Carl von Ossietzky University Oldenburg – Faculty V - Institute of Physics
Module Introductory Laboratory Course Physics – Part I

Fourier Analysis

Keywords:
FOURIER series (trigonometric series), FOURIER coefficients, FOURIER analysis (harmonic expansion, harmonic analysis), amplitude spectrum, phase spectrum, linear system, transfer function, fundamental and harmonics, EULER's formulas, sampling theorem.

Measuring program:
Aliasing at violation of the sampling theorem, spectra of photo detector signals, spectra of sound signals, spectrum of a beat signal and an amplitude modulated signal, spectra of a rectangular, a sawtooth, and a triangle signal, GIBBs phenomenon.

References:

/1/ HÄNSEL, H., NEUMANN, W.: „Physik – Mechanik und Wärmelehre", Spektrum Akademischer Verlag, Heidelberg among others
/2/ BRACEWELL, R. N.: „The Fourier Transform and its Applications", McGraw – Hill, London among others (advanced)
/3/ EICHLER, H.J., KRONFELDT, H.-D., SAHM, J.: „Das Neue Physikalische Grundpraktikum", Springer-Verlag, Berlin among others

1 Introduction

FOURIER analysis (after JEAN BAPTISTE JOSEPH DE FOURIER, Fig. 1) is an important tool in the area of signal analysis and processing. With its help, it can be determined which harmonic signals[1] – with different amplitudes, frequencies, and phases – a periodic signal consists of. In the following, we will limit ourselves to the analysis of *time signals*. These are signals, for example, like a voltage $U(t)$ or a current $I(t)$ which change in time t. Formally, physical values that change with location can be considered too, like the intensity $I(x)$ of light along the spatial coordinate x.

Fig. 1: Jean Baptiste Joseph de Fourier (1768-1830)[2].

We want to cite as an example of the application of FOURIER analysis its importance in *system theory* for the description of the behaviour of *linear systems*. The theory of linear systems has a great practical importance in physics. With it, the behaviour of many *physical systems* can be described without having to know how these systems are internally constructed in detail. We treat these systems as „black boxes" with unknown contents, from which we know that a certain input signal $e(t)$ will result in a certain output signal $E(t)$:

Input				**Output**
$e(t)$	$\rightarrow$	linear system	$\rightarrow$	$E(t)$

Linear systems fulfil the condition of *linearity* (hence the name): A sum of input signals lead to a corresponding sum of output signals:

	Input		**Output**
(1)	$f(t) = \sum_j e_j(t)$	linear system	$F(t) = \sum_j E_j(t)$

[1] *Harmonic signals* means *sinusoidal signals* in this text.
2 Source: GELLERT, W. et al. [Eds.]: „Kleine Enzyklopädie Mathematik", VEB Bibliographisches Institut, Leipzig, 1969

Examples of such linear systems are:

- in acoustics: the system microphone → amplifier → loud speaker
- in optics: the system objective → image detector,
- in electrical engineering: the system sender → transmission line → receiver

From the *FOURIER theorem*, which we will be detailed in Chap. 2, a *periodic* signal, including a *periodic input signal* $f(t)$ of a linear system, can be represented by an infinite sum of harmonic signals $h_n(t)$, $n \in \mathrm{N} \setminus \{0\}$), with differing angular frequencies ω_n, whose amplitudes c_n and phases ϕ_n may also differ (but not necessarily):

(2) $$f(t) = c_0 + \sum_{n=1}^{\infty} h_n(t) = c_0 + \sum_{n=1}^{\infty} c_n \sin(\omega_n t + \phi_n)$$ (c_0: constant[3])

Harmonic signals are transmitted undistorted from linear systems, i.e. the transmission definitely changes the amplitude and phase of the signal, but not the form. We are now making the assumption that we know how a system reacts to harmonic input signals with different frequencies, i.e. that for every harmonic input signal $h_n(t)$ we know the amplitude and phase of the corresponding harmonic output signal $H_n(t)$.

If the system changes the amplitude of all harmonic input signals in the same manner independently of their frequency (e.g. amplification by a factor of 2), and if all harmonic signals undergo a phase shift of $m\pi$ ($m \in \mathrm{N}$) we are dealing with an *ideal system*. From the linearity of the system (Eq. (1)) it follows immediately that a periodic input signal $f(t)$, which can be displayed as an infinite sum using the FOURIER theorem[4], will be transmitted undistorted through the system. The output signal $F(t)$ is only amplified by a constant factor (e.g. 2) compared to the input signal $f(t)$, however, it maintains its form.

As a rule, *real systems* behave differently. With these systems, depending on the angular frequency ω_n of the harmonic input signal, different *amplifications* $V(\omega_n)$ and different *phase shifts* $\Delta\phi(\omega_n)$ occur, leading to a distortion in the output signal $F(t)$ compared to the input signal $f(t)$.

$V(\omega_n)$ is called the *amplitude transfer function*, or *amplitude spectrum* and $\Delta\phi(\omega_n)$ the *phase transfer function*, or *phase spectrum* of the system. Together both functions describe the *frequency behaviour* of a real system.

According to our assumption above, the frequency behaviour is known for the investigated system. In practice this is often the case, e.g. because the manufacturer of the system supplied the corresponding data. Fig. 2 shows an example of an amplitude transfer function of PC sound card. From this we can gather that the card only has good transmission properties in the frequency regime between ν = 200 Hz and ν = 10 kHz because harmonic signals are amplified independent of the frequency by a constant factor here ($V(\nu)$ = const.). In comparison, outside this frequency regime the input and output signals undergo a frequency-dependent damping which inevitably leads to a signal distortion in case the input signal $f(t)$ comprises harmonic components with corresponding frequencies.
If we know the frequency behaviour of a linear system, we can *calculate* how the output signal $F(t)$ will look for *aperiodic input signal* $f(t)$. We only have to know, according to the FOURIER theorem, which harmonic signals $h_n(t)$ the signal $f(t)$ consists of. Then, knowing $V(\omega_n)$ and $\Delta\phi(\omega_n)$, we can state the corresponding output signals $H_n(t)$ for each of these signals $h_n(t)$ and then add the $H_n(t)$ to the output signal $F(t)$.
The necessary calculation of the parameters (amplitude, phase, frequency) of the *harmonic signals* which a *periodic signal* consists of is called *FOURIER analysis* or *harmonic analysis* or *harmonic expansion* and is the subject of this experiment.

[3] c_0 represents the time-independent DC component of $f(t)$, which does not contribute to information content of the signal.
[4] For formulations of this kind in this text, the DC component of the signal (c_0 in Eq. (2)) is always included.

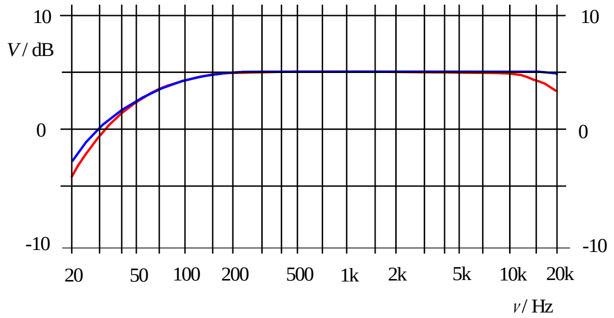

Fig. 2: Amplitude transfer functions of a PC sound card (YAKUMO sound card 16 MCD). Blue curve: playback, red curve: recording.[5]

2 Theory

In the following section, we are going to omit mathematical proofs which can be looked up in the given literature, and concentrate on the interpretation of the relationships necessary for the experiment.

2.1 Fourier Series and Fourier Coefficients

As already mentioned in the introduction, following the *FOURIER theorem*, a periodic signal *f*(*t*) with period *T* can be represented by a DC component and an infinite sum of harmonic signals whose angular frequencies are *integral multiples* of $\omega_o = 2\pi/T$. Harmonic signals with angular frequencies

(3) $$n\,\omega_0 := \omega_n\ ;\ n \in \mathrm{N} \setminus \{0\}$$

are called *harmonics* of the *fundamental* with the fundamental angular frequency ω_o, and the sum is denoted as a *trigonometric series* or a FOURIER *series*. It is given by:

(4) $$f(t) = c_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right]$$

The values c_o, $a_n = a(n\omega_o)$ and $b_n = b(n\omega_o)$ are called FOURIER *constants* or FOURIER *coefficients*. Determining these values is the subject of the FOURIER *analysis*. After a short calculation, it is found (cf. e.g. /1/) that they can be obtained from the signal *f(t)* as follows:

(5) $$c_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, dt$$

(6) $$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos(n\omega_0 t)\, dt \qquad n = 1, 2, 3, \ldots$$

(7) $$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin(n\omega_0 t)\, dt \qquad n = 1, 2, 3, \ldots$$

The constant c_0 is the average (DC component) of the signal *f*(*t*). If, e.g., *f*(*t*) is a temporally oscillating voltage *U*(*t*), c_0 corresponds to the DC voltage of the signal.
The representation of the FOURIER series in Eq. (4) can be simplified if the following relationship is used:

(8) $$a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) = c_n \sin(n\omega_0 t + \phi_n)$$

with

[5] The gain of an amplifier is often specified using the logarithmic decibel (dB) scale. This is detailed further in the experiment *"Operational amplifier"* in the SuSe. An amplification by *x* dB corresponds to a linear amplification by the factor $10^{x/20}$.

(9) $c_n = \sqrt{a_n^2 + b_n^2}$

and

(10) $\phi_n = \arctan\left(\frac{a_n}{b_n}\right)$

With this, Eq. (4) becomes the *spectral form* of the FOURIER series:

(11) $f(t) = c_0 + \sum_{n=1}^{\infty} c_n \sin(n\,\omega_0\,t + \phi_n)$

A periodic signal $f(t)$ can be described after FOURIER analysis with the following values[6]

(12)
c_0 : DC component (average of the signal $f(t)$, cf. Eq. (5))
$c_n = c_n(n\omega_0)$: amplitude spectrum
$\phi_n = \phi_n(n\omega_0)$: phase spectrum

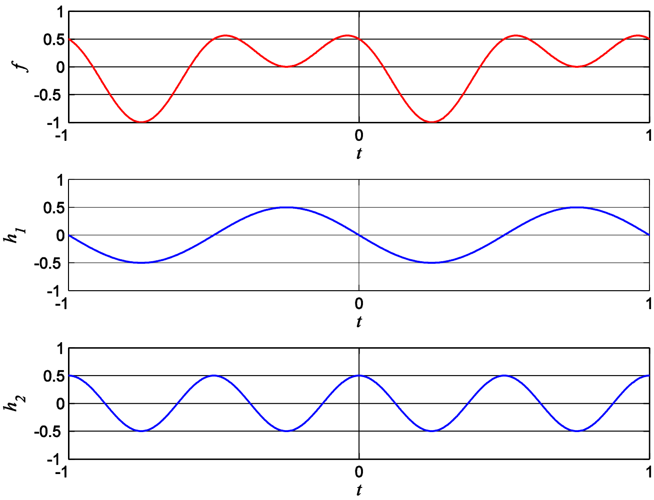

Fig. 3: Top (red): Anharmonic but periodic signal $f(t)$ with period $T = 1$ (in arbitrary units) with its harmonic components $h_1(t)$ (middle, blue) und $h_2(t)$ (bottom, blue).

Two examples will make the relations clear.
The first example shows a quite simple case. In Fig. 3 an anharmonic, but periodic, signal $f(t) = h_1(t) + h_2(t)$ with period $T = 1$ (in arbitrary units) is shown at the bottom. It is composed of the two harmonic signals shown beneath it in the figure: the fundamental $h_1(t)$ with the amplitude $c_1 = 0.5$ (in arbitrary units), the angular frequency $\omega_1 = 1\times\omega_0 = 2\pi/T$, and the phase $\phi_1 = \pi$ and the first harmonic $h_2(t)$ with the same amplitude $c_2 = 0.5$, the angular frequency $\omega_2 = 2\times\omega_0$, and the phase $\phi_2 = \pi/2$. A FOURIER analysis of the signal $f(t)$ would then produce:

DC component:	c_0	$= 0$	
amplitude spectrum:	$c_1 = c_1(\omega_0)$	$= 0.5$	
	$c_2 = c_2(2\omega_0)$	$= 0.5$	
	$c_m = c_m(m\omega_0)$	$= 0$	$\forall\ m \geq 3$
phase spectrum:	$\phi_1 = \phi_1(\omega_0)$	$= \pi$	
	$\phi_2 = \phi_2(2\omega_0)$	$= \pi/2$	
	$\phi_m = \phi_m(m\omega_0)$	$= 0$	$\forall\ m \geq 3$

The amplitude and phase spectra, that is $c_n(\omega_n)$ and $\phi_n(\omega_n)$, are represented in Fig. 4.

[6] The graphic representation of c_n over ω_n is called amplitude spectrum. The graphic representation of a_n over ω_n is called *cosine spectrum*; the representation of b_n over ω_n is called *sine spectrum*.

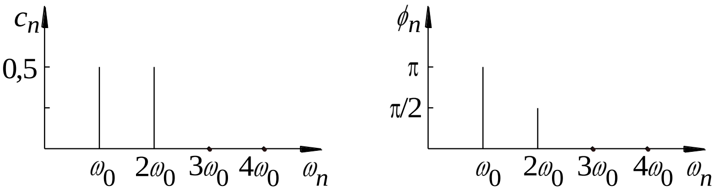

Fig. 4: Amplitude spectrum (left) and phase spectrum (right) of the signal shown at the top in Fig. 3. Generally, vertical *spectral lines* from the abscissa to the respective ordinate value are drawn instead of data points in such diagrams.

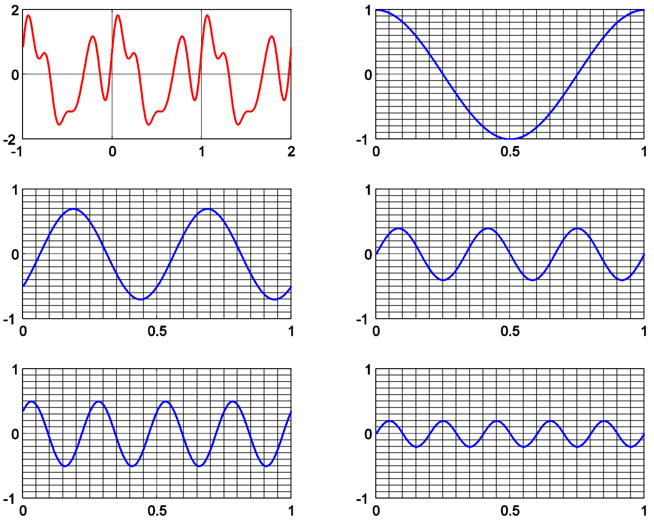

Fig. 5: Anharmonic, periodic signal $f(t)$ (upper left, red) with its five harmonic components (upper right as well as middle and below, blue). Abscissa: t, ordinate: $f(t)$ or $h_n(t)$, respectively, period $T = 1$ (t and $f(t)$ in arbitrary units).

Clearly the situation shown in Fig. 5 is more complex: in the upper left diagram an anharmonic, but periodic, signal $f(t)$ with period $T = 1$ (in arbitrary units) is shown. To its right the fundamental is shown with angular frequency $\omega_1 = 1\times\omega_0 = 2\pi/T$ and underneath four harmonics with the angular frequencies $\omega_n = n\omega_o$, $n = 2, 3, 4, 5$ which all have different amplitudes and phases. A FOURIER analysis would lead to the DC component of $c_0 = 0$ as well as five values of c_n for the amplitude spectrum and five values of ϕ_n for the phase spectrum.

Question 1:

- Try graphically to get the data necessary from Fig. 5 to sketch the amplitude and phase spectra analogously to Fig. 4.

2.2 Sampling and Sampling Theorem

We now know how the FOURIER coefficients a_0, a_n and b_n can be calculated, and from there, the values c_0, c_n and ϕ_n, i.e., the amplitude and phase spectra of periodic signals $f(t)$. In practise, a problem appears here: The signals under investigation are, in general, not analytically known signals , but rather *measured signals* having a complicated temporal course that were recorded by, e.g. a data acquisition board connected to a computer. Such recording systems yield discrete function values $y_i = f(t_i)$[7] at equidistant time points t_i (separation of Δt). It is also said that the signal $f(t)$ is *sampled* at the points t_i with *sampling angular frequency* $\omega_a = 2\pi/\Delta t$. The FOURIER analysis of a sampled signal is of course only an approximation – since the signal itself is only known approximately (i.e. only at the points t_i). How a FOURIER analysis is carried out in such a case is shown in the following.

Let us assume that from the signal $f(t)$ we have $2m$ measured points (sampling points) $y_i = (i = 1,\ldots,2m\text{-}1)$ at equidistant time points t_i. For the FOURIER coefficients we then get:

[7] Compare experiment *"Data Acquisition with the PC ..."*.

(13) $$c_0 = \frac{1}{2m} \sum_{i=0}^{2m-1} y_i$$

(14) $$a_n = \frac{1}{m} \sum_{i=0}^{2m-1} y_i \cos\left(\frac{2\pi\, n\, i}{2\, m}\right) \qquad n = 1, 2,..., m$$

(15) $$b_n = \frac{1}{m} \sum_{i=0}^{2m-1} y_i \sin\left(\frac{2\pi\, n\, i}{2\, m}\right) \qquad n = 1, 2, ..., m\text{-}1$$

From 2m *independent* function values we get m coefficients a_n, (m - 1) coefficients b_n, and a constant c_0. Together that is m + (m - 1) + 1 = 2m *independent* FOURIER coefficients. This is understandable from the point of view of the information content: the information content cannot be lost or increased through simple calculation.

The *sampling theorem* (SHANNON theorem[8]), with the help of Eqs. (13) – (15), answers the question of *the least number of* function values that are needed to reliably determine the angular frequency ω_s of a harmonic oscillation present in a signal $f(t)$. It says that an angular frequency ω_s can be reliably detected if the following holds for the sampling angular frequency ω_a:

(16) $$\omega_a > 2\, \omega_s \qquad \textit{sampling theorem}$$

In other words: The angular frequency ω_s of a harmonic signal can only be reliably determined, if more than two sampling values per period are available for the signal. If the condition given by the inequality (16) is violated, the signal with angular frequency ω_s is „under sampled" which leads to false results (*aliasing effects*). In this case, the FOURIER analysis produces the wrong angular frequency ω_f.

(17) $$\omega_f = \left|\omega_a - \omega_s\right|$$

The signal with the angular frequency ω_s hence shows up in the amplitude spectrum under the "wrong name" ω_f, and therefore the term „alias". For $\omega_s \le \omega_a \le 2\omega_s$, it appears in the spectrum reflected at the axis $\omega = \omega_a/2$.

If the sampling angular frequency ω_a is given, a harmonic signal can, according to Eq. (16), only be sampled correctly, if

(18) $$\omega_s < \frac{\omega_a}{2}$$

is true for its angular frequency ω_s. The angular frequency $\omega_a/2$ is also called NYQUIST frequency[9].

If the sampling theorem is met, the length $2\, m\, \Delta t$ of the time interval over which the measured signal was sampled determines the *frequency resolution* Δf, i.e. the accuracy with which signal frequencies can be measured:

(19) $$\Delta f \sim \frac{1}{2\, m\, \Delta t}$$

This aspect of the FOURIER analysis, however, cannot be discussed further in the introductory laboratory course.

2.3 Practical Hints

The calculations of the FOURIER coefficients, rather of the amplitude and phase spectra, are quite extensive. Today, however, they can be done very quickly with a personal computer, and in the case of large data sets the use of special processors can accelerate the calculations.

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8 CLAUDE ELWOOD SHANNON (1916 - 2001).

9 HARRY NYQUIST (1889 - 1976).

It was not so long ago that the intensive work had to be done by hand. In a mathematic handbook from the year 1969 the following tip is given (GELLERT, W. et al. [Eds.]: „Kleine Enzyklopädie Mathematik", VEB Bibliographisches Institut, Leipzig, 1969):

> *„A person practised in calculating, who is using an electronic calculation machine and applying a special method of calculation for harmonic analysis, needs about ½ an hour for 12 points, about 2 hours for 24 points, about 6 hours for 36 points, and about 16 hours for 72 points … A medium speed electronic calculation machine manages the calculation of 36 points in about 2 minutes. The time needed to print the results is usually longer than the calculation time."*

In the following investigation the FOURIER analysis will be performed with a few hundred to a few thousand points. Either don't make any plans for your semester break – or use the PC at your disposal and you will be finished without problems in one afternoon!
In practise, one is only interested in finding out which *amplitude* the harmonic signals have that are contained in a periodic measured signal. The phase of the single components is often unimportant. In other words: the amplitude spectrum is, in most cases, of considerably more practical importance than the phase spectrum. With the present investigation we will thus limit ourselves to the interpretation of the amplitude spectra.
For a non–periodic signal $f(t)$ defined in a time-interval of length τ (e.g. an output pulse from a fotodiode), the signal can be thought of as continuing periodically to the right and to the left of the given interval (with the „period" τ), and it can likewise be represented by a FOURIER series. It is the case that such a FOURIER series produces function values according to Eq. (4) which are situated outside of the definition interval τ, however, these values may simply be ignored for further considerations.

3 Experimental Procedure

Equipment:

Digital oscilloscope TEKTRONIX TDS 1012 / 1012B / 2012C / TBS 1102B - EDU, PC with DAQ device (NATIONAL INSTRUMENTS myDAQ) and accompanying BNC adapter, 2 function generators (TOELLNER 7401 and AGILENT 33120A / 33220A), addition amplifier, photo diode with integrated amplifier and pinhole diaphragm (diameter 1mm), AC filter for photo diode, incandescent lamp and fluorescent lamp in light-proof box, microphone with preamplifier, tuning fork, power supply (PHYWE (0 - 15 / 0 - 30) V).

3.1 General Hints

3.1.1 Operating the Data Acquisition Board

The switch of the BNC socket AGND, which is connected to the NI myDAQ device should also switched off here. The signal sources (function generator, microphone amplifier etc.) are connected via one of the BNC input channels AI 0 or AI 1; the switch above the BNC socket of ACH 0 or AI 0 has to be set to BNC. The corresponding input must then be selected in the software.

3.1.2 Input Voltage Range of the Data Acquisition Board

The maximum input voltage range that the data acquisition board can withstand is ± 10 V; this should not be exceeded. As a control, all of the input signals of the DAQ device are therefore simultaneously displayed on the oscilloscope.

3.1.3 Software

The following experiments are performed using the MATLAB-Scripts `Dateneingabe_FourierAnalyse.m` and `Dateneingabe_Rekonstruktion.m` respectively. The scripts announce themselves by self–explanatory screen messages. Text messages of the scripts such as tables with amplitudes and frequencies of FOURIER components appear in the Command Window. There, they can be marked and transferred to other applications by „Copy and Paste" (e.g. `Word,Notepad-Editor` among others).

3.1.4 Printing and Saving the Graphics

The graphics (`MATLAB` figures) can be sent to the printer in the lab using the key combination → File → Print. Via → File → Save as they can be stored in various well known graphics formats.

Details of graphics can be magnified by using the `Zoom` function in the Figure window.

3.2 Sampling Theorem

With the help of the AGILENT-function generator a sinusoidal time signal $U(t)$ without a DC component and with a frequency of 140 Hz and an amplitude of 4 V is generated (control of settings on the oscilloscope) and fed into one of the AI 0 or AI 1 inputs of the DAQ device. With the program `Dateneingabe_FourierAnalyse` 1,000 sampling values for each sampling frequency of (1000, 500, 300, 200, 150, 120) Hz should be read and FOURIER analysed. The results (time signals and amplitude spectra) are printed or stored, respectively.

Question 2:

- How can the results be interpreted when Eq. (16) to Eq. (18) are taken into consideration?

3.3 Spectra of the Signal of a Photo Detector

During the investigation of the oscilloscope we saw that the temporal course of the light intensity of a light bulb connected to the 230 V power supply system looked clearly different than the temporal course of the light intensity of a fluorescent lamp. We now want to quantitatively investigate this qualitative finding. To do this, a photo diode is illuminated by an incandescent lamp connected to the power supply system and then by a fluorescent lamp also connected to the 230 V power supply system. A suitable pinhole in front of the photo diode prevents saturation of the output signal of the photo diode amplifier (time signal $U(t)$). With the help of the AC filter the DC component of the measured signal is filtered out (control on the oscilloscope), and then the signal is fed into the AI 0 or AI 1 input channel of the DAQ device. With the help of the program `Dateneingabe_FourierAnalyse`, 5,000 sampling values should be read for both signals at a sampling frequency of 5 kHz and FOURIER analysed. The results (time signals and amplitude spectra) are printed and stored, respectively.

Question 3:

- What is the difference between the time signals and the difference between their amplitude spectra? (Statements about the *absolute* amplitudes of the spectral components are not of importance.)

3.4 Spectra of Sound Waves Recorded with a Microphone

Next the fundamental frequency of a tuning fork will be determined. For this, the tuning fork is struck and the sound waves produced are recorded with the help of a microphone by placing the foot end of the tuning fork on top of the microphone. The output from the microphone is amplified with the accompanying amplifier and its output signal $U(t)$) is fed into one of the AI 0 or AI 1 inputs of the DAQ device. With the program `Dateneingabe_FourierAnalyse` 10,000 sampling values are read at a sampling frequency of 5 kHz and FOURIER analysed. The result (time signal and amplitude spectrum) is printed or stored, respectively.

Question 4:

- Does the amplitude spectrum correspond with musical expectations?

In the second step, the note from the tuning fork (the a') should be sung and then hummed. For both cases the acoustic signals should be recorded using the microphone and an analysis like that for the tuning fork should follow.

Question 5:

- What does the result look like in comparison to the analysis of the tuning fork oscillation?

3.5 Spectrum of a Beat Signal

With the help of an addition amplifier the sinusoidal signals from two function generators (AGILENT and TOELLNER) are added. One generator is operated at 104 Hz, amplitude 1 V, no DC component, the other at 108 Hz, amplitude 0.75 V, no DC component (verify the settings with the digital oscilloscope). The output signal of the addition amplifier (time signal $U(t)$) is fed into the AI 0 or AI 1 input channel of the DAQ device and observed simultaneously with the oscilloscope. With the program `Dateneingabe_FourierAnalyse` 10,000 sampling values are read at a sampling frequency of 2 kHz and FOURIER analysed. The result (time signal and amplitude spectrum) is printed or stored, respectively, and the course of the amplitude spectrum is interpreted.

3.6 Spectrum of an Amplitude-Modulated Signal

Let us examine a harmonic voltage signal $U(t)$ of the form

(20) $$U(t) = U_T \sin(\omega_T t)$$

with the amplitude U_T and the angular frequency ω_T. If a time-dependent signal $U_M(t)$ is added to the constant amplitude U_T, then an *amplitude-modulated* signal is obtained[10]:

(21) $$U(t) = \left[U_T + U_M(t)\right] \sin(\omega_T t)$$

The signal from Eq. (20) is called *carrier signal* and ω_T is called *carrier angular frequency*. In the simplest case, $U_M(t)$ is a harmonic signal with the angular frequency ω_M and the amplitude U_{M0}. Hence, it follows:

(22) $$U(t) = \left[U_T + U_{M0} \sin(\omega_M t)\right] \sin(\omega_T t)$$

This equation can be converted to:

(23) $$U(t) = U_T \sin(\omega_T t) + \frac{U_{M0}}{2}\left[\cos\left((\omega_T - \omega_M)t\right) - \cos\left((\omega_T + \omega_M)t\right)\right]$$

Question 6:

- Draw the amplitude spectrum of the signal $U(t)$ according to Eq. (23) for the cases $U_T = 2U_{M0} = 1$ V, $\omega_T/2\pi = 750$ kHz and $\omega_M/2\pi = 15$ kHz.

With the AGILENT function generator an amplitude-modulated signal according to Eq. (22) with the following parameters is generated: $U_T = 2$ V, $\omega_T/2\pi = 1$ kHz, $U_{M0} = 1$ V, $\omega_M/2\pi = 200$ Hz (see footnote[11]!). The signal (time signal $U(t)$) is fed into one of the input channels AI 0 or AI 1 of the DAQ device and simultaneously observed with the oscilloscope. Using the program `Dateneingabe_FourierAnalyse` 10,000 sampling values are read at a sampling frequency of 10 kHz and FOURIER analysed. The result (time signal and amplitude spectrum) is printed or stored. The course of the amplitude spectrum is compared to the expectations according to Eq. (23).

3.7 Spectrum of a Square Signal, Gibbs Phenomenon

The square signal of a function generator (time signal $U(t)$; amplitude 4 V, frequency 50 Hz, no DC component) is fed into the AI 0 or AI 1 input channel of the DAQ device. Using the program `Dateneingabe_Rekonstruktion` 10,000 sampling values are read at a sampling frequency of 10 kHz and FOURIER analysed. The result (time signal and amplitude spectrum) is printed or stored and the course of the amplitude spectrum is compared to the theoretical expectations. For this comparison, the expected and measured amplitudes for the ten spectral components with the largest amplitudes are also plotted in tabular form.

Hint:

Descriptions of the FOURIER analysis for a square signal are found in almost every physics text book or, for example, in the *"Taschenbuch der Mathematik"* or in the online reference material from WOLFRAM RESEARCH[12] The appendix contains the corresponding formulas for calculating the theoretically expected voltage amplitudes for the individual signals The measured data required for the comparison are given in the `Matlab` command window and can be copied into a personal file from there.

Subsequently, the time signal is reconstructed by stepwise addition of its FOURIER components (FOURIER *synthesis*). Thus, it is clearly shown how the original square signal can be reconstructed piece by piece

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[10] The principle of amplitude modulation (*AM*) is used, e.g., for signal transmission in long-, medium- and short-wave broadcast. The current standard in the ultrashort-wave range is the frequency modulation (*FM*).

[11] These parameters were stored in the internal storage "1" in the function generator. They can be retrieved by clicking the button RECALL; first, there appears RECALL 0 with a blinking 0 in the display. By clicking the button ^ the 0 is raised to 1, then click on ENTER. Now, the function generator produces the required signal at the OUTPUT socket.

[12] cf. http://www.uni-oldenburg.de/en/physics/teaching/laboratory-courses/literature/ .

from its FOURIER components, if more and more harmonics are added to the fundamental during reconstruction. The result of the reconstruction is printed or stored, respectively.
Looking at the reconstructed square signal it becomes clear that over- and undershooting occur. This effect is called *GIBBS phenomenon*[13]. It occurs whenever the signal shows a discontinuity like the square signal at the transition point from the lower to the upper or from the upper to the lower signal level (cf. Fig. 6). The overshoots themselves are called *GIBBS humps*. The larger the number N of harmonics, which are used for the synthesis of the square signal, the closer the extrema of the under- and overshoots move together – their amplitudes, however, staying the same for large N. An exact but extensive calculation yields that the largest overshoot has a height of about 9% of the amplitude of the square signal, while the height of the largest undershoot amounts to about 4.8% of the amplitude.

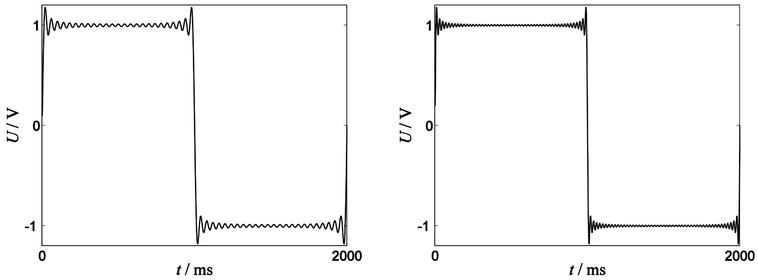

Fig. 6: GIBBS phenomenon for the FOURIER synthesis of a square signal with an amplitude of 1 V and a period of 2 s. Left N = 50, right N = 100.

3.8 Spectra of a Saw Tooth Signal and a Triangle Signal

The investigation described in Chap. 3.7 is repeated for a saw tooth signal and then for a triangle signal (amplitude of the signals always 4 V, frequency 50 Hz, no DC component; sampling frequency 10 kHz, 10,000 sampling values). The time signal and the amplitude spectra are printed or stored and the courses of the amplitude spectra are compared to the theoretical predictions. Presentations of the Fourier analysis of both signals can also be found in the following appendix.
Finally, both signals are reconstructed from their spectra. The results of the reconstruction are printed or stored, respectively.

Question7:
- For which signal is the GIBBS phenomenon noticeable, and how come?

4 Appendix

Fourier analysis of the signal of a square wave voltage:

(24) $$f(t) = \frac{4U_R}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\big((2n-1)\omega t\big)$$

mit der Spannungsamplitude U_R.

(25) $$U_n = \frac{4\,U_R}{(2n-1)\pi}$$

Fourier analysis of the signal of a saw tooth voltage:

(26) $$f(t) = \frac{2U_R}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\omega t)$$

(27) $$U_n = \frac{2U_R}{n\pi}$$

Fourier analysis of the signal of a triangular voltage:

(28) $$f(t) = \frac{8U_R}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin((2n-1)\omega t)$$

(29) $$U_n = \frac{8U_R}{(2n-1)^2\pi^2}$$

[13] JOSIAH WILLARD GIBBS (1839 - 1903)