

Moment of inertia - Steiner's theorem

Keywords:

Rotational motion, angular velocity, angular acceleration, moment of inertia, rotational moment, angular momentum, STEINER's theorem.

Measuring program:

Measurement of the moment of inertia of a circular disc, determination of the axis of gravity of an irregular shaped body.

References:

/1/ EICHLER, H. J., KRONFELDT, H.-D., SAHM, J.: „Das Neue Physikalische Grundpraktikum“, Springer-Verlag, Berlin, among others

1 Introduction

The aim of this experiment is to improve the understanding of the analogy between translational and rotational motion. For this purpose, a set-up is used which enables the measurement of moments of inertia of bodies with respect of optional axes.

First, the corresponding quantities of the translational and rotational motion are called to memory by means of Table 1.

Table 1: Comparison of translational and rotational motion.

Translational motion			Rotational motion		
Name	Symbol	Unit	Name	Symbol	Unit
Position vector	\mathbf{r}	m	Angle ¹	φ	rad
Velocity	$\mathbf{v} = \frac{d\mathbf{r}}{dt}$	m s ⁻¹	Angular velocity ¹	$\boldsymbol{\omega} = \frac{d\varphi}{dt}$	rad/s ⁻¹
Acceleration	$\mathbf{a} = \frac{d\mathbf{v}}{dt}$	m s ⁻²	Angular acceleration ¹	$\frac{d\boldsymbol{\omega}}{dt}$	rad/s ⁻²
Mass	m	kg	Moment of inertia ²	$I = \int R^2 dm$	kg m ²
Momentum	$\mathbf{p} = m\mathbf{v}$	kg m s ⁻¹	Angular Momentum	$\mathbf{L} = I\boldsymbol{\omega}$ $\mathbf{L} = \mathbf{r} \times \mathbf{p} = m \mathbf{r} \times \mathbf{v}$	kg m ² s ⁻¹
Force	$\mathbf{F} = m\mathbf{a} = \frac{d\mathbf{p}}{dt}$	N	Torque	$\mathbf{T} = I \frac{d\boldsymbol{\omega}}{dt} = \frac{d\mathbf{L}}{dt}$ $\mathbf{T} = \mathbf{r} \times \mathbf{F}$	N m

2 Theory

We consider a rotary disk D of the radius r , around which a thin thread has been wound according to Fig. 1. The thread is connected to a mass m via a pulley R. The disk is held at rest by the pin T of the magnet B. After closing the switch S, a current flows from the power supply U through the coil of the magnet. The holding pin T is pulled back by the resulting magnetic field, thereby unlocking the disc. The falling mass m then causes an accelerated rotation of the disk about the rotary axis H.

Now we require an equation by means of which we can calculate the moment of inertia I_D of the rotary disk from known or measurable quantities.

¹ The direction of the *axial* vectors $\boldsymbol{\varphi}$, $\boldsymbol{\omega}$ and $d\boldsymbol{\omega}/dt$ is by definition the direction of the axis of rotation. The sign obeys the right-hand rule: the incurved fingers show the direction of rotation, so the thumb shows the direction of $\boldsymbol{\varphi}$, $\boldsymbol{\omega}$ and $d\boldsymbol{\omega}/dt$. *Polar* vectors (normal vectors), as e.g. the position vector (\mathbf{r}) and the velocity vector (\mathbf{v}), change sign upon performing a point inversion of the coordinate system, whereas axial vectors (also called *pseudo-vectors*) do not.

² R is the distance of a mass element dm from the axis of rotation.

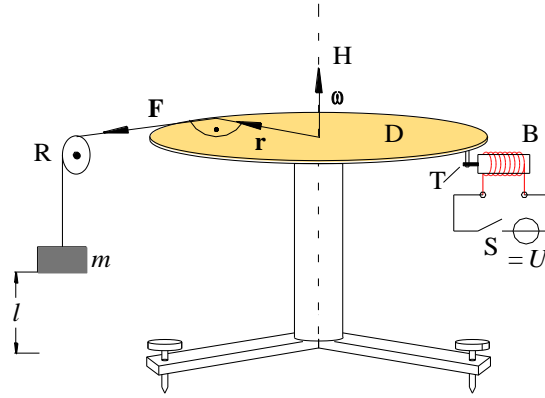


Fig. 1: Rotary disk for measuring moments of inertia. Refer to the text for labels.

For this purpose we first set up the equation of motion for the rotation of the rotary disk. It is very simple in this case: the rotary disk has the angular acceleration $d\omega/dt$ due to the rotational moment $\mathbf{r} \times \mathbf{F}$. In analogy to NEWTON's law $\mathbf{F} = m \mathbf{a}$ we thus obtain (cf. Table 1):

$$(1) \quad \mathbf{r} \times \mathbf{F} = I_D \frac{d\omega}{dt}$$

Then it follows from the chosen geometry ($\mathbf{r} \perp \mathbf{F}$) for the absolute values:

$$(2) \quad F = \frac{I_D}{r} \frac{d\omega}{dt}$$

In this equation we have to replace F and $d\omega/dt$ by known or measurable quantities. In order to find an expression for $d\omega/dt$, we first observe the motion of the mass m . If the time t is needed for falling through a distance l , we obtain for its acceleration a :

$$(3) \quad a = \frac{2l}{t^2}$$

Because m and the rotary disk are connected via the thread, this must also be the tangential acceleration of a mass point on the edge of the rotary disk. Based on the well-known relationship between tangential and angular acceleration with Eq. (3), we thus obtain for such a point:

$$(4) \quad \frac{d\omega}{dt} = \frac{a}{r} = \frac{2l}{r t^2}$$

Inserting Eq. (4) into Eq. (2) yields:

$$(5) \quad F = I_D \frac{2l}{r^2 t^2} = I_D \frac{a}{r^2}$$

We still need a relationship for the force F , which accelerates the disk, since it cannot be measured directly. For this we look at the net force acting on the set-up. The accelerating force of gravity $G = mg$ (g : gravitational acceleration) must accelerate the mass m , overcome frictional forces at pulley the R and the rotary disk D, and set the pulley and rotary disk into an accelerated rotation. For this, the following forces are necessary:

- F_m : Accelerating force for m
- F_{RR} : Frictional force at the pulley
- F_R : Accelerating force for the pulley
- F_{RD} : Frictional force at the rotary disk
- F : Accelerating force for the rotary disk

Thus we obtain:

$$(6) \quad G = mg = F_m + F_{RR} + F_R + F_{RD} + F$$

The force which accelerates m , $F_m = ma$, is therefore considerably smaller than the force of gravity $G = mg$.

To simplify matters we now assume that the force of friction and the accelerating force are replaced by *one* force acting on the pulley, which is necessary for the translational acceleration of an equivalent mass m_e (here: $m_e \approx 2.2$ g):

$$(7) \quad F_R + F_{RR} := m_e a$$

We therefore obtain for the required force F from Eq. (6):

$$(8) \quad F = mg - (m + m_e)a - F_{RD}$$

Inserting this equation into Eq. (5) we obtain:

$$(9) \quad mg - (m + m_e)a = I_D \frac{a}{r^2} + F_{RD}$$

For better readability we introduce a force

$$(10) \quad F_E := mg - (m + m_e)a$$

with the measurable quantities m and a and the known quantities m_e and g such that Eq. (9) becomes:

$$(11) \quad F_E = I_D \frac{a}{r^2} + F_{RD}$$

The unknown quantity F_{RD} which cannot be measured directly is still bothering us in this equation for determining I_D . If we assume, however, that the friction at the rotary disc is a rolling and sliding friction independent of the velocity (the so-called COULOMB *friction*), which only depends on the mass of the rotary table including the bodies on it, then F_{RD} can be considered a time-independent constant. In this case Eq. (11) represents a simple linear equation of the form

$$(12) \quad y = cx + b$$

with

$$(13) \quad y = F_E, \quad x = \frac{a}{r^2}, \quad c = I_D, \quad b = F_{RD}$$

Plotting the related quantity F_E (to be calculated according to Eq. (10)) against a/r^2 (with a from Eq. (3)) for constant r and different accelerating masses m (Eq. (11)), we obtain a line with the slope I_D . Thus we have found a way to measure the moment of inertia without knowing the quantity F_{RD} .

We now observe the case in which an additional body is placed on the rotary disk. Suppose I_K is the moment of inertia of this body (mass m_K) when it rotates about one of its gravity axes (*principal axis*); if this gravity axis corresponds with the rotary axis H of the rotary disk, then the overall moment of inertia I of the system rotary disk/body is:

$$(14) \quad I = I_D + I_K$$

If the axes H and C run parallel at a distance s we obtain according to STEINER's *theorem*³:

³ JAKOB STEINER (1796 - 1863)

$$(15) \quad I = I_D + I_K + m_K s^2$$

Eq. (11) then reads:

$$(16) \quad F_E = I \frac{a}{r^2} + F_{RD}$$

Using Eq. (3) it follows:

$$(17) \quad I = (F_E - F_{RD}) \frac{r^2}{a} = (F_E - F_{RD}) \frac{r^2}{2l} t^2$$

We can use this relationship to determine the position of a gravity axis running parallel to the rotary axis of the disk for a body of arbitrary shape lying on the rotary disc. We take the following steps: according to Eq. (15) I is minimal when $s = 0$, i.e., for the case that the gravity axis of the body is identical to that of the rotary axis of the disc. According to Eq. (17) a minimum of I is equivalent to a minimum of the fall time t and t^2 , respectively. Shifting the body on the rotary disc (varying s), the fall time t must therefore show a minimum at a certain position. The related function $t = f(s)$ describing this behaviour will now be determined. For this we insert Eq. (15) into Eq. (17), solve for t^2 and obtain for t as a function of s :

$$(18) \quad t^2 = \underbrace{\frac{(I_D + I_K)2l}{(F_E - F_{RD})r^2}}_{K_1} + \underbrace{\frac{2l m_k}{(F_E - F_{RD})r^2}}_{K_2} s^2$$

or in a clear way with the auxiliary quantities K_1 and K_2 :

$$(19) \quad t^2 = K_1 + K_2 s^2$$

Question 1:

- Which function (curve) does Eq. (19) represent? (Hint: Conic sections)

In order to determine the position of the required gravity axis C by means of Eq. (19) we proceed as follows: Choose a coordinate system XY on the rotary disc, the origin of which coincides with the axis of rotation H (cf. Fig. 2). A line of holes is created along the y -axis of the rotary disk. A pin is fixed at an optional point P on the body, for which we find the position of the gravity axis. The pin and line of holes are placed such that the body can be shifted in the Y direction on the rotary disk without changing its orientation with regard to the coordinate system XY (cf. remarks at the end of Chap. 3.2).

Let point P (the pin) have the coordinates $(0, y_P)$ after placing the body on the rotary disc. For the distance s of the gravity axis C from the rotary axis H we then obtain:

$$(20) \quad s = \sqrt{\Delta x^2 + (y_P - \Delta y)^2}$$

According to Eq. (19) the fall time t for the accelerating mass m has a minimum when s is minimal, which, according to Eq. (20) with fixed Δx , is the case for $y_P = \Delta y$.

If we shift the body in y direction on the rotary disk and plot the fall time t over the shift y_P , we can determine the quantity Δy by finding the minimum in the produced curve. In an analogous way, the quantity Δx can be determined and proceeding from the optional point P, we can state the position of the desired gravity axis.

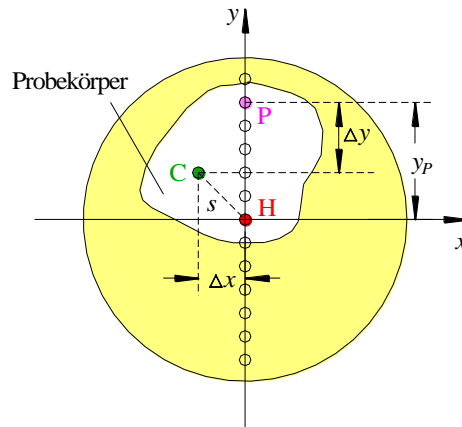


Fig. 2: Rotary disc (yellow) with sample body (white, top view). H is the axis of rotation, C the gravity axis of the sample body⁴ and P is the sample body's point of fixation along the vertical line of holes on the disc. s is the distance between C and H.

3 Experimental procedure

Equipment:

Rotary disc on tripod, 5 acceleration masses ($m = (1,00 \pm 0,01) \text{ g}$) with plate (m according to imprint, error negligible), brass disk with locking pins, irregularly shaped sample body with locking pins, power supply (PHYWE (0 - 15 / 0 - 30) V), magnetic holder, stand material for magnetic holder, switch, light barrier, electronic universal counter, digital oscilloscope TEKTRONIX TDS 1012 / 1012B / 2012C / TBS 1102B - EDU, precision spirit level (accuracy 0.1 mm on 1 m), balance, metal measuring tape, sliding calliper, deceleration rod, thread.

Attention:

The rotary discs have very sensitive precision bearings which are easy to damage through improper handling. Only move the rotary discs with careful fingers! Take care that the thread does not get entangled in the bearing by timely deceleration! Only decelerate the discs using the small rod available!

Hint:

Usually the rotary discs are levelled exactly by the technical assistant using a precision water level prior to the lab course.

3.1 Moment of inertia of a disc

The moment of inertia I_K of a brass disk (radius r_K , mass m_K) rotating about its symmetry axis C (Fig. 3) is to be determined by means of the set-up in Fig. 1. It is then calculated according to Eq. (14) as follows:

$$(21) \quad I_K = I - I_D$$

In order to obtain I_K , first the moment of inertia of the rotary disc (I_D) has to be determined by means of Eq. (11) and then the moment of inertia of the rotary- and brass disks together (I) by means of Eq. (16). For this purpose

- a) for the rotary disc
- b) for rotary disk with brass disc

the fall time t (mean value from at least four single measurements each) is measured for five different acceleration masses and for a predetermined distance l (to be measured!). The fall time is measured by means of an electronic universal counter. The counter is started by the impulse, which causes the release of the holding pin of the magnetic holder, which is responsible for keeping the rotary disk in the starting position. The stopping impulse for the universal counter is given by a light barrier, which the accelerated masses pass at the end of the specified distance l .

Subsequently F_E is plotted against a/r^2 for a) and b) according to Eq. (11) and Eq. (16) in *one* diagram and the regression lines are calculated (measure r *carefully* using a metal measuring tape)⁵. An error

⁴ Note that the white area represents the *top view* of the sample body. For this reason, C does not need to be located at the centre of gravity of the white area.

⁵ The acceleration a is in the order of magnitude of 10^{-2} ms^{-2} and thus small compared to g . Only small differences therefore arise for F_E (Eq. (10) in the cases a) and b).

analysis for the individual values of F_E and a/r^2 is not required. The friction forces F_{RD} on the rotary disc as well as the moments of inertia I_D and I are calculated from the parameters of the regression line (including error) and from that I_K according to Eq. (21) (also including error).

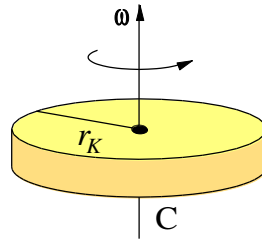


Fig. 3: Rotation of a disc of radius r_K and mass m_K about its symmetry axis C.

Question 2:

- How can the moment of inertia I of a disk with the mass m_K and the radius r_K rotating about its symmetry axis C (cf. Fig. 3) be calculated from the relationship $I = \int R^2 dm$ (cf. Chapter 1)? How large is the theoretically expected moment of inertia for the brass disk used (measure r_K and m_K)? What are the possible sources of deviations between theory and experiment?

3.2 Determining the position of a gravity axis of an irregularly shaped body

According to the explanations given for Eqs. (18) - (20) the position of a gravity axis C running parallel to the rotary axis H of an irregularly shaped sample body shall be determined. For this purpose the pin mounted on the body is put into ten different holes of the hole row along the y -axis of the rotary disk and the coordinate y_P is determined⁶. At each position, the average fall time t (mean of 4 single measurements) for a predetermined distance l is measured for *one* mass m each. Afterwards, t is plotted against y_P including error bars (standard deviation of the mean) and the value Δy is graphically determined, where t has a minimum.

Alternatively, the position of the minimum of t may be determined by a non-linear fit⁷. The target function is, according to Eq. (19), given by:

$$(22) \quad t = \sqrt{K_1 + K_2 (y_P - \Delta y)^2}$$

with the fit parameters K_1 , K_2 and Δy . This fit directly provides the value $y_P = \Delta y$ for which the fall time t is minimal.

Analogously, it would be possible to determine Δx and to state the position of the centre of gravity C in the xy -plane relative to the point P. In order to save time, however, we will confine ourselves to measuring only the distance Δy between P and C.

Remarks:

In order to make sure that the orientation of the sample body does not change when shifting along the y -axis, *two* pins are mounted on the body. Therefore, it has to be determined first, which of the two pins marks the position of point P.

⁶ The distance between two holes on the disc is 10 mm (error free).

⁷ Nonlinear fits are dealt with in part II of the basic laboratory course in the SoSe, see http://physikpraktika.uni-oldenburg.de/download/GPR/pdf/Nichtlineare_Fits.pdf. Here the application is optional.