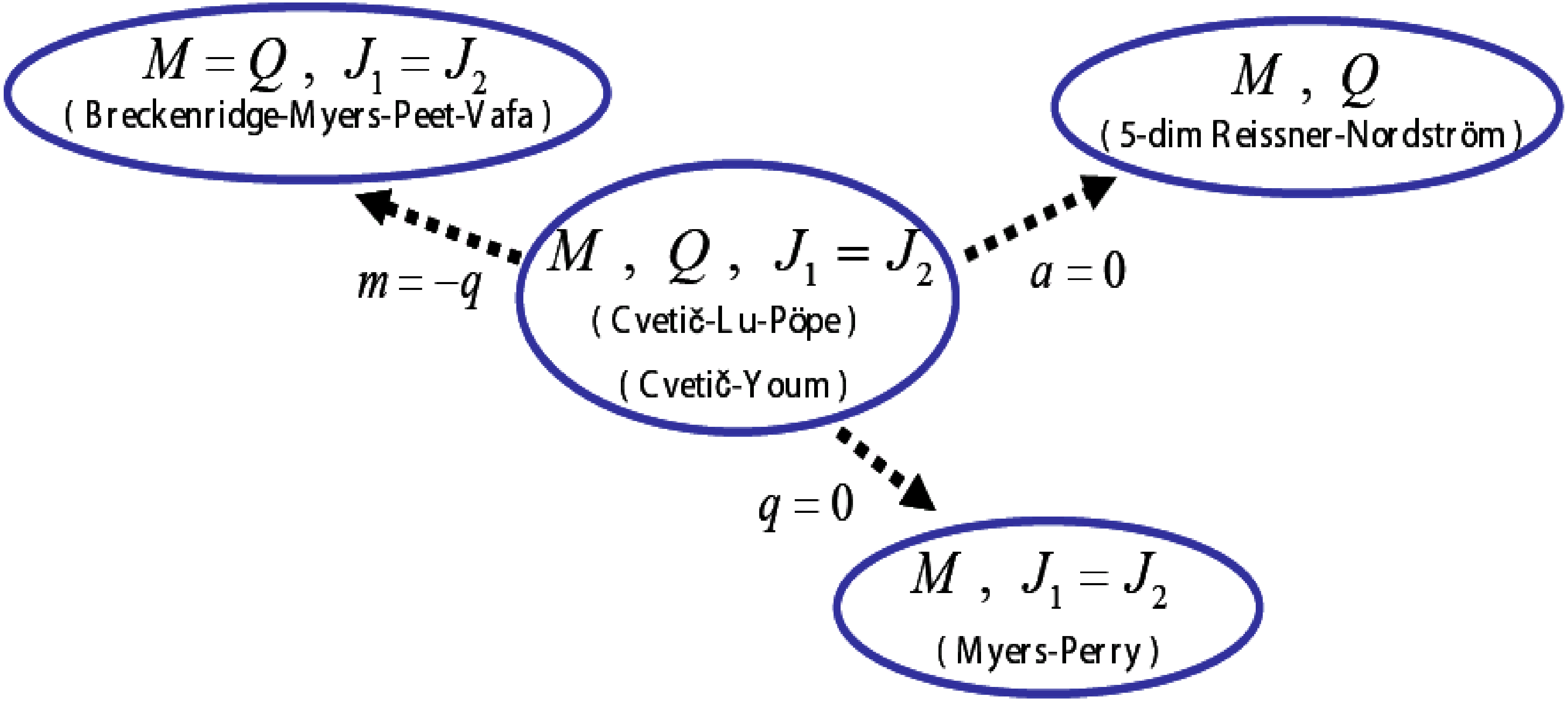
**Asymptotic Flat Black Holes**

# C-1 Black Holes in 5-dim.



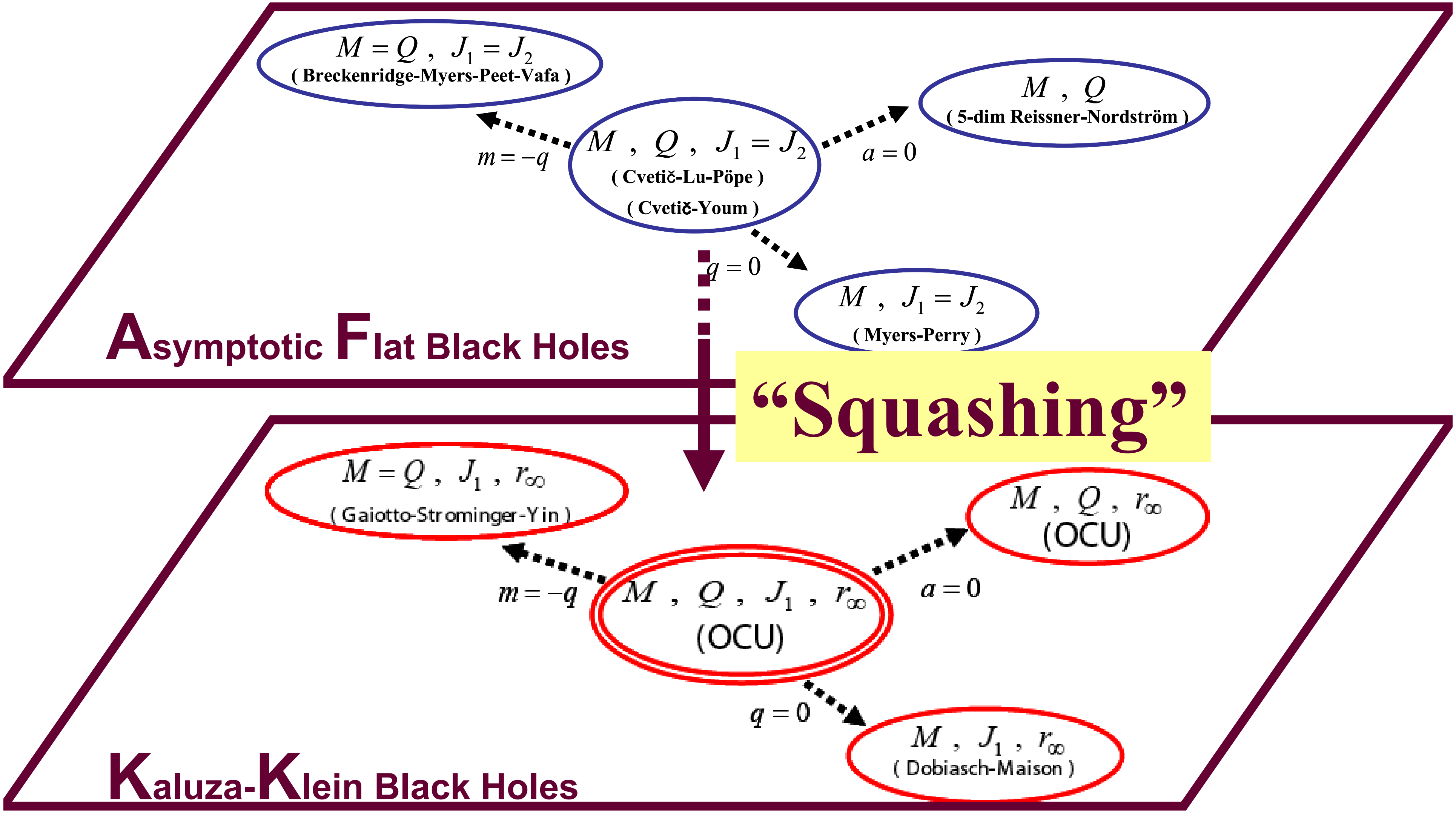

**Asymptotic Flat Black Holes**

# C-1 Black Holes in 5-dim.




**Asymptotic Flat Black Holes**

# C-1 Squashed Black Holes

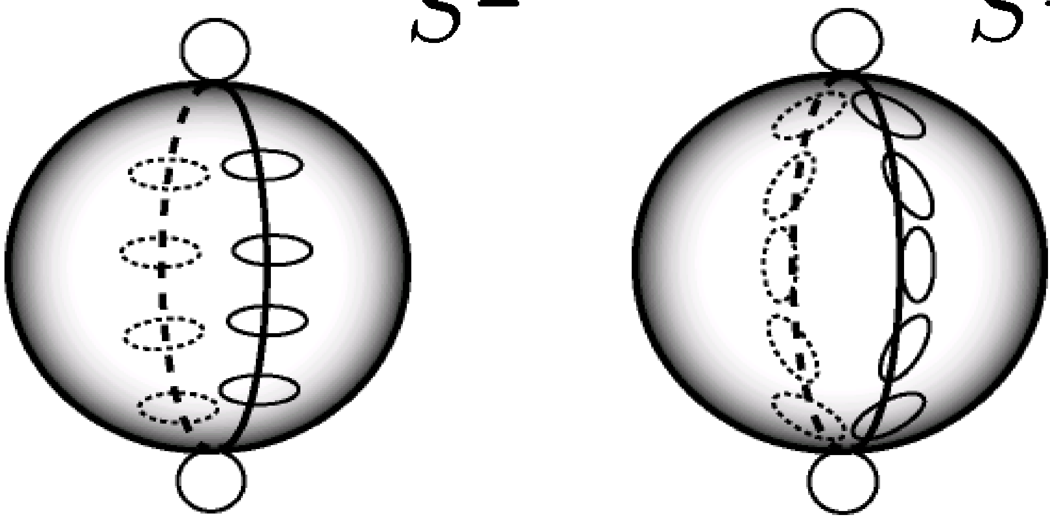


# 3. Squashing

# $S^3$ metric

$$\sigma_1 = -\sin\psi\, d\theta + \cos\psi \sin\theta\, d\phi$$
$$\sigma_2 = \cos\psi\, d\theta + \sin\psi \sin\theta\, d\phi$$
$$\sigma_3 = d\psi + \cos\theta\, d\phi$$

$$\begin{aligned} d\Omega^2_{S^3} &= \frac{1}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \\ &= \frac{1}{4}\Big(\underbrace{(d\theta^2 + \sin^2\theta\, d\phi^2)}_{S^2} + \underbrace{(d\psi + \cos\theta\, d\phi)^2}_{S^1}\Big) \end{aligned}$$



$S^2 \times S^1$ $\qquad$ $S^3$

# Squashed $S^3$

$$ds^2_\Sigma = \frac{1}{4}\left[\ k\ \ d\Omega^2_{S^2} + \sigma_3^2\right],$$

$$d\Omega^2_{S^2} = d\theta^2 + \sin^2\theta d\phi^2,$$

$$\sigma_3 = d\psi + \cos\theta d\phi.$$

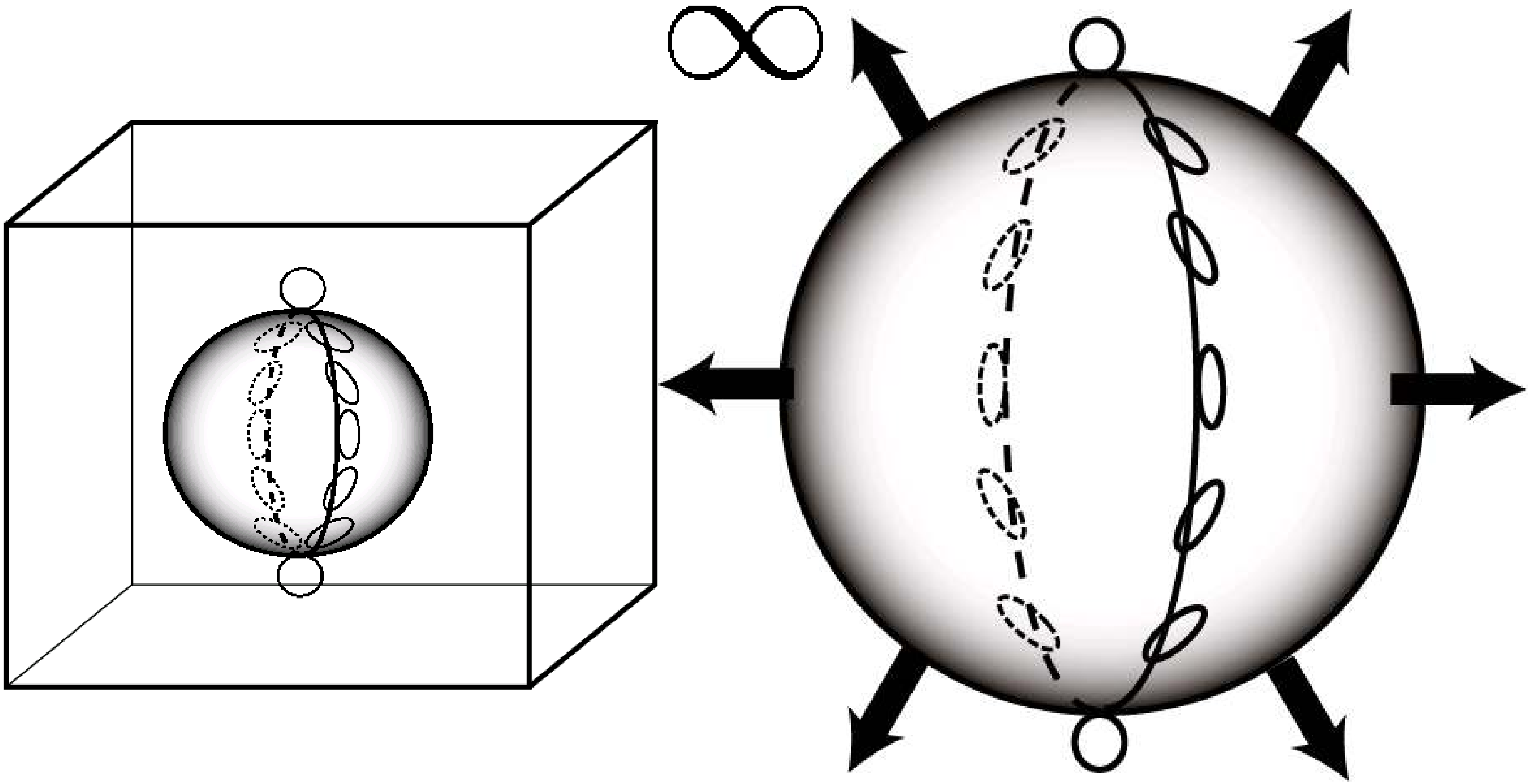
$S^3$ as a Hopf bundle $\begin{cases} \text{fiber} : S^1 \\ \text{base} : S^2 \end{cases}$

$k < 1$ prolate

$k = 1$ round $S^3$

$k > 1$ oblate.

# Squashing



# Example of squashed space

## Squashed Euclid space

$$ds^2 = k(r)^2 \ dr^2 + \frac{r^2}{4}[k(r) \ d\Omega_{S^2}^2 + \sigma_3^2]$$

$$k(r) = \frac{r_\infty^4}{(r_\infty^2 - r^2)^2}, \qquad r_\infty = const.$$

Euclidean Self-dual Taub-NUT (Ricci flat)

## Squashed Minkowski spacetime

$$ds^2 = -dt^2 + k(r)^2 \ dr^2 + \frac{r^2}{4}[k(r) \ d\Omega_{S^2}^2 + \sigma_3^2]$$

Gross-Perry-Sorkin monopole (Ricci flat)

# 4. Squashed Black Holes in 5-dimensions

# Action

$$S = \frac{1}{16\pi G}\int d^5x\sqrt{-g}\,(R - F_{\mu\nu}F^{\mu\nu})$$

$$+\frac{1}{16\pi G}\int F \wedge F \wedge A$$

Equation of motion

$$D_\nu F^{\mu\nu} + \frac{1}{2\sqrt{3}\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\nu\rho}F_{\sigma\lambda} = 0,$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2\left(F_{\mu\lambda}F_\nu^\lambda - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right).$$

# Squashed Black Holes (neutral static case)

## 5-dim. Schwarzschild B.H.

$$ds^2 = -\left(1 - \frac{r_g^2}{r^2}\right) dt^2 + \frac{dr^2}{1 - \frac{r_g^2}{r^2}} + \frac{r^2}{4}[\ d\Omega_{S^2}^2 + \sigma_3^2\ ]$$

## Squashed B.H.

$$ds^2 = -\left(1 - \frac{r_g^2}{r^2}\right) dt^2 + \frac{k^2(r)\ dr^2}{1 - \frac{r_g^2}{r^2}} + \frac{r^2}{4}[k(r)\ d\Omega_{S^2}^2 + \sigma_3^2\ ]$$

$$k(r) = \frac{(r_\infty^2 - r_g^2)r_\infty^2}{(r_\infty^2 - r^2)^2}$$

Charged case: Ishihara & Matsuno, Prog. Theor. Phys. 116, 417 ('06)

[arXiv:hep-th/0510094]

# Near Horizon

$$ds^2 = -\left(1 - \frac{r_g^2}{r^2}\right) dt^2 + \frac{k^2(r)\, dr^2}{1 - \frac{r_g^2}{r^2}} + \frac{r^2}{4}[k(r)\, d\Omega_{S^2}^2 + \sigma_3^2\ ]$$

In the region $r_g < r \ll r_\infty$

$$k(r) = \frac{(r_\infty^2 - r_g^2) r_\infty^2}{(r_\infty^2 - r^2)^2} \simeq 1$$

5-dm. Schwarzschild B.H.

# Asymptotic Infinity

Coordinate tr. $r \;\rightarrow\; \rho = \rho_0 \dfrac{r^2}{r_\infty^2 - r^2};$

$$ds^2 = -\left(1 - \frac{\rho_g}{\rho}\right) dT^2 + \frac{\left(1 + \frac{\rho_0}{\rho}\right)}{\left(1 - \frac{\rho_g}{\rho}\right)} d\rho^2$$

$$+ \left(1 + \frac{\rho_0}{\rho}\right) \rho^2 d\Omega_{S^2}^2 + \frac{r_\infty^2}{4} \left(1 + \frac{\rho_0}{\rho}\right)^{-1} \sigma_3^2,$$

$$ds^2 \overset{\rho\to\infty}{\longrightarrow} -dT^2 + d\rho^2 + \rho^2 d\Omega_{S^2}^2 + \frac{r_\infty^2}{4}(d\psi + \cos\theta d\phi)^2$$

# Squashed Black Hole $0 < r_g \ll r_\infty ,$

## Near horizon: fully 5-dim.

$$ds^2 \sim -\left(1-\frac{r_g^2}{r^2}\right)dt^2 + \frac{k_0^2\ dr^2}{\left(1-\frac{r_g^2}{r^2}\right)} + r^2 d\Omega^2_{\tilde{S}^3}$$

## Far zone: 4-dim. with twisted $S^1$

$$ds^2 \sim -\left(1-\frac{\rho_g}{\rho}\right)dT^2 + \frac{d\rho^2}{\left(1-\frac{\rho_g}{\rho}\right)} + \rho^2 d\Omega^2_{S^2} + \frac{r_\infty^2}{4}\sigma_3^2$$

**Stable circular orbits exist around 5-dimensional BH !**

# Extension of Squashed Black holes


T. Wang , Nucl.Phys.B756:86-99,2006.

S. S. Yazadjiev, Phys.Rev.D74:024022,2006.

Y. Brihaye, E. Radu, Phys.Lett.B641:212-220,2006.

S. Tomizawa, Y. Yasui, Y. Morisawa, arXiv:0809.2001 [hep-th]

D. V. Gal'tsov, N. G. Scherbluk, arXiv:0812.2336 [hep-th]

K. Murata, T. Nishioka, N. Tanahashi, arXiv:0901.2574 [hep-th]

Y. Brihaye, J. Kunz, E. Radu, arXiv:0904.1566 [gr-qc]

etc

# 5. Kaluza-Klein-Gödel Black Holes

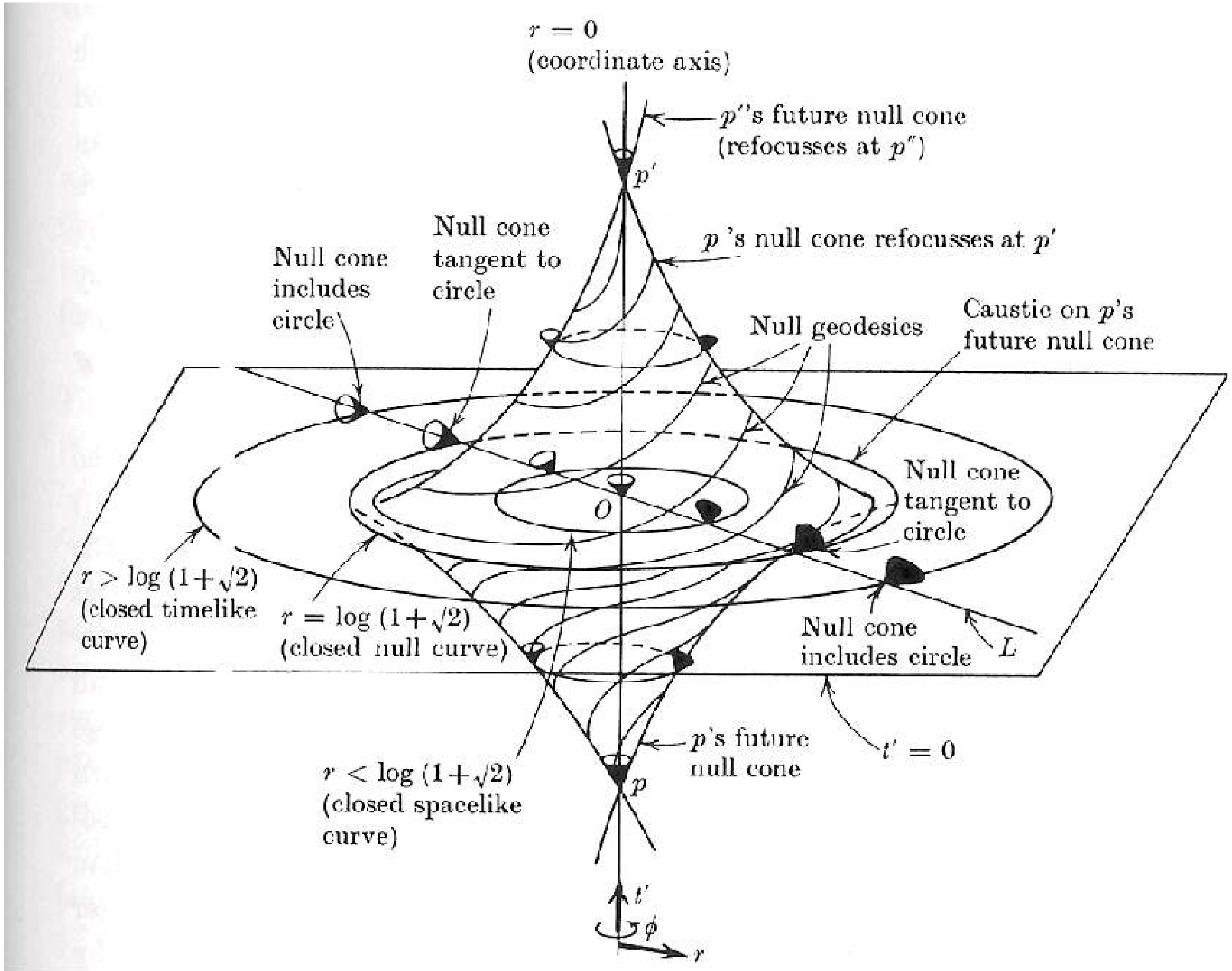
# Gödel Universe in 4-dim.

$$ds^2 = -(dt + \sqrt{2}\sinh^2 r \; d\phi)^2 + dr^2$$
$$+(\sinh^2 r + \sinh^4 r)d\phi^2 + dz^2$$

- Solution to Einstein+Cosm.const.+Perfect fluid
- Homogeneous Spacetime
- There exist Closed Timelike Curves!

$$r > \log(1 + \sqrt{2})$$

# Figure of Gödel Universe



# Gödel Spacetime in 5-dim.

Gauntlett,Gutowski,Hull,Pakis & Reall
Harmark & Takayanagi

$$ds^2 = -(dt + jr^2\sigma_3)^2 + dr^2 + \frac{r^2}{4}[d\Omega^2_{S^2} + \sigma_3^2]$$

$$A = \frac{\sqrt{3}}{2} jr^2\sigma_3$$

- Solution to Einstein-Maxwell-Chern-Simon
- Supersymmetric solution ( pp-wave solution)
- Homogeneous Spacetime
- There exist CTC! $r > 2j^{-1}$

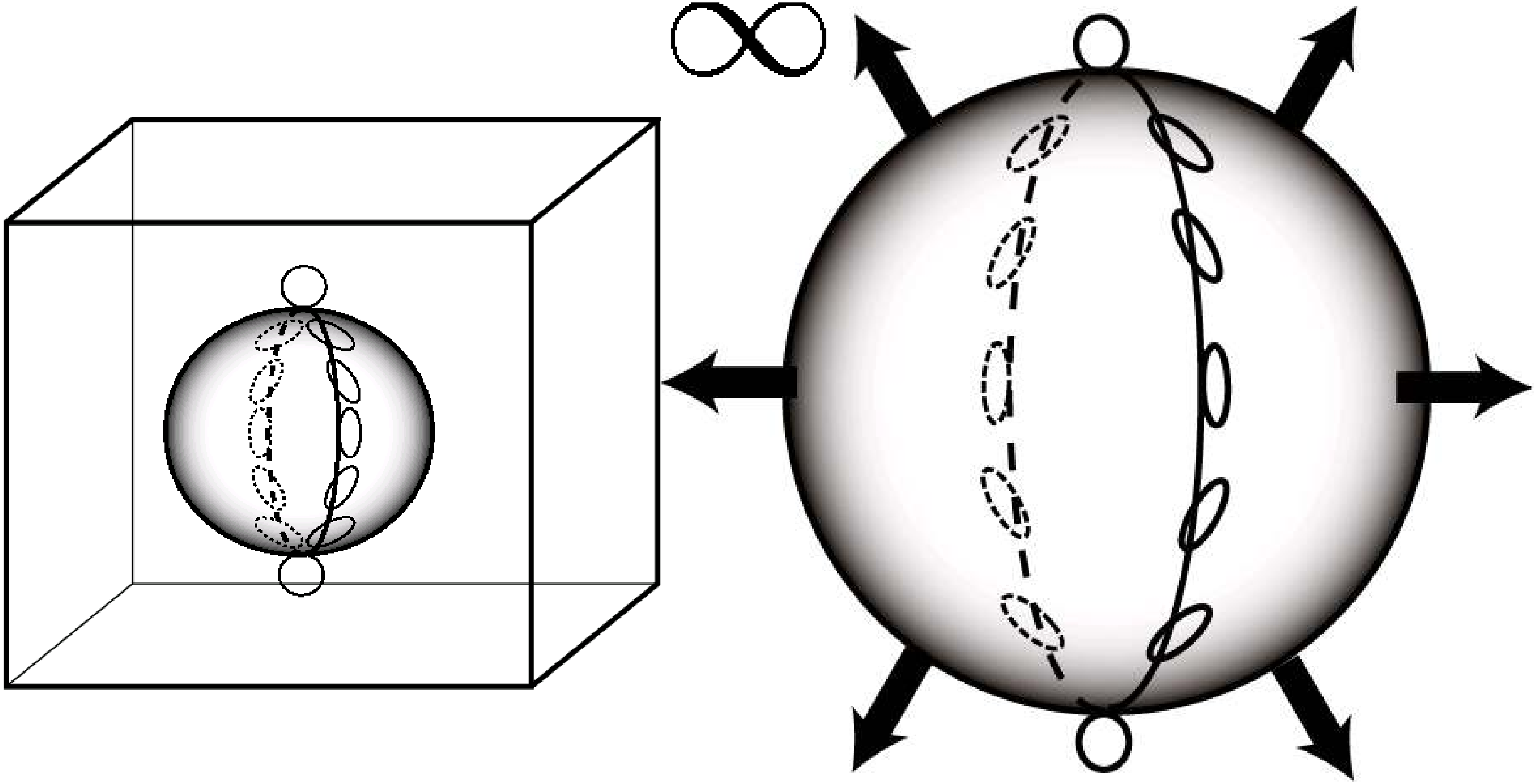
# Squashed Gödel Spacetime

$$ds^2 = -(dt + jr^2\sigma_3)^2 + k(r)^2 dr^2 + \frac{r^2}{4}[k(r)d\Omega_{S^2}^2 + \sigma_3^2]$$

$$k(r) = \frac{r_\infty^4}{(r_\infty^2 - r^2)^2}$$

- No CTC
- Rotating Gross-Perry-Sorkin monopole
- Ergo region appears

# Squashing



# Squashed Gödel Spacetime

$$ds^2 = -(dt + jr^2\sigma_3)^2 + k(r)^2 dr^2 + \frac{r^2}{4}[k(r)d\Omega^2_{S^2} + \sigma_3^2]$$

$$k(r) = \frac{r_\infty^4}{(r_\infty^2 - r^2)^2}$$

- No CTC if $j^2 r_\infty^4 < r_\infty^2/4$
- Rotating Gross-Perry-Sorkin monopole
- Ergo region appears

# Rotating GPS Monopole

Coordinate transformation $r \to \rho$

$$ds^2 = -\left[dt + 4j\rho_0^2\left(1+\frac{\rho_0}{\rho}\right)^{-1}\sigma_3\right]^2$$

$$+\left(1+\frac{\rho_0}{\rho}\right)\left[d\rho^2 + \rho^2 d\Omega_{S^2}^2\right] + \rho_0^2\left(1+\frac{\rho_0}{\rho}\right)^{-1}\sigma_3^2$$

At infinity $\rho \to \infty$

$$ds^2 \to -\left[dt + 4j\rho_0^2\sigma_3\right]^2 + \left[d\rho^2 + \rho^2 d\Omega_{S^2}^2\right] + \rho_0^2\sigma_3^2$$

No CTC $\text{if } \rho_0^2(1 - 16j^2\rho_0^2) > 0$

because squashing makes extra dim. finite

# Rest Frame at Infinity

$$ds^2 \to -\left[dt + 4j\rho_0^2\sigma_3\right]^2 + \left[d\rho^2 + \rho^2 d\Omega_{S^2}^2\right] + \rho_0^2\sigma_3^2$$

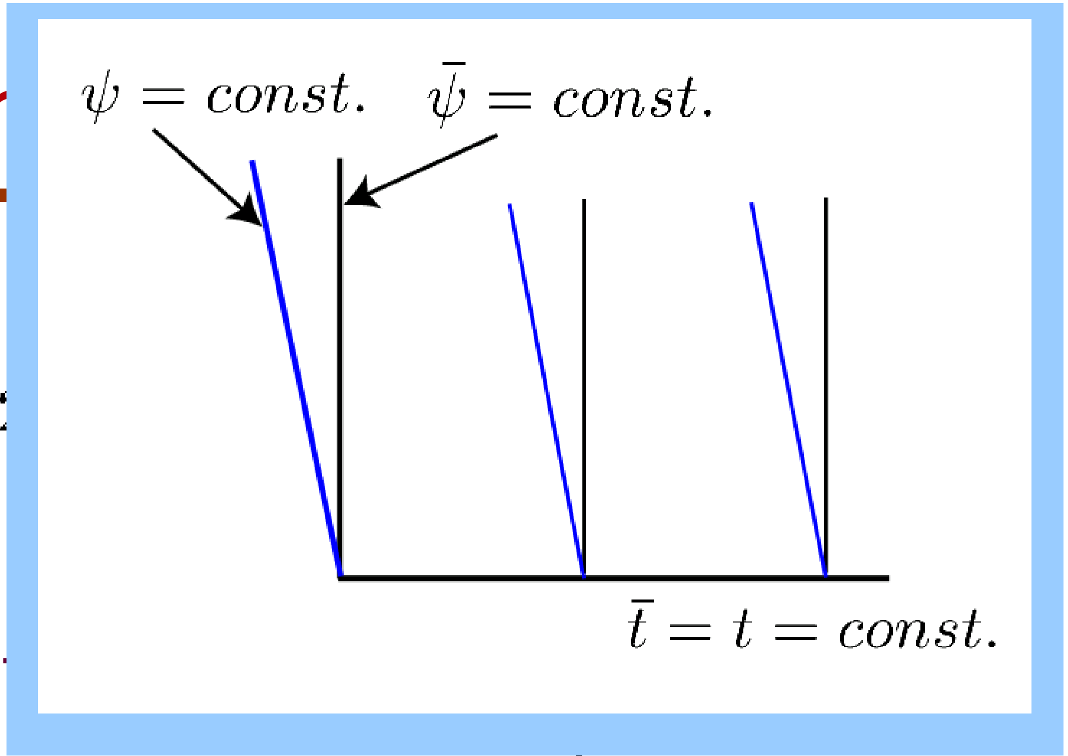
Coordinate transformation

$$\bar{\psi} = \psi - \frac{1}{1-16\rho_0^2}t, \quad \bar{t} = t\frac{1}{\sqrt{1-16\rho_0^2}}$$

$$ds^2 \to -d\bar{t}^2 + \left[d\rho^2 + \rho^2 d\Omega_{\bar{S}^2}^2\right] + \rho_0^2(1-16j^2\rho_0^2)\bar{\sigma}_3^2$$

# Rest Frame at Ir

$$ds^2 \to -\left[dt + 4j\rho_0^2\sigma_3\right]^2$$



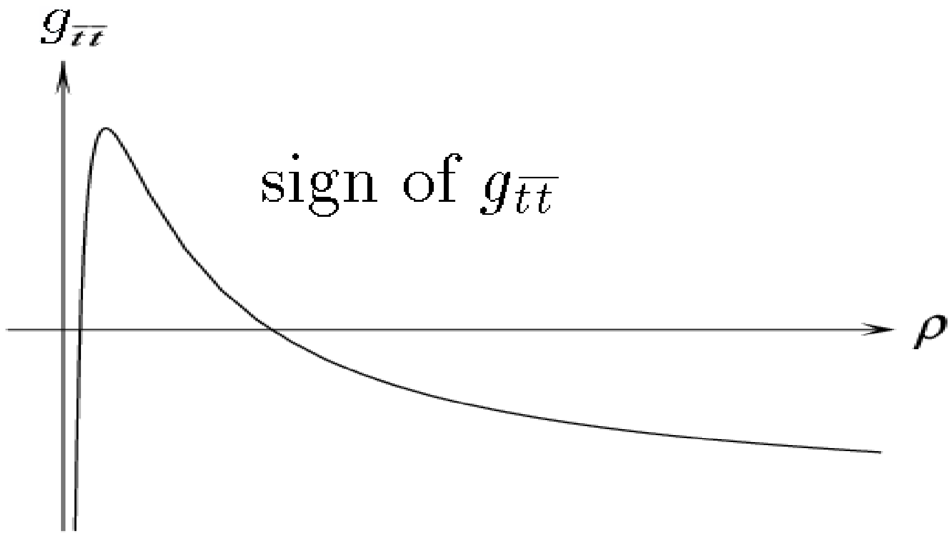

Coordinate transforma

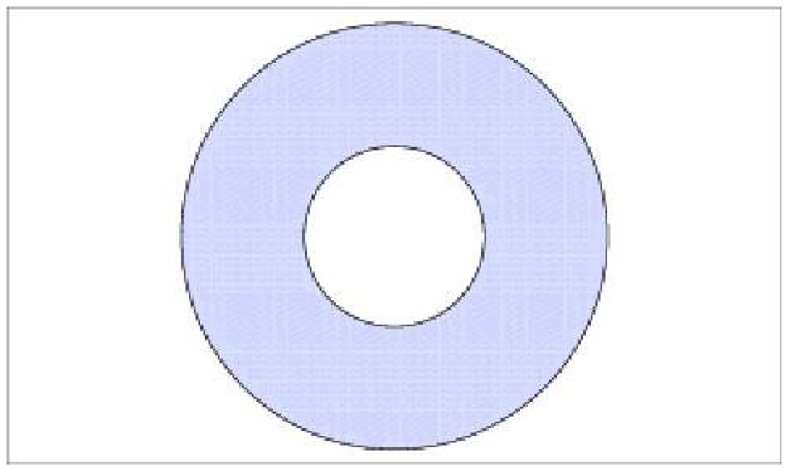
$$\bar{\psi} = \psi - \frac{1}{1-16\rho_0^2}t, \qquad \bar{t} = t\frac{1}{\sqrt{1-16\rho_0^2}}$$

$$ds^2 \to -d\bar{t}^2 + \left[d\rho^2 + \rho^2 d\Omega^2_{\bar{S}^2}\right] + \rho_0^2(1-16j^2\rho_0^2)\bar{\sigma}_3^2$$

# Coordinate Transformation

$$ds^2 = g_{\bar{t}\bar{t}}\, d\bar{t}^2 + 2 g_{\bar{t}\bar{3}}\, \bar{\sigma}_3\, d\bar{t} + g_{\rho\rho}\left[d\rho^2 + \rho^2 d\Omega^2_{\bar{S}^2}\right] + g_{\bar{3}\bar{3}}\, \bar{\sigma}_3^2$$



$$\frac{\partial}{\partial \bar{t}} \cdot \frac{\partial}{\partial \bar{t}} = g_{\bar{t}\bar{t}} > 0$$

Ergo Shell appears
in rotating GPS monopole

# Squashed Gödel Black Holes

Kerr-Gödel BH

Gimon & Hashimoto(2003)

Wu(2007)

↓ Squashing

Squashed Kerr-Gödel BH

No CTC,

Ergosphere and Ergoshell are possible

# Squashed Kerr-Gödel BH

$$ds^2 = -f(r)dt^2 - 2g(r)\sigma_3 dt + h(r)\sigma_3^2$$

$$+\frac{k(r)^2 dr^2}{V(r)} + \frac{r^2}{4}[k(r)(\sigma_1^2 + \sigma_2^2) + \sigma_3^2],$$

$$\boldsymbol{A} = \frac{\sqrt{3}}{2} j r^2 \sigma_3,$$

$$f(r) = 1 - \frac{2m}{r^2}, \quad h(r) = -j^2 r^2 (r^2 + 2m) + \frac{ma^2}{2r^2},$$

$$g(r) = jr^2 + \frac{ma}{r^2}, \quad V(r) = 1 - \frac{2m}{r^2} + \frac{8jm(a+2jm)}{r^2} + \frac{2ma^2}{r^4},$$

Tomizawa, Ishihara, Matsuno, & Nakagawa,

Prog. Thor. Phys 121 (2009) 823 , arXiv:0803.3873 [hep-th]

# Squashed Kerr-Gödel BH

$$ds^2 = -f(r)dt^2 - 2g(r)\sigma_3 dt + h(r)\sigma_3^2$$
$$+\frac{k(r)^2 dr^2}{V(r)} + \frac{r^2}{4}[k(r)(\sigma_1^2 + \sigma_2^2) + \sigma_3^2],$$

$$\boldsymbol{A} = \frac{\sqrt{3}}{2} j r^2 \sigma_3,$$

$$g(r) = jr^2 + \frac{ma}{r^2}$$

$\left\{ \begin{array}{l} \text{Kerr rotation} \\ \text{Gödel rotation} \end{array} \right.$

Tomizawa, Ishihara, Matsuno, & Nakagawa,

Prog. Thor. Phys 121 (2009) 823 , arXiv:0803.3873 [hep-th]

# Angular Momentum

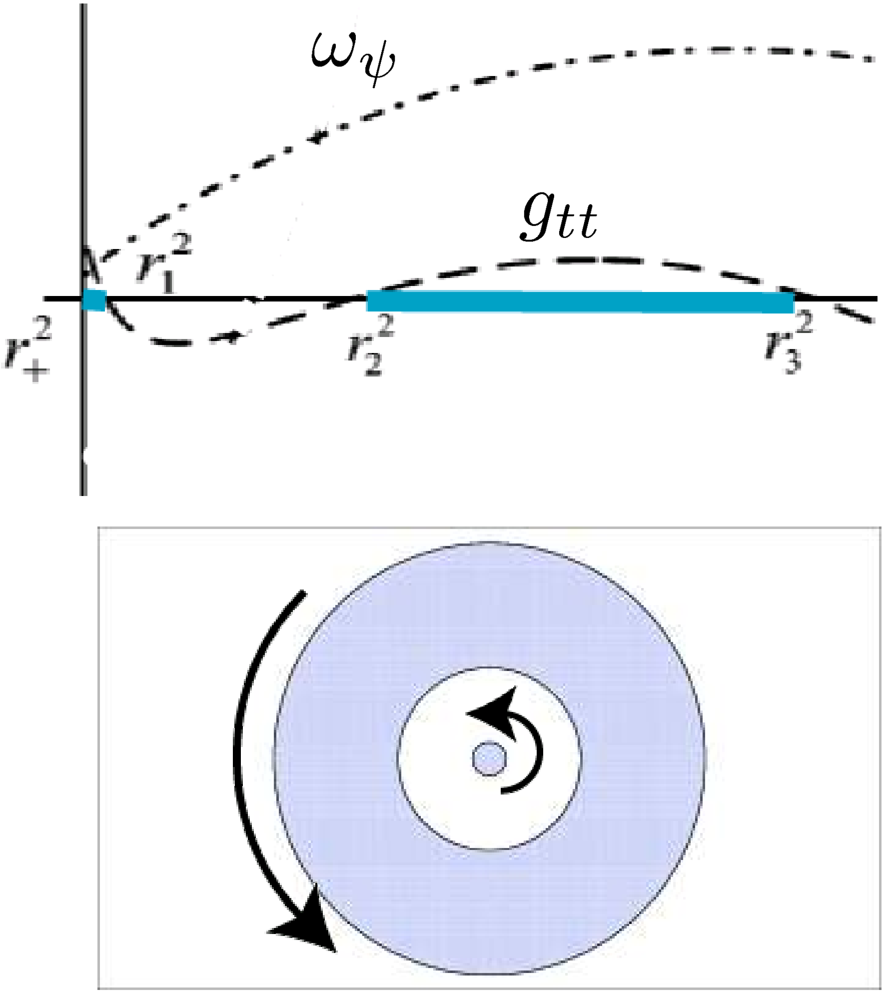
$$J_\psi := \int \epsilon_{abcde} \nabla^d (\partial_\psi)^e \qquad J_\phi := \int \epsilon_{abcde} \nabla^d (\partial_\phi)^e$$

$j = 0 \text{ case} \quad J_\psi = \frac{1}{2}\pi m a, \quad J_\phi = 0$ Kerr rotation

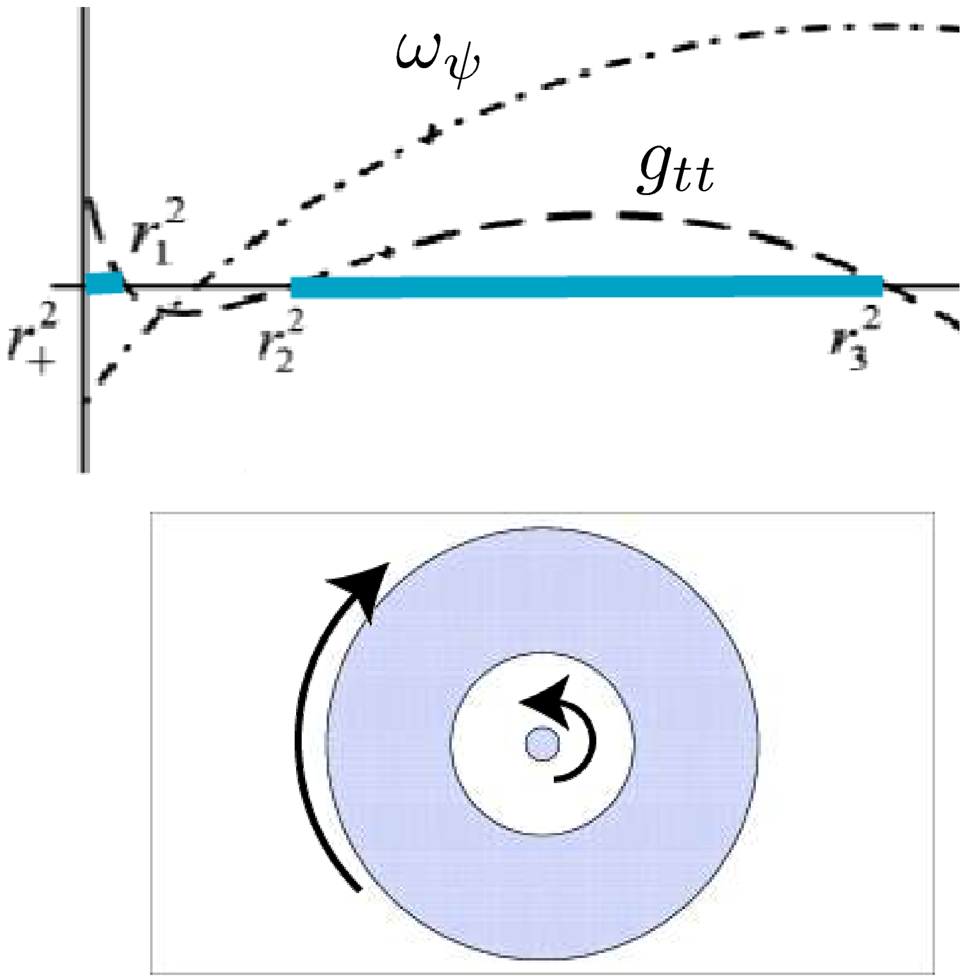
$a = 0 \text{ case} \quad J_\psi = -\pi r_\infty^6 j^3, \quad J_\phi = 0$ Gödel rotation

Two independent rotations around the same axis

# Gödel v.s. Kerr




Co-rotating Ergos




Counter-rotating Ergos

# Summary

- Cohomogeneity-one Black Holes are interesting simple objects in 5-dimensions.
- Cohomogeneity-one K-K Black Holes in 5-dimensions are generated by **squashing.**
- We have investigated Kaluza-Klein-Kerr-Gödel Black Holes.

  No closed timelike curve,
  Two rotations around one axis,
  Ergosphere and ergoshell are possible.