Existence of horizons in Robinson-Trautman spacetimes of arbitrary dimension

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OUTLINE

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Robinson–Trautman spacetime in D dimensions

Robinson–Trautman spacetimes (aligned pure radiation or vacuum with a cosmological constant Λ) in any dimension

$$\mathrm{d}s^2 = \frac{r^2}{P^2}\,\gamma_{ij}\,\mathrm{d}x^i\mathrm{d}x^j - 2\,\mathrm{d}u\mathrm{d}r - 2H\,\mathrm{d}u^2$$

where $2H \equiv g^{rr} = -g_{uu}$ is given by

$$2H = \frac{\mathcal{R}}{(D-2)(D-3)} - 2\,r(\ln P)_{,u} - \frac{2\Lambda}{(D-2)(D-1)}\,r^2 - \frac{\mu(u)}{r^{D-3}}$$

RT spacetimes defined as those possessing twistfree, shearfree and expanding null geodesic congruence.

(Podolský & Ortaggio)

The unimodular spatial metric $\gamma_{ij}(x)$ and the function $P(x,u)$ must satisfy the field equations for the metric $h_{ij} = P^{-2}\gamma_{ij}$:

$$\mathcal{R}_{ij} = \frac{\mathcal{R}}{D-2} h_{ij}$$

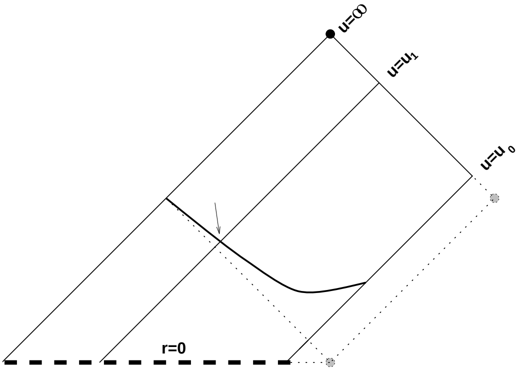
- In $D = 4$: previous equation satisfied always and $\mathcal{R}$ depends on x^i
- In $D > 4$: $\mathcal{R} = \mathcal{R}(u)$, but generally a huge variety of possible spatial metrics h_{ij} (e.g., for $\mathcal{R} > 0$ and $5 \leq D - 2 \leq 9$ infinite number of compact Einstein spaces were classified)

Apparent horizon

- Event horizon is a global property $\Rightarrow$ full spacetime evolution is necessary.
- Quasi-local horizon characterization:
  - apparent horizon (Hawking et al.)
  - trapping horizon (Hayward)
  - isolated or dynamical horizon (Ashtekar et al.)

  The basic *local* condition is the same:
  These horizons are sliced by marginally trapped surfaces $\Rightarrow$ vanishing expansion of outgoing (ingoing) null congruence orthogonal to the surface.

"Local horizons" are used in numerical relativity or black hole thermodynamics.



Past apparent horizon is a hypersurface $r = R(u, x^i)$ such that its intersection with each $u = u_1$ slice is an marginally (past) trapped $(D-2)$-surface.

Real D-ad (D-bein) formalism

$$g_{ab} = -2\, l_{(a} n_{b)} + m_{a\{i\}} m_{b\{j\}}\, \delta^{ij}$$

Null D-ad adapted to trapped hypersurfaces:

$$\begin{aligned} l^a &= (0, 1, 0, .., 0) \\ n^a &= (1,\, [-H + \tfrac{1}{2} g^{ij} R_{,i} R_{,j}]\,, \nabla R) \\ m^a_{\{i\}} &= (0,\, \tfrac{P}{r} R_{,i}\,,\, \tfrac{P}{r} \mathbf{w}_i) \end{aligned}$$

where $(D-2)$ vectors $\mathbf{w}_i$ diagonalize metric h and $\nabla R = \{R^{,x^1}, .., R^{,x^{D-2}}\}$.

- $\Theta_l = \frac{D-2}{r} \Rightarrow$ outgoing null congruence is diverging
- trapped hypersurface $R(u, x^i)$ is given by $\Theta_n = 0$

Generalized Penrose–Tod equation

Ingoing null congruence expansion

$$\Theta_n = n_{a;b}\, p^{ab}$$

$p^{ab} = g^{ab} + 2\, l_{(a} n_{b)}$ corresponds to the hypersurface projector.

From $\Theta_n = 0$ we get

$$\mathcal{R} - \frac{2(D-3)}{D-1}\Lambda R^2 - (D-2)(D-3)\frac{\mu}{R^{D-3}} - 2(D-3)\Delta(\ln R) -$$
$$-(D-4)(D-3)(\nabla \ln R)\cdot(\nabla \ln R) = 0$$

It is a nonlinear PDE, where both the Laplacian and scalar product in the last term correspond to the Einstein metric h_{ij}.

- $D = 3 \Rightarrow \mathcal{R} = 0$ (all 1-dimensional manifolds are locally isometric)

$D = 4$: Existence and uniqueness of the solution

RESULTS	$\Lambda = 0$	$\Lambda < 0$	$\Lambda > 0$
Existence	Always	Always	$\Lambda \leq \frac{4}{9\mu^2}$
Uniqueness	Always	Always	$R \leq \sqrt[3]{\frac{3\mu}{2\Lambda}}$

Spherical symmetry and $\Lambda > 0$ (Schwarzschild de–Sitter) :

- $\Lambda \leq \frac{4}{9\mu^2}$ rules out over-extreme case
- $R \leq \sqrt[3]{\frac{3\mu}{2\Lambda}}$ for extreme case $(9\Lambda m^2 = 1)$ reduces to $R \leq 3m$
  $\Rightarrow$ Uniqueness of BH horizon

$D > 4$: Existence of the solution

The methods used in $D = 4$ are not applicable when the equation is of the form (after substitution $R = Ce^{-u}$)

$$\Delta u = F(x, u, \nabla u)$$

and F is not linear in gradient.

$$\Downarrow$$

We will proceed by combining several steps (motivated by Kuo 2005).

Step 1 $Pu = -2(D-3)\Delta u + \rho u$, with $\rho > 0$
Maximum Principle $\Rightarrow$ $\mathrm{Ker} P = 0$

Step 2 $Pu = f$ for $f \in C^{0,\alpha}(M)$ has unique solution $u \in C^{2,\alpha}(M)$
(Fredholm alternative)

Step 3 $Pu = f(x, u, \nabla u)$
f_n - truncature of f by $\pm n$
$v \in C^{1,\beta}(M) \rightarrow f_n(x, v, \nabla v)$ is bounded and there exists unique $w \in C^{2,\alpha\beta}(M)$ solving $Pw = f_n(x, v, \nabla v)$

Step 4 The map $v \rightarrow w$ satisfies conditions of Schauder fixed point theorem $\Rightarrow$ for each n there exists $u_n \in C^{1,\beta}(M)$ solving $Pu_n = f_n(x, u_n, \nabla u_n)$ and $||u_n||_{L^\infty} \leq \frac{n}{\rho}$ (by Stokes theorem)

Step 5 Theorem by Boccardo, Murat & Puel: If the growth of f is at most $|\nabla u|^2$ and there exist sub- and super-solution $u^- \leq u^+$, then there is a L^∞-bounded subsequence $u_{\bar{n}}$ satisfying $u^- \leq u_{\bar{n}} \leq u^+$

Step 6 Thanks to elliptic estimate $||u_n||_{C^{2,\gamma}} \leq K(||u_n||_{C^0} + ||f||_{C^{0,\gamma}})$ it is even $C^{1,\beta}$-bounded

Step 7 Then there is a subsequence $u_{\tilde{n}} \to u$

NOTE: supersolution $Pu^+ \leq f(x, u^+, \nabla u^+)$

Constant sub- and super-solutions for $\mathcal{R}>0$ (assuming $u_{min}>0$):

$\Lambda \leq 0$

$$u_1^+ = \frac{1}{D-3}\ln\left[\frac{C^{D-3}}{(D-2)(D-3)\mu}(\mathcal{R}-\frac{2(D-3)}{D-1}\Lambda C^2)\right]$$

$$u_1^- = \frac{1}{D-3}\ln\left[\frac{C^{D-3}}{(D-2)(D-3)\mu}\mathcal{R}\right]$$

$\Lambda > 0$

$$u_2^+ = u_1^-$$

$$u_2^- = u_1^+$$

!!! but only when $\frac{2\mathcal{R}}{(D-1)(D-2)(D-3)\mu}\left(\frac{\mathcal{R}}{2\Lambda}\right)^{\frac{D-3}{2}} > 1$!!!

Constant sub- and super-solutions for $\mathcal{R} \leq 0$ (assuming $u_{min} \geq 0$):

$\Lambda < 0$

$$u^{+} = \frac{1}{D-3} \ln \left[-\frac{2C^{D-1}}{(D-1)(D-2)\mu} \Lambda \right]$$

$$u^{-} = 0$$

and select $C \geq C_{min}$

$$-\mathcal{R} + \frac{2(D-3)}{D-1} \Lambda C_{min}^{2} + (D-2)(D-3)\mu C_{min}^{-(D-3)} = 0$$

$\Lambda \geq 0$ impossible to find constant u^{-}

Uniqueness not resolved yet.

THANK YOU