Carl von Ossietzky University Oldenburg – Faculty V - Institute of Physics
Module Introductory Laboratory Course Physics – Part I

Measurement of Capacities, Charging and Discharging of Capacitors

Keywords:
Capacitor, parallel-plate capacitor, dielectric, *RC*-element, charge and discharge curves of capacitors, phase shift, KIRCHHOFF's laws, input and output impedances and capacitances

Measuring program:
Determination of the input resistance of an oscilloscope from the discharge curve of a capacitor, measurement of the capacitance of coaxial cables, measurement of the relative permittivity of PVC, determination of the phase shift between current and voltage in a RC-element.

References:
/1/ DEMTRÖDER, W.: „Experimentalphysik 2 – Elektrizität und Optik“, Springer-Verlag, Berlin among others
/2/ STÖCKER, H.: „Taschenbuch der Physik“, Harri Deutsch, Frankfurt
/3/ KORIES, R., SCHMIDT-WALTER, H.: „Taschenbuch der Elektrotechnik“, Harri Deutsch, Frankfurt

1 Introduction

In this experiment measuring methods are presented which can be used to determine the capacitance of a capacitor. Additionally, the behaviour of capacitors in alternating-current circuits is investigated. These subjects will be treated in more detail in the experimental physics lecture of the second semester. Simple basics, as covered here, need to be known in advance, in order to understand the behaviour of capacitors in the electrical circuits used in this laboratory course.

2 Theory

2.1 Capacitance of a Capacitor

Every set-up of two electric conductors separated by a certain distance represents a capacitor. Hence, two wires lying beside each other (e.g. laboratory cables) are just as much a capacitor as two parallel metal plates or a wire surrounded by a wire mesh at a certain distance (coaxial cable).

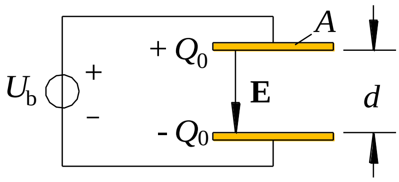

Fig. 1: Scheme of a parallel-plate capacitor. For the labels, please refer to the text.

Let us exemplarily study a capacitor of a particularly simple structure, the *parallel-plate capacitor*, consisting of two electrically conductive plates, each with an area A, set up in parallel at a distance *d* (Fig. 1). If such a capacitor is connected with a voltage source with the operating voltage U_b (terminal voltage in the unloaded state) there is a short-time *charge current*: the voltage source pulls electrons from the one plate and transfers them to the other plate, i.e., it causes a *shift* of a *charge* Q from one plate to the other one. This charge displacement causes an electric field **E** to be built between the plates, the value of which is given by $E = U/d$, U being the instantaneous voltage across the capacitor. This voltage reaches its maximum $U = U_b$ after a certain time period. This is when the capacitor is completely charged; one plate then has the charge $+Q_0$, the other one, the charge $-Q_0$.

U_b and Q_0 are proportional. The proportionality coefficient

$$C = \frac{Q_0}{U_b} \tag{1}$$

is termed the *capacitance* of the capacitor. Its unit is FARAD F[1]:

[1] Named after MICHAEL FARADAY (1791 - 1867)

(2) $$[C] = \mathrm{F} = \frac{\mathrm{A \cdot s}}{\mathrm{V}} = \frac{\mathrm{C}}{\mathrm{V}}$$ (1 C = 1 COULOMB[2])

For a parallel-plate capacitor *in a vacuum* the capacitance is exclusively determined by the geometry of its arrangement. It is directly proportional to the area A of the plate and inversely proportional to the distance d between the plates:

(3) $$C \sim \frac{A}{d}$$

Question 1:

- How can the proportionality $C \sim 1/d$ be illustrated? (Hint: Consider the electric field E and the voltage U in a charged parallel-plate capacitor that is separated from the voltage source following charging and whose plates are pulled apart afterwards. See to it that the charge remains constant.)

Applying the proportionality coefficient ε_0 we obtain:

(4) $$C = \varepsilon_0 \frac{A}{d}$$ (in a vacuum)

ε_0 is called the *electric field constant* (*permittivity of vacuum*). It is calculated from two internationally determined constants, namely the *speed of light c* (in vacuum) and the *magnetic field constant* (*permeability of vacuum*) μ_0, and can therefore be stated with an optional precision (cf. back page of the cover of this script). We confine ourselves to four digits here:

(5) $$\varepsilon_0 := \frac{1}{\mu_0 c^2} = 8{,}8541 \cdot 10^{-12} \, \frac{\mathrm{As}}{\mathrm{Vm}}$$

By putting an electric insulator (*dielectric*) between the plates of the capacitor the capacitance is increased by the factor $\varepsilon_r \geq 1$:

(6) $$C = \varepsilon_0 \varepsilon_r \frac{A}{d}$$ (in matter)

ε_r is termed *relative permittivity* (*relative dielectric constant*), the product $\varepsilon = \varepsilon_0 \varepsilon_r$ is called *permittivity* (*dielectric constant*). ε_r is a numerical value dependent on the insulating material used. It is, e.g. for air at 20° C and normal pressure (101,325 Pa): $\varepsilon_r \approx 1.0006$, for water at 20° C: $\varepsilon_r \approx 81$, for different kinds of glass: $\varepsilon_r \approx 5$ - 16, and for ceramics (depending on kind): $\varepsilon_r \approx 50$ – 1,000. In a vacuum $\varepsilon_r = 1$.[3]

Question 2:

- How can we explain the increase in capacitance due to the dielectric? (Hint: Attenuation of the electric field.)

Many different types of capacitors are commonly available in retail. They come in a variety of casings, and their capacitances span several orders of magnitude. Fig. 2 shows some examples.

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[2] CHARLES AUGUSTIN DE COULOMB (1736 - 1806)

[3] In an alternating current circuit, ε_r is dependent on the frequency of the fed voltage. The mentioned values are approximate values for the case of low frequencies within a range below 1 kHz.

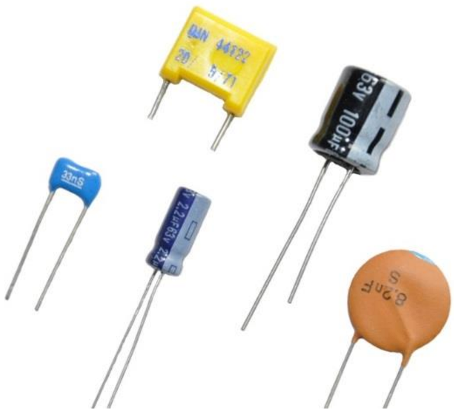

Fig. 2: Common retail versions of capacitors of different types and casings. The capacitances of the depicted types vary between several picofarad (pF) and several microfarad (μF).

2.2 Charging and Discharging of a Capacitor

2.2.1 Discharging

Let us first take a look at the discharging of a capacitor. We are particularly interested in knowing how long the discharging takes and how it develops with time. For this purpose we examine a charged capacitor with capacitance *C* according to Fig. 3 which is discharged via a resistance *R*. Such an arrangement is called resistance-capacitance element. At an optional time *t* after closing the switch S we obtain (cf. Eq. (1)):

(7) $$Q(t) = C \cdot U(t)$$

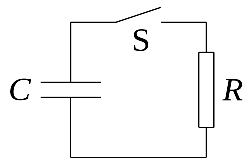

Fig. 3: Discharging of a capacitor via a resistor.

Q(*t*) is the momentary charge of the capacitor and *U*(*t*) the momentary voltage across the capacitor. According to KIRCHHOFF's law this voltage equals the voltage across the resistance *R*, so that we obtain with the momentary current *I*(*t*):

(8) $$U(t) = R \cdot I(t)$$

The current *I*(*t*) is caused by the *decreasing* (hence the minus sign) charge of the capacitor with time. Hence,

(9) $$I(t) = -\frac{dQ(t)}{dt}$$

Eqs. (7), (8), and (9) combine to yield the differential equation for the discharging of the capacitor:

(10) $$Q(t) = -RC \cdot \frac{dQ(t)}{dt}$$

The solution of this differential equation under the initial condition $Q(t = 0) = Q_0$ reads:

(11) $$Q(t) = Q_0 \cdot e^{-\frac{t}{RC}}$$

The product RC has the unit $[RC] = \Omega \cdot F = (V/A) \cdot (As/V) = s$. Thus RC represents a time period τ, the so-called *time constant* τ which has the following meaning: at a time $t = \tau = RC$ the charge has decreased to a value Q_0/e, which is about the 0.368-fold of the initial value:

$$(12) \qquad t = \tau = RC \quad \rightarrow \quad Q(t) = Q(\tau) = \frac{Q_0}{e} \approx 0.368 \cdot Q_0$$

For the time $t = T$ (*half-life time*), within which the charge has decreased to *half* of the initial value, we obtain:

$$(13) \qquad Q(t = T) = \frac{Q_0}{2} \quad \rightarrow \quad T = \ln 2 \cdot RC \approx 0.693 \cdot RC$$

If a discharge process shall be observed it is easier to look at the decreasing *voltage* across the capacitor instead of observing the decreasing *charge* of the capacitor according to Eq. (11). Applying Eqs. (1) and (7), Eq. (11) yields:

$$(14) \qquad U(t) = U_0 \cdot e^{-\frac{t}{RC}}$$

The voltage drop, which can be very easily measured using, for example, an oscilloscope, has the same temporal variation as the decrease in charge. Hence, Eq. (14) yields an important relation for measuring capacitances in practice. Measuring the voltage $U(t)$ at two different times t_1 and t_2, we obtain (cf. Fig. 4):

$$(15) \qquad \begin{aligned} U(t_1) &:= U_1 = U_0 \, e^{-\frac{t_1}{RC}} \\ U(t_2) &:= U_2 = U_0 \, e^{-\frac{t_2}{RC}} \end{aligned}$$

The natural logarithm of Eq. (15) yields[4]:

$$(16) \qquad \begin{aligned} \ln(U_1) &= \ln(U_0) - \frac{t_1}{RC} \\ \ln(U_2) &= \ln(U_0) - \frac{t_2}{RC} \end{aligned}$$

Hence, it follows:

$$(17) \qquad \ln(U_1) - \ln(U_2) = \ln\left(\frac{U_1}{U_2}\right) = \frac{t_2 - t_1}{RC}$$

and finally:

$$(18) \qquad C = \frac{t_2 - t_1}{R \, \ln\left(U_1 / U_2 \right)}$$

- The equation above is the basis for all capacitance measurements in this laboratory session.[5]

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[4] In order to be stringent, it would be necessary to replace $\ln(U_1)$ by $\ln(\{U_1\})$ (likewise for U_0, U_1, etc.) in equation (16) and the following, since the logarithm is only defined for a numerical argument (e.g. $\{U_1\}$), but not for quantities having an associated unit (e.g. U_1). To simplify the presentation we omit the curly brackets, silently implying the numerical value of the given physical quantity.

[5] Many multimeters employ this principle for measuring capacities.

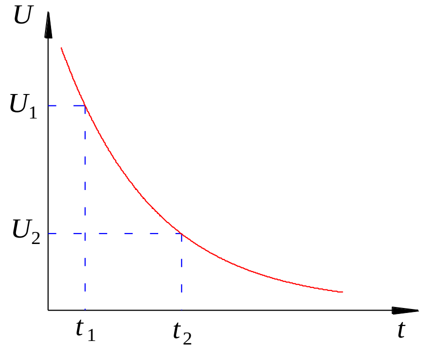

Fig. 4: Course of discharge of a capacity.

2.2.2 Charging

Let us now observe the *charging* of a capacitor with the capacitance C with the help of a real voltage source according to Fig. 5. The real voltage source can be considered an ideal voltage source G in series with the source voltage U_0 and a resistance R (the internal resistance of a real voltage source). According to KIRCHHOFF's law we obtain an optional time t after closing the switch S ($I(t)$ is the charging current):

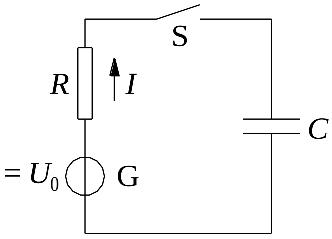

Fig. 5: Charging of a capacitor via a real voltage source.

(19) $$U_0 = U_R(t) + U_C(t) = R \cdot I(t) + \frac{Q(t)}{C} = R\frac{\mathrm{d}Q(t)}{\mathrm{d}t} + \frac{Q(t)}{C}$$

Hence it follows with $Q_0 = C\,U_0$:

(20) $$Q(t) + RC\frac{\mathrm{d}Q(t)}{\mathrm{d}t} - Q_0 = 0$$

The solution of this differential equation reads:

(21) $$Q(t) = Q_0\left(1 - \mathrm{e}^{-\frac{t}{RC}}\right)$$

The time constant τ=RC states the time period within which the capacitor is charged to the (1 - 1/e)-fold of its maximum charge Q_0.

Analogous to the *discharging* of the capacitor, for the easily observable voltage increase of the capacitor we can write:

(22) $$U(t) = U_0\left(1 - \mathrm{e}^{-\frac{t}{RC}}\right)$$

Question 3:

- Plot the development of Eqs. (14) and (22) for the time interval [0; 5τ] for the values R = 1 kΩ, C = 4.7 nF and U_0 = 1 V using `Matlab`.

2.3 Interconnection of Several Capacitors

The total capacitance of an arrangement consisting of several capacitors can be calculated by applying KIRCHHOFF's laws. For a *series connection* of n capacitors with the capacitances C_i we obtain (cf. Fig. 6 for $n = 2$):

(23) $$\frac{1}{C}=\sum_{i=1}^{n}\frac{1}{C_i}$$

For a *parallel connection* one obtains (cf. Fig. 7 for $n = 2$):

(24) $$C=\sum_{i=1}^{n}C_i$$

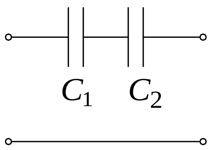

Fig. 6: Series connection of capacitors.

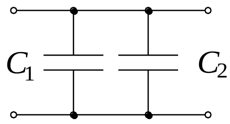

Fig. 7: Parallel connection of capacitors.

2.4 Cosinusoidal Excitation of a *RC* Element

So far we have studied the behaviour of a capacitor which is charged or discharged *once* via a resistance. In order to understand the behaviour of capacitors in alternating circuits we will now observe the reaction of a *RC* element, which means a set-up consisting of resistance and capacitor, upon cosinusoidal excitation. We look at a set-up according to Fig. 8. An ideal voltage source provides the alternating voltage $U_G(t)$ [6] with the angular frequency ω:

(25) $$U_G(t)=U_0\cos(\omega t)$$

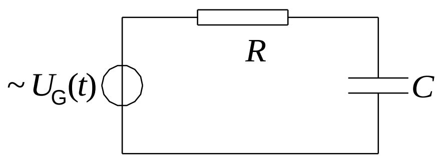

Fig. 8: RC element with cosinusoidal excitation.

Analogous to Eq. (19) it follows from KIRCHHOFF's voltage law:

(26) $$U_G\left(t\right)=U_0\cos(\omega\,t)=U_R(t)+U_C(t)=R\frac{\mathrm{d}Q(t)}{\mathrm{d}t}+\frac{Q(t)}{C}$$

Hence it follows:

(27) $$Q(t)+RC\frac{\mathrm{d}Q(t)}{\mathrm{d}t}-CU_0\cos(\omega t)=0$$

It is our aim to determine the temporal development of $U_C(t)$. For this purpose, it is sufficient, according to Eq. (7), to find the temporal development of $Q(t)$. From the considerations presented in Chapter 2.2 we know that the capacitor cannot be charged or discharged infinitely rapidly. This means that the course of charging $Q(t)$ cannot follow the voltage $U_G(t)$ instantaneously, but rather with a certain temporal delay. Therefore, we expect a *phase shift* φ of $Q(t)$ compared to $U_G(t)$. Thus, we try to solve the differential equation (27) by setting:

(28) $$Q(t)=Q_0\cos(\omega t+\varphi)$$

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[6] Of course, the ansatz $U_G(t) = U_0 \sin(\omega\, t)$ would also achieve its purpose; however, the form with the cos-function has become established in physics.

By inserting Eq. (28) into Eq. (27) we now have to determine the unknown quantities Q_0 and φ. Following some calculations (which are most easily done using complex quantities, see appendix in Chap. 4) we obtain for the *maximum charge* Q_0 of the capacitor:

(29) $$Q_0 = \frac{CU_0}{\sqrt{(\omega RC)^2 + 1}}$$

and for the *phase shift* φ between $Q(t)$ or $U_C(t)$ and $U_G(t)$:

(30) $$\varphi = \arctan(-\omega RC)$$ and

(31) $$\tan\varphi = -\omega RC ,$$

respectively.

From Eq. (30) we learn that φ *is always negative*. The charge $Q(t)$ always lags behind the voltage $U_G(t)$. For the limit $\omega \to 0$ we obtain $\varphi \approx 0°$ and for the limit $\omega \to \infty$ it follows: $\varphi = -90°$.

With the relationship:

(32) $$\cos\varphi = \frac{1}{\sqrt{\tan^2\varphi + 1}} = \frac{1}{\sqrt{(\omega RC)^2 + 1}}$$

we obtain by inserting Eq. (32) into Eq. (29):

(33) $$Q_0 = CU_0 \cos\varphi$$

Comparing Eqs. (1) and (33) we learn that the maximum charge of the capacitor is lower by a factor of cos φ under a cosinusoidal excitation than under a direct voltage of magnitude U_0. For the limit $\omega \to 0$ we obtain $Q_0 \approx CU_0$ and for the limit $\omega \to \infty$ it follows that $Q_0 = 0$.

Question 4:
- How can these extreme cases be illustrated?

We will now calculate the temporal course of the current $I(t)$ through the loop according to Fig. 8. We have:

(34) $$I(t) = \frac{\mathrm{d}Q(t)}{\mathrm{d}t}$$

Inserting Eq. (28) into (34) and performing the differentiation yields:

(35) $$I(t) = -\omega Q_0 \sin(\omega t + \varphi) = \omega Q_0 \cos\left(\omega t + \varphi + \frac{\pi}{2}\right) = I_0 \cos(\omega t + \theta)$$

with the current amplitude I_0:

(36) $$I_0 = \omega Q_0 = \frac{U_0}{\sqrt{R^2 + \frac{1}{(\omega C)^2}}}$$

and the phase shift θ between the current $I(t)$ and the voltage $U_G(t)$:

(37) $$\theta = \varphi + \frac{\pi}{2}$$

Using the relationship $\tan(\varphi + \pi/2) = -1/\tan\varphi$, we obtain from Eqs. (37) and (31):

(38) $$\tan\theta = \frac{1}{\omega RC}$$

Eq. (38) shows that in the case $\omega \to 0$ the current $I(t)$ precedes the voltage $U_G(t)$ by 90° ($\theta = \pi/2$). In the case $\omega \to \infty$, however, current and voltage are in phase ($\theta \approx 0°$). With increasing frequency the phase shift between current and voltage decreases from 90° to 0°.

2.5 Impedance

The *impedance* (or *apparent resistance*) is an important parameter for the description of electrical circuits. It will be treated in more detail in the experimental physics lecture of the second semester. For this reason, we will restrict ourselves here to a few remarks on impedance.

The impedance Z is defined as the *total resistance*[7] an electrical circuit poses to an alternating voltage of angular frequency ω. It follows that $Z = Z(\omega)$. The unit of impedance is Ohm:

$$[Z] = \Omega$$

An impedance in an AC circuit will, in general, influence the *amplitude* and the *phase* of the current in a circuit. Thus it is practical to represent it as a complex quantity:

(39) $$Z = \mathrm{Re}(Z) + i\,\mathrm{Im}(Z)$$

Fig. 9 shows Z as a pointer in the plane of complex numbers. The real part of Z is the (ohmic) *resistance R* of a circuit:

(40) $$R = \mathrm{Re}(Z)$$

The imaginary part of Z is called *reactance X*[8]:

(41) $$X = \mathrm{Im}(Z)$$

Thus, we can write for Z (according to equation (39)):

(42) $$Z = R + i\,X$$

The magnitude of Z (i.e. the length of the arrow in Fig. 9) is given by:

(43) $$|Z| = \sqrt{R^2 + X^2}$$

and the phase, meaning the angle of the arrow with the Re-axis is given by:

(44) $$\varphi = \arctan\left(\frac{X}{R}\right)$$

With the above, Z from Eq. (39) or (42) can be written in *polar form* as:

(45) $$Z = |Z|\,\mathrm{e}^{i\varphi}$$

In analogy to Ohm's law $|Z|$ is given by the ratio of the voltage amplitude U_0 to the current amplitude I_0. For the *RC*-element in Chap. 2.4 it follows (I_0 given by Eq. (36)):

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7 In general, the *total resistance* is not a pure ohmic resistance!

8 In an AC circuit with capacitor C and coil L, the reactance X is composed of an *inductive* component caused by L, and a *capacitive* component caused by C. More about this in the second semester.

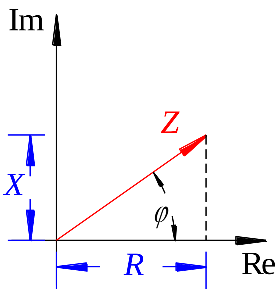

Fig. 9: Impedance Z as a pointer in the plane of complex numbers

(46) $$|Z| = \frac{U_0}{I_0} = \sqrt{R^2 + \frac{1}{(\omega\, C)^2}}$$

Comparison of Eq. (46) with Eq. (43) shows that Z is composed of an ohmic *resistance* R and a capacitive *reactance* $X = 1/(\omega C)$. In the case $\omega \rightarrow 0$ we have $1/(\omega C) \rightarrow \infty$, i.e., Z is mainly determined by the capacitor which ”blocks“ the circuit in this case. For $\omega \rightarrow \infty$, however, the situation is inverse: In that case $1/(\omega C) \rightarrow 0$, i.e., the capacitor does not block and Z is mainly determined by the ohmic resistance R.

3 Experimental Procedure

Equipment:

Digital oscilloscope TEKTRONIX TDS 1012 / 1012B / 2012C / TBS 1102B - EDU, function generator (TOELLNER 7401, output resistance $R \approx 50\ \Omega$), Multimeter (AGILENT 34405A), voltage supply, stopwatch, resistor decade, single capacitors on mounting plate (approx. 10 µF, approx. 10 nF), plate capacitor (aluminium; $A \approx 0.20 \cdot 0.17\ \mathrm{m}^2$) with dielectric (PVC plates of variable thickness, $d \approx (1, 2, 3)$ mm), 5 coaxial cables of different length, switch, metal measuring tape, tape measure, calliper gauge.

Hint:

In the following circuit diagrams those components are drawn in ***red*** whose quantities (capacitance or resistance) are to be measured (Fig. 10 - Fig. 12) or above which signals are measured (Fig. 15). The dashed frames surround the equivalent circuit diagrams of the instruments which are used to measure the required quantities, such as the function generator or the oscilloscope. Besides the input and output resistances and the capacitances of the instruments, often another capacitor C_K is drawn into the circuit diagram. C_K represents the capacitance of all cables required for the measurement setup (capacitance of *connecting cables*).

In order to simplify the text we will often use the terms „input capacitance“ C_O, the „capacitance of connecting cables“ C_K, the capacitor C etc. when we mean „capacitors of the capacitances“ C_O, C_K or C etc.

3.1 Determining the Input Resistance of an Oscilloscope from the Discharge Curve of a Capacitor

The input resistance R_O of an oscilloscope is to be determined from the discharge curve of a capacitor with the capacitance C (Fig. 10). For this purpose, C is charged via the internal resistance R_S of a voltage source (voltage supply; initial voltage ≈ 5 V), then C is separated from the voltage source (open switch S) and the discharge of C via R_O is observed.

The input capacitance C_O of the oscilloscope, the capacitance of connecting cables C_K and the capacitance C are in parallel. We choose $C >> C_O + C_K$, so we can neglect C_O and C_K (here, $C \approx 10$ µF, measure with multimeter AGILENT 34405A). According to Eq. (18) the time difference $\Delta t = t_2 - t_1$ is measured ten times using a stopwatch within which the voltage U decreases from the value U_1 to the value U_2 (measure U_1 and U_2). The input resistance of the oscilloscope, including the maximum error, is determined from the mean value of Δt according to Eq. (18). The values for U_1 and U_2 may be assumed to be error free (exact) for this purpose.

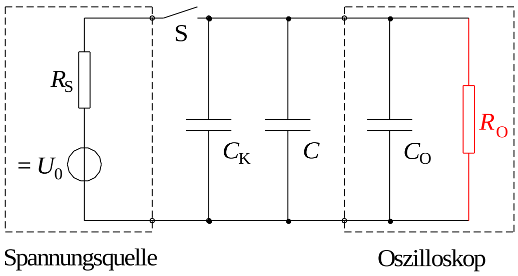

Fig. 10: Equivalent circuit for voltage supply, capacitor C, connecting cables (with capacity C_K), and oscilloscope with the input resistance R_O to be measured.

3.2 Measuring Capacitances

3.2.1 Description of the Measuring Method

The procedure applied in experiment 3.1 to measure the time difference $\Delta t = t_2 - t_1$ is well suited if the time constant $\tau = RC$ is large. For small time constants it is ideal to *periodically* charge and discharge the capacitor and to measure the time difference $\Delta t = t_2 - t_1$ by direct observation of the discharging curve with an oscilloscope. Periodic charging and discharging can be achieved by connecting the capacitor with a function generator (FG) and providing a periodic square-wave voltage U_{FG} with an amplitude U_0 (e.g. $U_0 = 4$ V). The FG then serves as a voltage source with an incorporated „electronic switch". Fig. 11 shows the related equivalent circuit diagram.

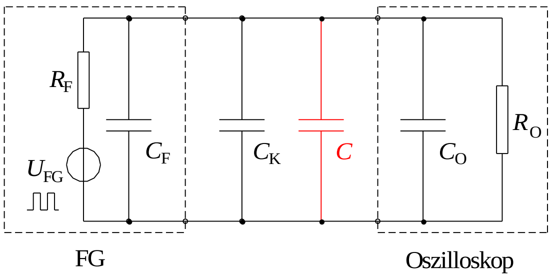

Fig. 11: Equivalent circuit for function generator FG, connecting cables (with capacity C_K) capacitance C to be measured, and oscilloscope. Refer to the text for other labels.

A comparison with Fig. 10 shows two differences:

a) Besides the capacitance of the connecting cables (C_K), the input capacitance of the oscilloscope (C_O) and the capacitance C to be measured the "output capacitance"[9] C_F of the FG has to be taken into account. These three capacities together form the total capacitance C_A of the *measuring set-up*:

(47) $$C_A = C_O + C_K + C_F$$

b) The FG as an „electronic switch" does not separate the voltage source with resistance R_F ($\approx 50\ \Omega$) from the circuit (like the switch S in Fig. 10), but only causes a periodic charge reversal of the capacities C_A and C.[10] Due to $R_F << R_O$ the charge reversal is performed via R_F. Therefore R_F determines the time constant τ of the RC element together with C_A and C. In this case, Eq. (18) therefore reads:

(48) $$C_A + C = \frac{t_2 - t_1}{R_F \ln\left(U_1 / U_2 \right)}$$

[9] A real square-wave signal from a FG never has edges with slope ∞. Rather, e.g. the falling edge resembles the discharging curve of a capacitor with capacitance C_F. This quantity is described as output capacitance according to an equivalent circuit here.

[10] It is of no importance to the measurement, whether the capacitor is charged and then discharged or periodically commutated, as in this case. This does not influence the time response.

Eq. (48) provides the possibility to determine an unknown capacitance C by measuring U_1, U_2 and $\Delta t = t_2 - t_1$, provided that R_F and C_A are known.

For the function generators used in the laboratory course $R_F \approx 50\ \Omega$. This results in a small value of the time constant τ of the capacitor discharge, leading to a small (and hence difficult to measure) time difference $\Delta t = t_2 - t_1$. For this reason, an external resistance $R_D \approx 1\ \mathrm{k}\Omega$ from the resistance decade is placed in series with R_F in a set-up according to Fig. 12 and Fig. 13 in order to achieve a total resistance of

$$(49) \qquad R_G = R_F + R_D$$

thus increasing the time difference Δt. Eq. (48) then becomes:

$$(50) \qquad C_A + C = \frac{t_2 - t_1}{R_G \ln\left(U_1 / U_2 \right)}$$

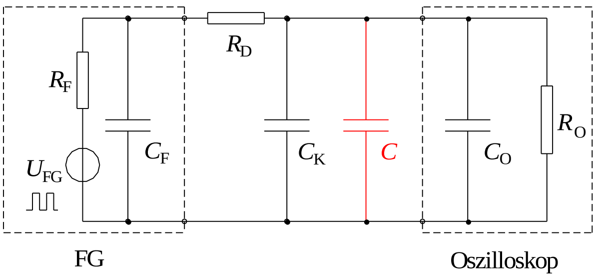

Fig. 12: Circuit from Fig. 11 with added resistor R_D.

Fig. 13: Picture of the circuit from Fig. showing the function generator on the left, the oscilloscope on the right, and the resistance decade with resistor R_D in the centre. R_D is located between the two black terminals of the resistance decade. The yellow terminal is a support contact without an electrical connection to R_D. A BNC-T connector is inserted in the cable connecting the resistance decade and the oscilloscope in order to connect the capacitor for which the capacitance C is to be determined.

From this follows that the capacitance C is given by:

$$(51) \qquad C = \frac{t_2 - t_1}{R_G \ln\left(U_1 / U_2 \right)} - C_A$$

3.2.2 Preliminary Measurements

In order to determine an unknown capacitance C from Eq. (51), the value of the total capacitance C_A of the circuit needs to know in addition to the resistance R_G. C_A is determined by setting up the circuit according to Fig. 12 with $C = 0$ (i.e. *without* the capacitance C to be measured). A BNC-T piece is included in the circuit (Fig. 13) to connect the capacitance C which is to be determined for each subsequent measurement. C_A can now be determined using Eq. (50). For this purpose, the discharge curve of C_A is displayed on the oscilloscope and the time difference $\Delta t = t_2 - t_1$ associated with the voltage drop from U_1 to U_2 is measured. For measuring these quantities, the digital oscilloscope can be operated in the mode → Acquisition → Mean value. In this operation mode, the influence of signal noise is minimized.

U_1 and U_2 may be taken as exact values for calculating the maximum error of C_A. For R_G, a maximum error of $0.01 \times R_G$, in accordance with the accuracy of the resistance decade, may be used.

Once these preparations have been made, unknown capacitances C added to the circuit can be measured.

Hint:

Eqs. (18) and (51) hold for the discharge of a charged capacitor from an initial voltage U_0 to 0 V. The voltage levels U_1 and U_2 are positive at all times t. If, however, a rectangular voltage with amplitude U_0 is applied to the capacitor, it follows that the maximum voltage is $+ U_0$ and the minimum voltage is $- U_0$ (Fig. 14, left ordinate). Hence, the resulting reloading curve may include negative voltage values. In this case, Eqs. (18) and (51) cannot be applied, since the logarithm function is only defined for arguments having a positive value.

This problem can be solved by recognising that the temporal evolution of a reloading curve from the voltage $+ U_0$ to $- U_0$ has the same shape as the discharge curve of a capacitor having an initial voltage of $2\,U_0$ and a minimum voltage of 0 V (Fig. 14, right ordinate). Thus, adding the amplitude U_0 to all voltage values recorded from the oscilloscope ensures that U_1 and U_2 are always positive, and hence Eqs. (18) and (51) can be used.

This method requires, that the rectangular voltage signal does not have any DC component (DC-Offset knob on the FG must be set to OFF) and that its amplitude U_0 is known. It follows that U_0 must be measured once. To facilitate reading the voltage levels off the oscilloscope, it is recommended to place the signal symmetrically about the centre (horizontal) line of the scale (“0” in Fig. 14, left ordinate). In this case, U_1 and U_2 can be determined simply by reading the scale marks on the oscilloscope’s screen and Δt can be determined by using the time cursors.

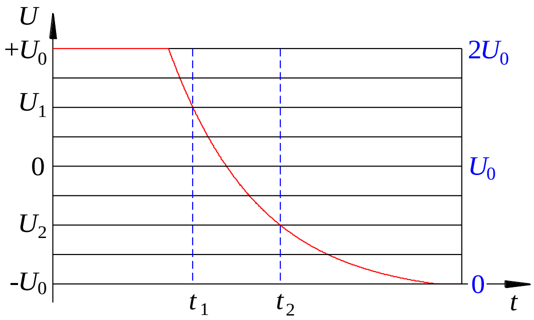

Fig. 14: Charge reversal curve of the capacitor upon applying a rectangular voltage of amplitude U_0 without DC-offset (left ordinate). The same temporal course results for a rectangular voltage with amplitude U_0 *and* DC-offset U_0 (right ordinate, blue). The horizontal lines indicate the scale ticks of the oscilloscope.

3.2.3 Determination of the Capacitance of Coaxial Cables

In this part of the experiment the capacitance C of coaxial cables *added* to the existing (coaxial-) cables (having a total capacitance C_K), is to be measured. The simplest method to achieve this is to connect the extraneous cables to the BNC-T connector (Fig. 13). C is thus connected parallel to C_A.
Five coaxial cables of different lengths $L \geq 1$ m (measure the lengths!) are connected in turn to the BNC-T-piece. For each cable, the quantities U_1, U_2, t_1 and t_2 are measured and the capacitance C is calculated according to Eq. (51). Stating the errors for the individual values of C may be omitted.
As a result the mean value of the capacitance of the coaxial cable per meter including the standard deviation of the mean is to be stated and to be compared with the value from literature for coaxial cables of the type RG 58 C/U (101 pF/m).

3.2.4 Determining the Relative Permittivity of PVC

Following the method described in Chapter 3.2.3 the capacitance of a plate capacitor with the dielectric PVC between its plates is to be determined. The objective is to determine the relative permittivity ε_r of PVC from a series of capacitance measurements with varying thickness d of the dielectric.
The plate capacitor consists of two equal aluminium plates of the area A with a PVC plate of equal size and thickness d between them. The capacitor is connected between function generator and oscilloscope in addition and in parallel to the existing connecting cables. It is connected to the BNC-T-piece by a coaxial

cable having laboratory plugs on the other end[11]. One of the aluminium plates is put on the laboratory bench and connected to the „negative pole" of the function generator (outer contact of the BNC-connector). The PVC plate is put on this plate and the second aluminium plate is put on top of it and connected to the other pole of the function generator.

Measurements are done for PVC plate sizes of $d \approx (3, 4, 5, 6)$ mm (measure d with a calliper gauge and A with a metal measuring tape). C is determined for each size (Eq. (51)). For further analysis, C is plotted over $1/d$. ε_r can be determined (Eq. (6)) from the slope of the regression line and can be compared with the literature value (Eq. (6)).[12]

3.3 Phase Shift Between Current and Voltage in an *RC* Element

Using a set-up according to Fig. 15 the phase shift θ between the cosinusoidal output voltage U_{FG} of the function generator and the charge and discharge current I of the capacitor with dependence on the angular frequency ω is to be measured. We can neglect the internal resistances as well as input and output capacitances of the function generator and the oscilloscope for this experiment.
The output voltage U_{FG} of the function generator can be measured directly using the oscilloscope (symbolized by the "voltmeter" V_1 in Fig. 15). The current I is measured via a small detour: I causes a voltage drop, $U_R = R\,I$ at R, that is in phase with I and can also be measured with the oscilloscope (V_2).

The measurement of θ is carried out for an RC element with $R \approx 1$ kΩ and $C \approx 10$ nF (measure both values with multimeter AGILENT 34405A) at frequencies of $f = (1, 5, 10, 20, 30, 40, 50, 100)$ kHz. The amplitude of U_{FG} shall amount to approx. 5 V at $f = 10$ kHz.

θ is plotted vs. ω with maximum error for θ. Into the same diagram the theoretical expected values for θ are plotted too and are compared with the measured data.

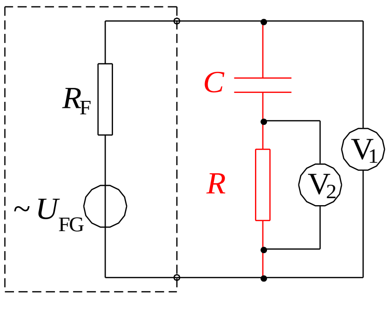

Fig. 15: Set-up for measuring the phase shift between $U_G(t)$ and $I(t)$ in a *RC* element.

Practical hints:

- When carrying out the experiment it should be considered that the reactance $X = 1/(\omega C)$ of the capacity is a function of ω so that the voltage amplitudes also vary with ω.
- The phase shift θ can best be determined by measuring the time difference Δt of the passages through zero by both voltages $U_G(t)$ and $U_R(t)$ (compare with the experiment "*Oscilloscope*...").
- Consider at the connecting of the cables for the measurement of $U_G(t)$ and $U_R(t)$ that the outer contacts of the BNC sockets of the oscilloscope are on the same potential! Consequently this also applies to the outer contacts of the BNC plugs at the coaxial cables!

Question 5:

- How large is the phase shift between the voltage at the capacitor (U_C) and the current I? How can the phase shift be measured?

[11] This additional cable increases the total capacity C_K of the connecting cables in the experimental setup. It is thus necessary to (re-)measure the total capacity C_A of the measuring apparatus prior to connecting the parallel plate capacitor.
[12] Literature value according to /3/: $\varepsilon_r = 3.1 \ldots 3.5$ (without stating frequency).

4 Appendix

Calculating with complex quantities, Eqs. (29) and (30) are easy to derive. In a complex form the formulas in Eqs. (25) and (28), respectively, can be written as:

(52) $$U_{\mathrm{G}}(t) = U_0\ \mathrm{e}^{i\omega t}$$

(53) $$Q(t) = Q_0\ \mathrm{e}^{i(\omega t + \varphi)}$$

Inserting both equations into Eq. (26) and performing the differentiation we obtain after division by $\mathrm{e}^{i\omega t}$:

(54) $$U_0 = i\omega R Q_0\ \mathrm{e}^{i\varphi} + \frac{1}{C} Q_0\ \mathrm{e}^{i\varphi}$$

Hence it follows:

(55) $$Q_0\ \mathrm{e}^{i\varphi} = \frac{U_0}{\frac{1}{C} + i\omega R}$$

The left side of Eq. (55) is *one* common way to represent a complex number (*polar notation*) z of *modulus* $|z|$ and the phase angle (*argument*) φ:

(56) $$z := |z|\ \mathrm{e}^{i\varphi} \qquad \text{here: } z = Q_0\ \mathrm{e}^{-\varphi},\ |z| = Q_0$$

The modulus of z is given by

(57) $$|z| = \sqrt{z\, z^*}$$

z^* being the complex conjugated to z which is obtained by changing the sign of the imaginary unit i ($i \to -i$ and $-i \to i$). For the modulus Q_0 we thus obtain:

(58) $$Q_0 = \sqrt{\frac{U_0}{\frac{1}{C} + i\omega R}\ \frac{U_0}{\frac{1}{C} - i\omega R}} = \sqrt{\frac{U_0^2}{\frac{1}{C^2} + (\omega R)^2}} = \frac{U_0 C}{\sqrt{1 + (\omega RC)^2}}$$

This is the result given in Eq. (29) .

We use a *second* common method to represent complex numbers to calculate the phase angle, namely

(59) $$z = \mathrm{Re}(z) + i\ \mathrm{Im}(z) := \alpha + i\beta$$

where α is the *real part* (Re) and β the *imaginary part* (Im) of z. From these quantities the phase angle φ can be calculated as

(60) $$\varphi = \arctan\left(\frac{\beta}{\alpha}\right) \begin{Bmatrix} +\pi & \Leftrightarrow & \alpha < 0\ \wedge\ \beta \geq 0 \\ -\pi & \Leftrightarrow & \alpha < 0\ \wedge\ \beta < 0 \end{Bmatrix}$$

In order to apply Eq. (60), we have to convert Eq. (55) into the form of Eq. (59), that is we must separate the real and the imaginary part from each other. For this purpose we have to eliminate i from the denominator, for which the fraction is appropriately extended. Eq. (55) then becomes:

$$(61) \qquad Q_0\, e^{i\varphi} = \frac{U_0\left(\frac{1}{C} - i\omega R\right)}{\left(\frac{1}{C} + i\omega R\right)\left(\frac{1}{C} - i\omega R\right)} = \frac{\frac{U_0}{C}}{\frac{1}{C^2} + \omega^2 R^2} - i\,\frac{U_0\, \omega\, R}{\frac{1}{C^2} + \omega^2 R^2} := \alpha + i\beta$$

From Eq. (61) we can read off α and β:

$$(62) \qquad \alpha = \frac{\frac{U_0}{C}}{\frac{1}{C^2} + \omega^2 R^2} \qquad\qquad \alpha = \frac{\frac{U_0}{C}}{\frac{1}{C^2} + \omega^2 R^2}$$

Attention must be paid to the fact that there is a positive sign in the definition equation (59). Thus, the negative sign of i in Eq. (61) belongs to the imaginary part β. By inserting Eq. (62) into Eq. (60) we obtain:

$$(63) \qquad \varphi = \arctan\left(\frac{\beta}{\alpha}\right) = \arctan\left(-\omega RC\right)$$

This is the result given in Eq. (30).