Black Lenses

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Black-hole uniqueness theorems in 4D

- Stationary, asymptotically flat vacuum black holes are uniquely determined by their mass and angular momentum
- They coincide with the Kerr black hole, which has a spherical S^2 horizon topology

Current situation in 5D

- Known asymptotically flat vacuum solutions:
  - Myers–Perry rotating black hole with horizon topology S^3
  - Emparan–Reall rotating black ring with horizon topology $S^1 \times S^2$ (generalised by Pomeransky & Sen'kov [hep-th/0612005] to two independent rotations)
- Not fully determined by their asymptotic mass and angular momenta
- Different horizon topologies allowed

A 5D uniqueness theorem of Hollands and Yazadjiev [arXiv:0707.2775]

- Stationary, asymptotically flat vacuum black holes with two commuting axial symmetries are uniquely determined by their mass, angular momentum, and so-called rod structure
- In particular, the rod structure determines the topology of the horizon, and only three possibilities are allowed:
  - a sphere: S^3
  - a ring: $S^1 \times S^2$
  - a lens space: $L(p, q)$

Definition of a lens space

- A lens space $L(p,q)$ is defined as a $\mathbb{Z}_p$ quotient space of S^3 as follows:
  - Write S^3 as $\{(z_1,z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$
  - Consider the $\mathbb{Z}_p = \{0,1,\ldots,p-1\}$ action on S^3 given by

$$m \cdot (z_1,z_2) = (e^{\frac{2\pi im}{p}} z_1, e^{\frac{2\pi imq}{p}} z_2)$$

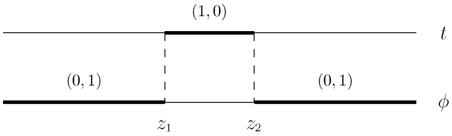
  - Then $L(p,q) \equiv S^3/\mathbb{Z}_p$
- Special cases:
  - $L(0,1) = S^1 \times S^2$
  - $L(1,q) = S^3$
  - $L(2,1) = \mathbb{R}P^3$

Rod structure of stationary, axisymmetric solutions [Harmark, hep-th/0408141]

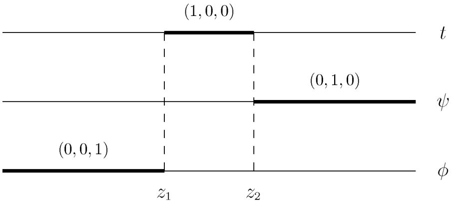
- Consider a D-dimensional space-time with $D-2$ commuting Killing vector fields $\partial/\partial x^i$
- Change coordinates to so-called Weyl coordinates (x^i, ρ, z), in which the Einstein equations simplify considerably
- The metric $g_{ij}(\rho, z)$ becomes degenerate along the axis $\rho = 0$, with a sequence of rod-like sources (with differing directions) along it
- Time-like rods correspond to horizons, while space-like rods correspond to axes of the space-time

Rod structure examples

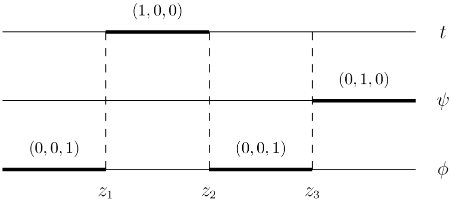
- 4D Schwarzschild:



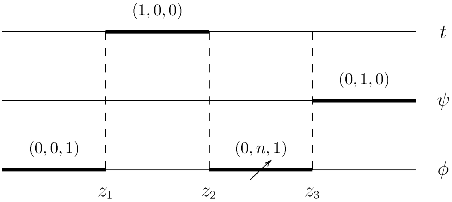
- 5D Schwarzschild:



- 5D static black ring:



- Simplest possible rod structure of a 5D static 'black lens':



Horizon has topology $L(n, 1)$; the value 1 is chosen so that there is no orbifold singularity at z_3

$\mathbb{Z}_n$ orbifold singularity or A_{n-1} singularity

- Consider flat $\mathbb{R}^4$ in spherical polar coordinates:

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2 + \cos^2\theta\, d\psi^2)$$

  where ϕ and ψ have period 2π

- If the constant-r S^3 surfaces were replaced by $L(n,1)$ surfaces (through appropriate identifications of ϕ and ψ), then there will be a $\mathbb{Z}_n$ orbifold singularity at the origin $r = 0$

A static black lens

- The local metric for this was found by Ford, et al. [arXiv:0708.3823] and Lu, et al. [arXiv:0804.1152], but a black lens interpretation was not made
- We have found a new and simple form of this solution (in so-called C-metric coordinates):

$$\begin{aligned} ds^2 = & -\frac{1+cy}{1+cx}\,dt^2 \\ & + \frac{2\kappa^2(1+cx)}{(1-a^2)(x-y)^2H(x,y)}\Bigg\{\frac{H(x,y)^2}{1-c}\left(\frac{dx^2}{G(x)}-\frac{dy^2}{G(y)}\right) \\ & \quad +(1-x^2)\left[(1-c-a^2(1+cy))d\phi - ac(1+y)d\psi\right]^2 \\ & \quad -(1-y^2)\left[(1-c-a^2(1+cx))d\psi - ac(1+x)d\phi\right]^2\Bigg\} \end{aligned}$$

- Here, $0 < c < 1$, $-1 < a < 1$, and

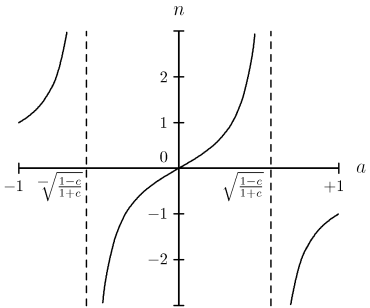
$$G(x) = (1 - x^2)(1 + cx)$$
$$H(x, y) = (1 - c)^2 - a^2(1 + cx)(1 + cy)$$

- Asymptotically flat if ψ and ϕ are identified with period 2π
- Horizon at $y = -1/c$ has topology $L(n, 1)$, provided we identify:

$$\frac{2ac}{1 - c - a^2(1 + c)} = n \qquad (1)$$

- Special cases:
  - Static black ring: $a = 0$
  - 5D Schwarzschild black hole: $a = \pm\sqrt{(1 - c)/(1 + c)}$

- Solve the condition (1):



- Two possible ranges for a:
  - Range I: $0 < a < \sqrt{(1-c)/(1+c)}$
  - Range II: $-1 < a < -\sqrt{(1-c)/(1+c)}$

Properties of the static black lens

- Range I:
  - There always exists a conical (strut) singularity along the finite axis $x = 1$
  - No CTCs outside the horizon
- Range II:
  - The conical singularity can be eliminated for certain parameter values
  - However, there will always exist a naked singularity with S^3 topology surrounding z_3
  - CTCs exist inside this naked singularity
- Range I appears to be more relevant physically. Can the conical singularity be eliminated by introducing rotation (as in the case of black rings)?

Check horizon topology of the static black lens

- The horizon metric is homeomorphic to (modulo the possible presence of a conical singularity):

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\tilde{\phi}^2 + \cos^2\theta\, \mathrm{d}\tilde{\psi}^2$$

where $\tilde{\phi}$ and $\tilde{\psi}$ have period 2π. Here,

$$\tilde{\phi} = \phi - \frac{1}{n}\psi\,, \qquad \tilde{\psi} = \frac{1}{n}\psi$$

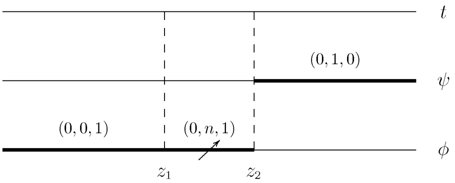
- Since ψ has period 2π, we must make the following identifications:

$$\tilde{\phi} \to \tilde{\phi} - \frac{2\pi}{n}\,, \qquad \tilde{\psi} \to \tilde{\psi} + \frac{2\pi}{n}$$

- These are precisely the identifications of S^3 required to turn it into the lens space $L(n, 1)$

Background limit of the static black lens

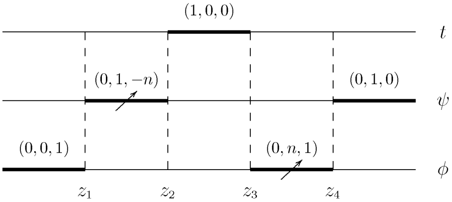
- Rod structure:



- There exists an orbifold singularity at z_1
- There also exists a conical singularity along the finite axis
- This conical singularity can be eliminated (as in Range II), but then there will be a spherical naked singularity surrounding z_2

Evslin's static black lens [arXiv:0806.3389]

- Slightly more complicated rod structure:



- Horizon has more restrictive topology $L(n^2+1,1)$
- Conical singularities were eliminated, but there exists spherical naked singularities surrounding z_1 and z_4 (similar to our Range II)

A single-rotating black lens

$$
\begin{aligned}
\mathrm{d}s^2 \;=\; & -\frac{H(y,x)}{H(x,y)}(\mathrm{d}t - \omega_\psi\,\mathrm{d}\psi - \omega_\phi\,\mathrm{d}\phi)^2 - \frac{F(x,y)}{H(y,x)}\,\mathrm{d}\psi^2 + 2\frac{J(x,y)}{H(y,x)}\,\mathrm{d}\psi\mathrm{d}\phi \\
& + \frac{F(y,x)}{H(y,x)}\,\mathrm{d}\phi^2 + \frac{\kappa^2 H(x,y)}{2(1-a^2)(1-b)^3(x-y)^2}\left(\frac{\mathrm{d}x^2}{G(x)} - \frac{\mathrm{d}y^2}{G(y)}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\omega_\psi \;=\; & \frac{2\kappa}{H(y,x)}\sqrt{\frac{2b(1+b)(b-c)}{(1-a^2)(1-b)}}\,(1-c)(1+y)\{2[1-b-a^2(1+bx)]^2(1-c) \\
& \qquad - a^2(1-a^2)b(1-b)(1-x)(1+cx)(1+y)\} \\
\omega_\phi \;=\; & \frac{2\kappa}{H(y,x)}\sqrt{\frac{2b(1+b)(b-c)}{(1-a^2)(1-b)}}\,a(1-c)(1+x)^2(1+y)[a^4(1+b)(b-c) \\
& \qquad + a^2(1-b)(-b+cb+2c) - (1-b)^2c]
\end{aligned}
$$

$$
\begin{aligned}
G(x) &= (1-x^2)(1+cx) \\
H(x,y) &= 4(1-b)(1-c)(1+bx)\{(1-b)(1-c)-a^2[(1+bx)(1+cy)+(b-c)(1+y)]\} \\
&\quad +a^2(b-c)(1+x)(1+y)\{(1+b)(1+y)[(1-a^2)(1-b)c(1+x)+2a^2b(1-c)] \\
&\quad -2b(1-b)(1-c)(1-x)\} \\
F(x,y) &= \frac{2\kappa^2}{(1-a^2)(x-y)^2}\Big[4(1-c)^2(1+bx)[1-b-a^2(1+bx)]^2G(y) \\
&\quad -a^2G(x)(1+y)^2\Big([1-b-a^2(1+b)]^2(1-c)^2(1+by)-(1-a^2)(1-b^2)\times \\
&\quad \times(1+cy)\{(1-a^2)(b-c)(1+y)+[1-3b-a^2(1+b)](1-c)\}\Big)\Big] \\
J(x,y) &= \frac{4\kappa^2 a(1-c)(1+x)(1+y)}{(1-a^2)(x-y)}[1-b-a^2(1+b)][(1-b)c+a^2(b-c)]\times \\
&\quad \times[(1+bx)(1+cy)+(1+cx)(1+by)+(b-c)(1-xy)]
\end{aligned}
$$

Derivation using the Inverse Scattering Method (ISM)

- First derive the Emparan–Reall black ring using the ISM with a one-soliton transformation, negative-density rod to the right of the hoirzon [Iguchi & Mishima, hep-th/0604050; Tomizawa & Nozawa, hep-th/0604067]
- To rotate the finite space-like rod to the correct orientation:
  - Remove solitons from z_1 and z_3, with BZ vector $(0, 0, 1)$
  - Add back the two solitons with more general BZ vectors $(C_2, 0, 1)$ and $(0, C_1, 1)$, respectively
- Change coordinates to bring to asymptotically flat form, and choose C_2 to ensure the finite space-like rod has no time-like component

Properties of the single-rotating black lens

- Asymptotically flat if ψ and ϕ are identified with period 2π
- Asymptotically rotating only in the ψ direction, although the horizon has non-zero angular velocity in both the ψ and ϕ directions
- Horizon has topology $L(n,1)$, provided we identify:

$$\frac{2a[(1-b)c+a^2(b-c)]}{[1-b-a^2(1+b)](1-c)}=n \tag{2}$$

- Special cases:
  - Static black lens: $b=c$
  - Emparan–Reall black ring: $a=0$
  - Single-rotating Myers–Perry black hole: $a=\pm\sqrt{(1-b)/(1+b)}$

- Solve the condition (2) to obtain two possible ranges for a:
  - Range I: $0 < a < \sqrt{(1-b)/(1+b)}$
  - Range II: $-1 < a < -\sqrt{(1-b)/(1+b)}$
- Range I:
  - There always exists a conical (strut) singularity along the finite axis $x = 1$
  - No CTCs outside the horizon (from numerical evidence)
- Range II:
  - The conical singularity can be eliminated for certain parameter values
  - However, there will always exist a naked singularity with S^3 topology surrounding z_3
  - CTCs exist inside and also just outside this naked singularity

A double-rotating black lens

- Can be constructed using the ISM but too complicated to write down here!
- Special cases:
  - Single-rotating black lens
  - Pomeransky–Sen'kov black ring
- Unfortunately, it does not seem possible to eliminate the conical and naked singularities simultaneously, while at the same time maintaining a positive ADM mass for the black lens (from numerical evidence)

Summary

- If a new type of black hole in 5D exists, it would have to have a lens-space horizon topology
- We have constructed (using the ISM) a black lens solution with one rotation; however, it necessarily has a conical singularity to prevent it from collapsing under its own gravitation
- Making it doubly rotating does not seem to cure this problem: perhaps some other way (e.g., adding a gauge field) might
- Or perhaps a completely regular black lens does not exist at all? (This may be related to the fact that the background space-time cannot be regular)