Dilatonic black holes in heterotic string theory: perturbations and stability

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Leading α' corrections

- Effective action in the Einstein frame

$$\frac{1}{2\kappa^2}\int\sqrt{-g}\left[\mathcal{R}-\frac{4}{d-2}\left(\partial^\mu\phi\right)\partial_\mu\phi+\mathrm{e}^{\frac{4}{2-d}\phi}\frac{\lambda}{2}\mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma}\right]\mathrm{d}^dx,$$

$$\lambda=\frac{\alpha'}{2},\frac{\alpha'}{4}\text{ (bosonic, heterotic).}$$

- Field equations

$$\mathcal{R}_{\mu\nu}+\lambda\mathrm{e}^{\frac{4}{2-d}\phi}\left(\mathcal{R}_{\mu\rho\sigma\tau}\mathcal{R}_{\nu}^{\ \rho\sigma\tau}-\frac{1}{2(d-2)}g_{\mu\nu}\mathcal{R}_{\rho\sigma\lambda\tau}\mathcal{R}^{\rho\sigma\lambda\tau}\right)=0;$$

$$\nabla^2\phi-\frac{\lambda}{4}\mathrm{e}^{\frac{4}{2-d}\phi}\left(\mathcal{R}_{\rho\sigma\lambda\tau}\mathcal{R}^{\rho\sigma\lambda\tau}\right)=0.$$

General setup

- Metric of the type

$$d\,s^2 = -f(r)\,d\,t^2 + g^{-1}(r)\,d\,r^2 + r^2\,d\,\Omega^2_{d-2};$$

- Variation of the metric

$$h_{\mu\nu} = \delta g_{\mu\nu};$$

- Variation of the Riemann tensor:

$$\begin{aligned}\delta\mathcal{R}_{\rho\sigma\mu\nu} &= \frac{1}{2}\Big(\mathcal{R}_{\mu\nu\rho}{}^{\lambda}h_{\lambda\sigma} - \mathcal{R}_{\mu\nu\sigma}{}^{\lambda}h_{\lambda\rho} \\ &- \nabla_\mu\nabla_\rho h_{\nu\sigma} + \nabla_\mu\nabla_\sigma h_{\nu\rho} - \nabla_\nu\nabla_\sigma h_{\mu\rho} + \nabla_\nu\nabla_\rho h_{\mu\sigma})\,.\end{aligned}$$

Perturbations on the $(d-2)$-sphere

- General tensors of rank at least 2 on the $(d-2)$-sphere can be uniquely decomposed in their *tensorial, vectorial and scalar* components.
- One can in general consider perturbations to the metric and any other physical field of the system under consideration.

Tensorial perturbations of the metric

- We consider only the tensorial part of $h_{\mu\nu}$:

$$h_{ij} = 2r^2 H_T(r,t)\mathcal{T}_{ij}\left(\theta^i\right),\; h_{ia} = 0, h_{ab} = 0$$

with

$$\left(\gamma^{kl} D_k D_l + k_T\right)\mathcal{T}_{ij} = 0,\; D^i\mathcal{T}_{ij} = 0,\; g^{ij}\mathcal{T}_{ij} = 0.$$

- D_i: $(d-2)$-sphere covariant derivative, associated to the metric γ_{ij}.
- $\mathcal{T}_{ij}$ are the eigentensors of D^2 on S^{d-2}
- $-k_T = 2 - \ell\left(\ell + d - 3\right)$ are the eigenvalues of D^2 on S^{d-2}, where $\ell = 2, 3, 4, \ldots$.

Tensorial perturbations of $\mathcal{R}_{\rho\sigma\mu\nu}$

$$
\begin{aligned}
\delta\mathcal{R}_{ijkl} &= \left[(3g-1)\,H_T + rg\partial_r H_T\right]\left(g_{il}\mathcal{T}_{jk} - g_{ik}\mathcal{T}_{jl} - g_{jl}\mathcal{T}_{ik} + g_{jk}\mathcal{T}_{il}\right) \\
&+ r^2 H_T\left(D_i D_l \mathcal{T}_{jk} - D_i D_k \mathcal{T}_{jl} - D_j D_l \mathcal{T}_{ik} + D_j D_k \mathcal{T}_{il}\right); \\
\delta\mathcal{R}_{itjt} &= \left[-r^2\partial_t^2 H_T + \frac{1}{2} f f' r^2 \partial_r H_T + f f' r H_T\right]\mathcal{T}_{ij}; \\
\delta\mathcal{R}_{itjr} &= \left(-r^2\partial_t\partial_r H_T - r\partial_t H_T + \frac{1}{2} r^2 \frac{f'}{f}\partial_t H_T\right)\mathcal{T}_{ij}; \\
\delta\mathcal{R}_{irjr} &= \left(-r\frac{g'}{g} H_T - \frac{1}{2} r^2 \frac{g'}{g}\partial_r H_T - 2r\partial_r H_T - r^2\partial_r^2 H_T\right)\mathcal{T}_{ij}.
\end{aligned}
$$

All other tensorial perturbations are 0.

Perturbations of the field equations

$$
\begin{aligned}
\delta\nabla^2\phi \quad &- \quad \frac{\lambda}{4}\mathrm{e}^{\frac{4}{2-d}\phi}\delta\left(\mathcal{R}_{\rho\sigma\lambda\tau}\mathcal{R}^{\rho\sigma\lambda\tau}\right)+\frac{\lambda}{d-2}\mathrm{e}^{\frac{4}{2-d}\phi}\mathcal{R}_{\rho\sigma\lambda\tau}\mathcal{R}^{\rho\sigma\lambda\tau}\delta\phi=0,\\
\delta\mathcal{R}_{ij} \quad &+ \quad \lambda\mathrm{e}^{\frac{4}{2-d}\phi}\left[\delta\left(\mathcal{R}_{i\rho\sigma\tau}\mathcal{R}_{j}^{\ \rho\sigma\tau}\right)-\frac{1}{2(d-2)}\mathcal{R}_{\rho\sigma\lambda\tau}\mathcal{R}^{\rho\sigma\lambda\tau}h_{ij}\right.\\
&- \quad \left.\frac{1}{2(d-2)}g_{ij}\delta\left(\mathcal{R}_{\rho\sigma\lambda\tau}\mathcal{R}^{\rho\sigma\lambda\tau}\right)\right]+\frac{4}{d-2}\mathcal{R}_{ij}\delta\phi=0.
\end{aligned}
$$

- Spherical symmetry, $\partial_k\phi=0,\ (a,b=r,t)$:

$$
\begin{aligned}
\delta\nabla^2\phi \quad &= \quad g^{ab}\partial_a\partial_b\delta\phi-g^{ab}\Gamma_{ab}^{\ \ c}\partial_c\delta\phi+g^{ij}\partial_i\partial_j\delta\phi-g^{ij}\Gamma_{ij}^{\ \ k}\partial_k\delta\phi\\
&- \quad g^{ij}\Gamma_{ij}^{\ \ a}\partial_a\delta\phi.
\end{aligned}
$$

- Using $\delta\left(\mathcal{R}_{\rho\sigma\lambda\tau}\mathcal{R}^{\rho\sigma\lambda\tau}\right)=0$, we can set $\delta\phi=0$.

Perturbed graviton field equation

$$
\left(1-2\lambda\frac{f'}{r}\right)\frac{r^2}{f}\partial_t^2 H_T - \left(1-2\lambda\frac{g'}{r}\right) r^2 g\,\partial_r^2 H_T -
$$

$$
-\left[(d-2)rg+\frac{1}{2}r^2\left(f'+g'\right)+4\lambda(d-4)\frac{g\,(1-g)}{r}-4\lambda g g'-\lambda r\left(f'^2+g'^2\right)\right]\partial_r H_T +
$$

$$
+\left[\ell\,(\ell+d-3)\left(1+\frac{4\lambda}{r^2}\,(1-g)\right)+2(d-2)-2(d-3)g-r\left(f'+g'\right)\,+\right.
$$

$$
\left.+\,\lambda\left(8\frac{1-g}{r^2}+2\,(d-3)\,\frac{(1-g)^2}{r^2}-\frac{r^2}{d-2}\left[f''+\frac{1}{2}\left(\frac{f'g'}{g}-\frac{f'^2}{f}\right)\right]^2\right)\right]H_T=0
$$

can be written in the form

$$
\partial_t^2 H_T - F^2(r)\,\partial_r^2 H_T + P(r)\,\partial_r H_T + Q(r)\,H_T = 0.
$$

The Master Equation

The perturbation equation can be written as a "master equation"

$$\frac{\partial^2\Phi}{\partial x^2} - \frac{\partial^2\Phi}{\partial t^2} =: V_T\Phi.$$

- $dx/dr = 1/\sqrt{fg}$ ("tortoise" coordinate);
- $\Phi = k(r)H_T$ ("master" variable);
- V_T : potential for tensor-type gravitational perturbations. In classical EH gravity it is the same as the potential for scalar fields (Ishibashi, Kodama);
- $k(r) =$

$$\frac{1}{\sqrt[4]{fg}} \exp\left(\int \frac{(d-2)rg+\frac{1}{2}r^2(f'+g')+4\lambda(d-4)\frac{g(1-g)}{r}-4\lambda gg'-\lambda r\left(f'^2+g'^2\right)}{2fg} dr \right)$$

The string-corrected tensor potential

$$
\begin{aligned}
V_{\mathsf{T}}[f(r),g(r)] \quad = \quad & \frac{1}{r^4 fg}\Bigg(\ell(\ell+d-3)r^2f^2g+\frac{1}{4}(d-2)(d-4)r^2f^2g^2 \\
+ \quad & \frac{1}{4}(d-6)r^3f^2gf' + r^3fg^2f' + \frac{1}{16}r^4f^2f'^2 + \frac{3}{16}r^4g^2f'^2 \\
+ \quad & \frac{1}{4}(d-2)r^3f^2gg' - \frac{1}{8}r^4f(g+f)f'g' - \frac{1}{4}r^4fg(g-f)f''\Bigg) \\
+ \quad & \frac{\lambda}{r^4fg}\big(4\ell(\ell+d-3)(1-g)gf^2 + 2(d-4)(d-5)(1-g)g^2f^2 \\
+ \quad & (d-4)rf^2gf' + 2r\ell(\ell+d-3)f^2gf' + (d-3)(d-4)rf^2g^2f' \\
+ \quad & \frac{1}{2}(d-6)r^2f^2gf'^2 + 2r^2fg^2f'^2 + (d-4)rf^2gg' - 5(d-4)rf^2g^2g' \\
+ \quad & \left(d-\frac{7}{2}\right)r^2f^2gf'g' + \frac{1}{4}r^3f^2f'^2g' - \frac{1}{2}(d-1)r^2f^2gg'^2 - \frac{1}{2}r^3f^2f'g'^2 \\
+ \quad & \frac{1}{4}r^3f^2g'^3 + (d-2)r^2f^2g^2f'' + \frac{1}{2}r^3f^2gg'f'' - 2r^2f^2g^2g'' \\
+ \quad & \frac{1}{2}r^3f^2gf'g'' - r^3f^2gg'g''\Big)
\end{aligned}
$$

Study of the stability

- That was the potential for tensor–type gravitational perturbations of any kind of static, spherically symmetric $\mathcal{R}^2$ string–corrected black hole in d–dimensions.
- Solutions of the form $\Phi(x,t) = e^{i\omega t}\phi(x)$;
- The master equation is then written in the Schrödinger form,

$$\left[-\frac{d^2}{dx^2} + V\right]\phi(x) =: A\phi(x) = \omega^2\phi(x);$$

- A solution to the field equation is then stable if the operator A has no negative eigenvalues (Ishibashi, Kodama; Dotti, Gleiser).

"S-deformation" approach

Stability means positivity (for every possible ϕ) of the following inner product:

$$\begin{aligned}\langle\phi, A\phi\rangle &= \int_{-\infty}^{+\infty} \overline{\phi}(x)\left[-\frac{d^2}{dx^2}+V\right]\phi(x)\,dx \\ &= \int_{-\infty}^{+\infty}\left[\left|\frac{d\phi}{dx}\right|^2+V\left|\phi\right|^2\right]\,dx \\ &= \int_{-\infty}^{+\infty}\left[\left|D\phi\right|^2+\widetilde{V}\left|\phi\right|^2\right]\,dx\end{aligned}$$

with $D=\frac{d}{dx}+S$, $\widetilde{V}=V+\sqrt{fg}\frac{dS}{dr}-S^2$.

"S-deformation" approach (cont.)

- Taking $S = -\frac{\sqrt{fg}}{k}\frac{dk}{dr}$ we are left with

$$\langle \phi, A\phi \rangle = \int_{-\infty}^{+\infty} |D\phi|^2 \; dx + \int_{-\infty}^{+\infty} \frac{Q(r)}{\sqrt{fg}} |\phi|^2 \; dx,$$

with

$$Q = \frac{\ell(-3+d+\ell)f\left(r^2+4\lambda(1-g)\right)+r^3(g-f)f'}{r^3\left(r-2\lambda f'\right)}$$

(after using equations of motion).

Stability condition

- The second term of $\langle \phi, A\phi \rangle$ can be written as

$$\int_{R_H}^{+\infty} Q(r) \frac{|\phi|^2}{\sqrt{fg}} dr.$$

- For $r > R_H$, $f(r), g(r) > 0$.
- This condition keeps valid with α' corrections as long as the black hole in consideration is *large*, i.e. $R_H \gg \sqrt{\lambda}$, which is true in string perturbation theory.
- This way the perturbative stability of a given black hole solution, with respect to tensor–type gravitational perturbations, follows if and only if one has $Q(r) > 0$ for $r \geq R_H$.

The Callan-Myers-Perry black hole

- The only free parameter is the horizon radius R_H (secondary hair), which is not changed;
- $f(r) = g(r) =$ $\left(1 - \left(\frac{R_H}{r}\right)^{d-3}\right) \left[1 - \lambda \frac{(d-3)(d-4)}{2} \frac{R_H^{d-5}}{r^{d-1}} \frac{r^{d-1} - R_H^{d-1}}{r^{d-3} - R_H^{d-3}}\right]$;
- $\alpha' = 0$: Schwarzschild-Tangherlini solution;
- α'-corrected ADM black hole mass:

$$m = \left(1 + \frac{(d-3)(d-4)}{2} \frac{\lambda}{R_H^2}\right) \frac{(d-2)\, A_{d-2}}{2\kappa^2} R_H^{d-3}$$

- dilaton vanishes classically and only gets α'-corrections (1988).

α'-corrections to the dilaton

- Dilaton field equation in the background of a CMP black hole:

$$((r^{d-2} - rR_H^{d-3})\phi')' = \lambda \frac{(d-2)^2(d-3)(d-1)}{4} \frac{R_H^{2(d-3)}}{r^d}$$

- First integration:

$$\frac{\phi'}{\lambda} = -\frac{(d-2)^2(d-3)R_H^{2(d-3)}}{4r^d\left(r^{d-3} - R_H^{d-3}\right)} + \frac{c}{r\left(r^{d-3} - R_H^{d-3}\right)}$$

- For each d it is always possible to choose

$$c = \frac{(d-3)(d-2)^2}{4} R_H^{d-5},$$

such that ϕ' is regular at the horizon.

α'-corrected dilaton solution

- Solution:

$$\begin{aligned}\frac{\phi(r)}{\lambda} &= -\frac{(d-3)(d-2)^2}{8(d-1)r^{d+2}}\left[(d-1)+2\left(\frac{R_H}{r}\right)^{d-3}\right.\\ &- 2\frac{d-1}{d-3}\left(\frac{r}{R_H}\right)^2 B\left(\left(\frac{R_H}{r}\right)^{d-3};\frac{2}{d-3},0\right)\Bigg]\\ &+ \frac{(d-2)^2}{4R_H^2}\ln\left(1-\left(\frac{R_H}{r}\right)^{d-3}\right)\end{aligned}$$

with

$$B(x;\,a,b)=\int_0^x t^{a-1}\,(1-t)^{b-1}\,dt.$$

Dilatonic BH and compactified strings

- Metric in $d_s = 10$ (or 26) dimensions of the type

$$d\,s^2 = -f(r)\,d\,t^2 + g^{-1}(r)\,d\,r^2 + r^2\,d\,\Omega_{d-2}^2 + h(\phi)\,g_{mn}(y)\,d\,y^m\,d\,y^n;$$

- Solution:

$$\begin{aligned} h(\phi) &= (1 - \frac{2}{d_s - 2}\phi)^2; \\ g(r) &= \left(1 - \left(\frac{R_H}{r}\right)^{d-3}\right)\left(1 - \frac{(d-3)(d-4)}{2}\,\frac{\lambda}{R_H^2}\left(\frac{R_H}{r}\right)^{d-3}\,\frac{1-\left(\frac{R_H}{r}\right)^{d-1}}{1-\left(\frac{R_H}{r}\right)^{d-3}}\right) \end{aligned}$$

(Callan-Myers-Perry);

Dilatonic BH and compactified strings

$$
\begin{aligned}
f(r) &= g(r) + 4\left(1-\left(\frac{R_H}{r}\right)^{d-3}\right)\frac{d_s-d}{(d_s-2)^2}\left(\phi - r\phi'\right) \\
&= \left(1-\left(\frac{R_H}{r}\right)^{d-3}\right)\left(1-\frac{(d-3)(d-4)}{2}\,\frac{\lambda}{R_H^2}\left(\frac{R_H}{r}\right)^{d-3}\,\frac{1-\left(\frac{R_H}{r}\right)^{d-1}}{1-\left(\frac{R_H}{r}\right)^{d-3}}\right. \\
&- \frac{(d-2)^2}{2}\,\frac{d_s-d}{(d_s-2)^2}\,\frac{\lambda}{R_H^2}\left[(d-3)\left(\frac{R_H}{r}\right)^2 + 2\frac{d-3}{d-1}\left(\frac{R_H}{r}\right)^{d-1}\right. \\
&- \left.2B\left(\left(\frac{R_H}{r}\right)^{d-3};\ \frac{2}{d-3},0\right)\right] \\
&+ (d-2)^2\,\frac{d_s-d}{(d_s-2)^2}\,\frac{\lambda}{R_H^2}\ln\left(1-\left(\frac{R_H}{r}\right)^{d-3}\right) \\
&- \left.(d-2)^2(d-3)\frac{d_s-d}{(d_s-2)^2}\,\frac{\lambda}{R_H^2}\left(\frac{R_H}{r}\right)^{d-3}\,\frac{1-\left(\frac{R_H}{r}\right)^{d-1}}{1-\left(\frac{R_H}{r}\right)^{d-3}}\right).
\end{aligned}
$$

Stability of solutions with secondary hair

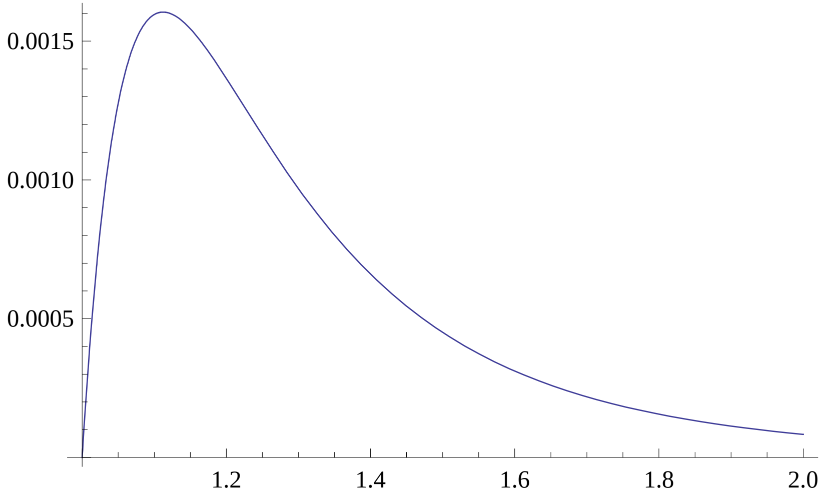
- For *any* string theory corrected, spherically symmetric, static solution, which has *no* dilaton field at the classical level, one has

$$
\begin{aligned}
Q(r) \;&\simeq\; \frac{\ell(-3+d+\ell)f + r(g-f)f'}{r^2} \\
&+\; 2\lambda \frac{\left(\ell(-3+d+\ell)f\left(\frac{2(1-g)}{r} + f'\right) + r(g-f)f'^2\right)}{r^3}.
\end{aligned}
$$

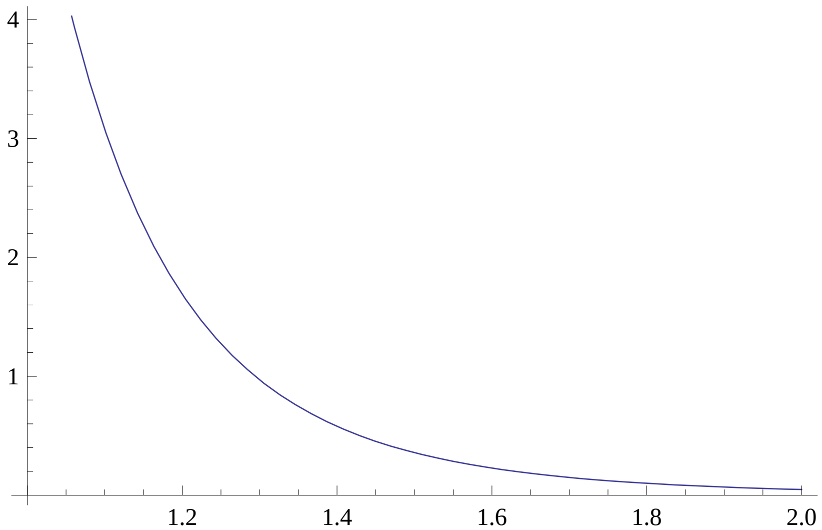
- One will have $Q(r) \geq 0$ for $r \geq R_H$, in any spacetime dimension, as long as

$$
(g-f)f' > 0\,,\; 2\frac{1-g(r)}{r} + f'(r)\Big|_{\lambda=0} > 0.
$$

Typical behavior of $g - f$ and f'



$g \qquad - \qquad f$



f'

Stability of dilaton-coupled BH

- At the classical level, the solution is unique (Tangherlini, Myers, Perry) and one has

$$2\frac{1-g(r)}{r} + g'(r)\Big|_{\lambda=0} = (d-1)\frac{R_H^{d-3}}{r^{d-2}},$$

which is positive for any $r > R_H$.

- This proves stability under tensor–type gravitational perturbations of any spherically symmetric static solution with no dilaton at $\lambda = 0$ for any $d > 4$.

Scattering Theory

- The equation describing gravitational perturbations to a black hole solution allows for a study of scattering in this spacetime geometry.
- Classical result in EH gravity: for *any* spherically symmetric black hole in arbitrary dimension, the absorption cross–section of minimally–coupled massless scalar fields equals the area of the black hole horizon (Das, Gibbons, Mathur, 1997).
- Universality of the low–frequency absorption cross–sections of generic black holes in EH gravity (Halmark, Natário, Schiappa, 2007)?
- Work that needs to be done: trying to extend such result with the inclusion of higher–derivative corrections.

Conclusions

- We found out the dilatonic black hole solution with $\mathcal{R}^2$ corrections in d dimensions;
- We extended the perturbation theory to $\mathcal{R}^2$ stringy gravity;
- We studied the stability of black hole solutions under tensor type gravitational perturbations, and proved the perturbative stability of the dilatonic $\mathcal{R}^2$ black hole for any space-time dimension.