BLACK HOLE UNIQUENESS THEOREMS IN HIGHER DIMENSIONS

S. Hollands[1], S. Yazadjiev[2]

[1] *School of Mathematics, Cardiff University*
Cardiff, CF24 4AG, UK,

[2] *Department of Theoretical Physics, Sofia University*
5 J. Bourchier Blvd., Sofia 1164, Bulgaria

*

S.Hollands, S.Yazadjiev

A uniqueness theorem for 5-dimensional black holes with two axial Killing fields,
Comm. Math. Phys. (2008), gr-qc/0707.2775

S.Hollands, S.Yazadjiev

A uniqueness theorem for 5-dimensional Einstein-Maxwell black holes,
Class. Quant. Grav (2008), gr-qc/0711.1722

S.Hollands, S.Yazadjiev,

A uniqueness theorem for stationary Kaluza-Klein black holes,
arXiv: 0812.3036 [gr-qc]

*

THE STORY IN FOUR DIMENSIONS

Black holes in 4 dimensions are fully specified by their conserved asymptotic charges. – [**Israel (1967), Muller zum Hagen (1974), Carter (1971), Robinson (1975), Mazur (1982); Bunting (1983)**],

Theorem [**Mazur (1982); Bunting (1983)**] The Kerr-Newman solution with parameters M, $a = J/M$ and Q is the unique black hole solution with regular event horizon $(M^2 > a^2 + Q^2)$ and stationary, axisymetric and asymptotically flat domain of outer communications.

This uniqueness (classification) theorem is a key result in the theory of the 4D black holes !

*

SURPRISE IN HIGHER DIMENSIONS – BLACK RINGS AND ... NONUNIQUENESS

Black Rings [**Emparan and Reall (2001)**]–asymptotically flat black 5D solutions with $S^2 \times S^1$ horizon topology

$$ds^2 = -\frac{F(y)}{F(x)}\left(dt - CR\frac{1+y}{F(y)}d\psi\right)^2 + \frac{R^2 F(x)}{(x-y)^2}\left[-\frac{G(y)}{F(y)}d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2\right]$$

$$-1 \le x \le 1, \quad -\infty < y \le -1$$

$$F(y) = 1 + \lambda y, \;\; G(y) = (1-y^2)(1+\nu y), \;\; C = \sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}}, \quad 0 < \nu \le \lambda < 1$$

$$\lambda = \frac{2\nu}{1+\nu^2}. \tag{1}$$

$S^2 \times S^1$ horizon at $y = -1/\nu$, $\quad S^2 \times S^1$ ergosurface at $y = -1/\lambda$

Reduced dimensionless quantities

$$j^2 = \frac{27\pi J^2}{32 M^3} = \frac{(1+\nu)^3}{8\nu} \qquad \textit{and} \qquad a_H = \frac{3}{16}\sqrt{\frac{3}{\pi}}\frac{A_H}{M^{3/2}} = 2\sqrt{\nu(1-\nu)}, \tag{2}$$

$$a_H = a_H(j). \tag{3}$$

For the Myers-Perry black hole the corresponding relation is

$$a_H(MP) = 2\sqrt{2(1-j^2)}. \tag{4}$$

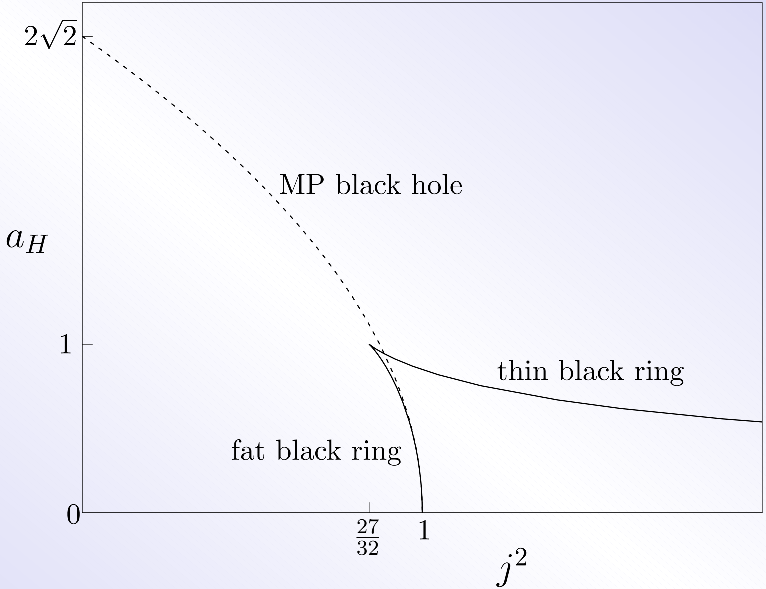

Figure 1: Horizon area a_H vs j^2, for given mass.

*

Conclusions

1) In 5 dimensions, the horizon topology can be different from S^3 !

2) In 5 dimensions, the black objects are not uniquely specified by their conserved asymptotic charges !

*** Black Hole Uniqueness Problem In 5D**

The question is whether some kind of uniqueness theorem could be formulated in higher dimensions. The most difficult question, however, is what kind of suitable further parameters associated with the black solutions should be specified in addition to the conserved asymptotic charges.

*** Resolution Of The 5D Black Hole Uniqueness Problem**

We will show that the black solutions in 5D are uniquely determined in terms of their angular momenta and a datum called "**INTERVAL (ROD) STRUCTURE**" . The rod structure encodes information about the relative position of various axis and the horizon, and gives a measure of their lengths. Another very important fact is that the rod structure determines the topology of the horizon. Actually, the topology of the horizon can be either S^3, $S^2 \times S^1$, or a Lens-space $L(p,q)$.

INTERVAL (ROD) STRUCTURE

* Consider spacetimes with an isometry group $R \times U(1)^2$

* Black object exterior M_{ext} and topological censorship theorem

We assume that the exterior M_{ext} is globally hyperbolic. By the topological censorship theorem [**Galloway et.al.**] the exterior M_{ext} is a simply connected manifold with boundary $\partial M_{ext} = H$.

* Spacetime symmetries and the factor space $M_{ext}/\mathcal{G}$

Due to the symmetries of the spacetime the natural space to work on is the orbit (factor) space $\hat{M} = M_{ext}/\mathcal{G}$, where $\mathcal{G} = R \times U(1)^2$. Since the spacelike Killing fields in general have zeros, the factor space normally has singularities. That is why we need to know the structure of the factor space in detail.

THE FACTOR SPACE $\hat{M}$ IN 5 DIMENSIONS

Theorem [S. Hollands and S.Y. (2007)] Let (M_{ext}, g_{ab}) be the exterior of a 5-dimensional stationary, asymptotically flat, vacuum black hole spacetime with 2 mutually commuting independent axial Killing fields ψ_1^a and ψ_2^2. Then the orbit space $\hat{M} = M_{ext}/\mathcal{G}$ by the isometry group $\mathcal{G} = \mathcal{R} \times U(1)^2$ is a simply connected, 2-dimensional manifold with boundaries and corners. Points in the interior of $\hat{M}$ correspond to point in M where all Killing fields t^a, ψ_1^a, ψ_2^a are linearly independent. Points on the i-th 1-dimensional boundary segment of $\partial\hat{M}$ correspond to either the horizon H, or points where a linear combination $\nu_i^1\psi_1^a + \nu_i^2\psi_2^a = 0$, where $v_i = (\nu_i^1, \nu_i^2)$ is a vector of integers. Points in the corners of $\partial\hat{M}$ correspond to points in M_{ext} where $\psi_1^a = \psi_2^a = 0$. The boundary $\partial\hat{M}$ is connected.

Since $\hat{M}$ is a simply connected 2-dimensional analytic manifold with boundaries and corners, we may map it analytically to the upper complex half plane $\{\zeta = z + ir \in \mathcal{C};\ Im\zeta = r > 0\}$ by the Riemann mapping theorem. The boundary $\partial\hat{M}$ corresponds to $r = 0$.

In the realization of $\hat{M}$ as the upper complex half plane, the line segments of $\partial\hat{M}$ correspond to intervals

$$(-\infty, z_1), (z_1, z_2), \ldots, (z_k, z_{k+1}), (z_{k+1}, \infty) \tag{5}$$

of the real axis forming the boundary of the upper half plane. Evidently, if the horizon is connected as we assume, precisely one interval (z_h, z_{h+1}) corresponds to the horizon. The other intervals correspond to rotation-axis, while the points z_j correspond to the intersection points of the axis, except for the boundary points of the interval (z_h, z_{h+1}) representing the horizon. The coordinate z is defined in a diffeomorphism invariant way in terms of the solution up to shifts by a constant. Consequently, the k positive real numbers

$$l_1 = z_1 - z_2, \quad l_2 = z_2 - z_3, \quad \ldots \quad l_k = z_k - z_{k+1} \tag{6}$$

are invariantly defined, i.e., are the same for any pair of isometric stationary black hole space-times of the type we consider. Thus, they may be viewed as global parameters ("moduli") characterizing the given solution in addition to the mass m and the two angular momenta J_1, J_2.

Furthermore, with each l_j, there is associated a label in $\{v_j \in \mathbb{Z}^2, H\}$ according to whether we are on the horizon, or which rational linear combination $v_j^1 \psi_1^a + v_j^2 \psi_2^a$ vanishes. The labels corresponding to the "outmost" intervals $(-\infty, z_1)$ and (z_k, ∞) must be $(0,1)$ respectively $(1,0)$, because this is the case for Minkowski spacetime, and we assume that our solutions are asymptotically flat.

For 4 dimensional black holes, there is only the trivial rod structure $(-\infty, z_1), (z_1, z_2), (z_2, \infty)$, with the middle interval corresponding to the horizon, and the first and third corresponding to single axis of rotation of the Killing field. Furthermore, the rod length l_1 may be expressed in terms of the global parameters m, J of the solution. By contrast, in 5 dimensions, the rod structure can be non-trivial, and in fact differs for the Myers-Perry and Black Ring solutions. For these cases, the rod structure is summarized in the following table:

	Rods	Rod Vectors	H Topology
Myers-Perry BH	$(-\infty, z_1), (z_1, z_2), (z_2, \infty)$	$(1,0), H, (0,1)$	S^3
Black Ring	$(-\infty, z_1), (z_1, z_2), (z_2, z_3), (z_3, \infty)$	$(1,0), H, (1,0), (0,1)$	$S^2 \times S^1$
Flat Spacetime	$(-\infty, z_1), (z_1, \infty)$	$(1,0), (0,1)$	—

*

CLASSIFICATION OF THE POSSIBLE HORIZON TOPOLOGIES IN 5D

Theorem [**S. Hollands and S.Y. (2007)**] In a black hole spacetime of dimension 5 with 2 commuting, independent axial Killing fields, the horizon cross section H must be topologically either a ring $S^1 \times S^2$, a sphere S^3, or a Lens-space $L(p,q)$, with $p, q \in Z$ where $p = \det(v_{h-1}, v_{h+1})$. For $p = 0$ the topology is $H = S^2 \times S^1$, $H = S^3$ for $p = \pm 1$ and $H = L(p,q)$ for other values of p.

Remark The Lens-spaces $L(p,q)$ are the spaces obtained by glueing the boundaries of two solid tori together in such a way that the meridian of the first goes to a curve on the second which wraps around the longitude p-times and which wraps around the meridian q-times. A Lens-space may also be obtained by factoring the unit sphere S^3 in $\mathcal{C}^2$ by the group action $(z_1, z_2) \to (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2)$ $(p \neq 0, q \neq 0)$. If $q = 0$ modulo pZ, then $p = \pm 1$ and $L(\pm 1, 0\, modulo\, pZ) = S^3$. If $p = 0$ modulo qZ, then $q = \pm 1$ and $L(0,\, modulo\, qZ, \pm 1) = S^2 \times S^1$.

*

***Possible existence of new type black objects–Black Lenses**

	Rods	Rod Vectors (Labels)	Horizon Topology
Black Lens	$(-\infty, z_1), (z_1, z_2), (z_2, z_3), (z_3, \infty)$	$(1,0), (0,0), (1,n), (0,1)$	$L(n,1)$

The known exact black lens solutions are singular [**Evslin (2008), Chen and Teo (2008)**]

*

UNIQUENESS THEOREM

Since the rod structure is a diffeomorphism invariant datum constructed from the given solution, two given stationary black hole solutions with 2 axial Killing fields cannot be isomorphic unless the rod structures and the masses and angular momenta coincide. The main purpose here is to point out the following converse to this statement:

Theorem [**S. Hollands and S.Y. (2007)**] Consider two stationary, asymptotically flat, vacuum non-degenerate black hole spacetimes of dimension 5, having two commuting axial Killing fields that commute also with the time-translation Killing field. Assume that both solutions have the same interval (rod) structure, and the same values of angular momenta J_1, J_2. Then they are isometric.

The Idea of the Proof:

* Reduced Einstein equations

On M_{ext}, we define $R \times U(1) \times U(1)$-invariant 1-forms

$$\omega_1 = \star(\Psi_1 \wedge \Psi_2 \wedge d\Psi_1) \tag{7}$$
$$\omega_2 = \star(\Psi_1 \wedge \Psi_2 \wedge d\Psi_2) \tag{8}$$

$R_{ab} = 0 \Longrightarrow d\omega_1 = d\omega_2 = 0$

Naturally projected 1-forms $\hat{\omega}_1$ and $\hat{\omega}_2$ on $\hat{M} = \{\zeta \in \mathcal{C}, Im\zeta > 0\}$, $d\hat{\omega}_1 = d\hat{\omega}_2 = 0$

$\pi_1(\hat{M}) = 0 \Longrightarrow$ On $\hat{M}$, there exist globally defined potentials $\hat{\omega}_1 = d\chi_1$, $\hat{\omega}_2 = d\chi_2$

We define the 3×3 matrix field Φ by [**Maison (1979)**]

$$\Phi = \begin{pmatrix} f^{-1} & -f^{-1}\chi_1 & -f^{-1}\chi_2 \\ -f^{-1}\chi_1 & G_{11} + f^{-1}\chi_1\chi_1 & G_{12} + f^{-1}\chi_1\chi_2 \\ -f^{-1}\chi_2 & G_{21} + f^{-1}\chi_2\chi_1 & G_{22} + f^{-1}\chi_2\chi_2 \end{pmatrix}, \quad f = G_{11}G_{22} - G_{12}G_{21}\,. \tag{9}$$

$$G_{ij} = g_{ab}\Psi_i^a \Psi_j^b \qquad (10)$$

The matrix satisfies $\Phi^T = \Phi$, det $\Phi = 1$, and is positive semi-definite, meaning that it may be written in the form $\Phi = S^T S$ for some matrix S of determinant 1.

The vacuum Einstein equations are reduced to the sigma-model equation

$$\hat{D}^a[r\Phi^{-1}\hat{D}_a\Phi] = 0 \qquad (11)$$

on $\hat{M}$.

Consider now the exterior of two black solutions denoted by (M, g_{ab}), and $(\tilde{M}, \tilde{g}_{ab})$, The corresponding matrices are Φ and $\tilde{\Phi}$.

M as a manifold with a $\mathcal{G}$-action is uniquely determined by the rod structure modulo diffeomorphisms preserving the action of $\mathcal{G}$. Therefore, since the rod structures $\{\tilde{l}_j, \tilde{v}_j\}$ and $\{l_j, v_j\}$ are the same, M and $\tilde{M}$ are isomorphic as manifolds with a $\mathcal{G}$ action, and we may hence assume that $M = \tilde{M}$, and that $\tilde{t}^a = t^a, \tilde{\psi}_i^a = \psi_i^a$ for $i = 1, 2$. It follows in particular that $\tilde{g}_{ab}$ and g_{ab} may be viewed as being defined on the same analytic manifold, M, and we may also assume that $\tilde{r} = r$ and $\tilde{z} = z$. Consequently, it is possible to combine the divergence identities for the two solutions into a single identity on the upper complex half plane. This key identity is called the "Mazur identity" [**Mazur (1982)**] and is given by:

$$\hat{D}_a(r\hat{D}^a \mathrm{Tr}\,\Psi) = r\,\hat{g}^{ab}\mathrm{Tr}\left[\Delta\mathcal{J}_a^T\tilde{\Phi}\Delta\mathcal{J}_b\Phi^{-1}\right] \tag{12}$$

where

$$\Psi = \tilde{\Phi}\Phi^{-1} - 1, \quad \Delta\mathcal{J}_a = \tilde{\Phi}^{-1}\hat{D}_a\tilde{\Phi} - \Phi^{-1}\hat{D}_a\Phi\,. \tag{13}$$

Using now the identities $\Phi = S^T S$ and $\tilde{\Phi} = \tilde{S}^T\tilde{S}$, the Mazur identity can be presented in the form

$$\hat{D}_a(r\hat{D}^a \mathrm{Tr}\,\Psi) = r\,\hat{g}^{ab}\mathrm{Tr}\left[N_a^T N_b\right] \tag{14}$$

where $N_a = S^{-1}\Delta\mathcal{J}_a\tilde{S}$. The key point about the Mazur identity is that on the left side we have a total divergence, while the term on the right hand side is non-negative. This structure is now exploited in the following way

$$\hat{\Delta}\mathrm{Tr}\,\Psi \geq 0, \quad on\; R^3/\{z - axis\} \tag{15}$$

where $\hat{\Delta}$ is the ordinary Laplacian on R^3.

Using now the boundary conditions, one can show that $\mathrm{Tr}\,\Psi$ is globally bounded on R^3. It then follows from the maximum principle that $\mathrm{Tr}\,\Psi = 0$ everywhere. This in turn implies that $N_b = 0$ and therefore $\Phi\tilde{\Phi}^{-1} = const$. At infinity $\Phi\tilde{\Phi}^{-1} = 1$ which gives $\Phi = \tilde{\Phi}$ everywhere. In other words the two solutions coincide i.e. they are isomorphic.

*The idea of the proof can be applied to the extremal case

* The long standing problem of the uniqueness of the extremal 4D black holes was solved very recently by applying the described approach [**Amsel, Horowitz, Marolf, Roberts, (2009); arXiv:0906.2367 [gr-qc]**]

* The described approach was also used for proving the uniqueness of the 5D extremal black holes [**Figueras and Lucietti, (2009)**]; **arXiv:0906.5565[hep-th]**

*

A uniqueness theorem for 5D Einstein-Maxwell gravity

1. About the *spacetime metric* we assume that one of the axial Killing fields, say ψ_1^a, is orthogonal to the other Killing fields, $g_{ab}\psi_1^a\psi_2^b = 0 = g_{ab}t^a\psi_1^b$, and that it is hypersurface orthogonal, $\psi_{1[a}\nabla_b\psi_{1c]} = 0$.

2. About the *Maxwell field* we assume that there is a 1-form ξ_a orthogonal to the Killing fields such that $F_{ab} = \xi_{[a}\psi_{1b]}$. It can easily be shown that, if the Maxwell field arises from a vector potential $F_{ab} = 2\nabla_{[a}A_{b]}$ which is invariant under the Killing fields, then this will be the case if and only if A^a is proportional to ψ_1^a at each point in M. Note, however that we do *not* assume the existence of such a vector potential here.

Theorem [**S. Hollands and S.Y. (2007)**] Consider two stationary, asymptotically flat, Einstein-Maxwell black hole spacetime of dimension 5, having one time-translation Killing field and two axial Killing fields. We also assume that there are no points with discrete isotropy subgroup under the action of the isometry group in the exterior of the black hole, and we assume that the Killing and Maxwell fields satisfy the assumptions 1) and 2) above, implying that $\mathbf{v}_i = (1, 0)$ or $(0, 1)$, and $\mathcal{H} = S^3$ or $S^1 \times S^2$, and $Q_{\rm E} = 0 = J_1$ for the solutions. If the two solutions have the same interval structures, the same values of the mass m, same angular momentum J_2, and same magnetic charges $Q_{\rm M}[C_l]$ for all 2-cycles C_l, then they are isometric.

The magnetic (dipole) charges are defined by

$$Q_{\rm M}[C_l] = \frac{1}{2\pi} \int_{C_l} F \qquad (16)$$

where C_l form a basis of the homology group $H_2(M)$.

The ideas of the proof can be easily generalized to the case of

- ⋆ **Einstein-Maxwell-dilaton gravity**

- ⋆ **5D minimal supergravity**

*

Uniqueness theorem for Kaluza-Klein black holes

Consider asymptotically Kaluza-Klein spacetimes $M(\infty) = R^{s,1} \times T^{D-s-1}$, $s = 3 \ \mathit{or} \ 4$ with an isometry group $R \times T^{D-3}$.

Theorem [**S. Hollands and S.Y. (2008)**] There can be at most one stationary, asymptotically Kaluza-Klein spacetime (M, g) with D-3 axial Killing fields, satisfying some natural technical assumptions, for a given interval structure {a(Ij), l(Ij)} and a given set of angular momenta {Ji}, i = 1,...,D - 3.

Topological Classification

The topology of the horizon can be either $S^2 \times T^{k-3}$, $S^3 \times T^{k-4}$, $L(p,q) \times T^{k-4}$ or T^{k-1}, for Kaluza-Klein spacetimes with asymptotics $R^{k-1,1} \times T^{D-k}$ where $k = 4 \; or \; 5$.

*

UNSOLVED PROBLEM: "UNIQUENESS THEOREM" FOR ASYMPTOTICALLY FLAT BLACK HOLES IN SPACETIMES WITH $D > 5$.

*

THANK YOU!