Carl von Ossietzky University Oldenburg – Faculty V - Institute of Physics
Module Introductory Laboratory Course Physics – Part I

Viscosity and Reynolds Numbers

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Friction, frictional force, buoyancy, viscosity, laminar and turbulent flow, REYNOLDS number, STOKES law, BERNOULLI's equation, HAGEN-POISEUILLE law.

Measuring program:
Measurement of the viscosity with the falling ball method, measurement of the kinematic viscosity with the capillary viscometer, determination of the REYNOLDS number for the transition from laminar to turbulent pipe flow.

References:

/1/ DEMTRÖDER, W.: „Experimentalphysik 1 - Mechanik und Wärme“, Springer-Verlag, Berlin among others.
/2/ SCHENK, W., KREMER, F. (HRSG.): „Physikalisches Praktikum“, Vieweg + Teubner-Verlag, Wiesbaden
/3/ WALCHER, W.: „Praktikum der Physik“, Teubner Studienbücher, Teubner-Verlag, Stuttgart

1 Introduction

NEWTON's law “force is proportional to acceleration“ seems to contradict many every-day experiences. Observing, for example, the motion of bodies under the influence of friction, the description „force is proportional to velocity“ rather gets to the core of the matter, e.g., in order to keep a constant speed when riding on a bicycle, strength has to be used indefinitely. If you want to travel at a faster speed indefinitely, then you have to pedal more vigorously which indefinitely requires more strength.
Actually, many mechanical processes in which friction plays a role can be satisfactorily described with the ansatz „force ~ speed“. This is true, for example, for the influence of friction on falling balls in fluids and gases. Two important examples for such falling processes are the deposition of dust particles or water droplets (fog) from the air and the motion of minute oil droplets as used in the MILLIKAN experiment for determining the elementary electronic charge.
The present experiment aims at determining the viscosity of a liquid by observing such falling processes, as well as the flow of liquids through capillary tubes. Additionally, the transition from laminar to turbulent flows will be analysed and the corresponding REYNOLDS number will be determined.

2 Theory

2.1 Determining Viscosity Using the Falling-Ball Method According to Stokes

According to Fig. 1 we observe a ball with the radius *r*, led through an infinitely extended liquid at the rate *v*. Frictional forces have to be overcome to move the ball. They result from the fact that the liquid layer adjacent to the ball adheres to the ball, and therefore, has to be moved as well. The moved layer drags its neighbouring layers along, which, in their turn, drag their neighbouring layers along etc. As a result of this friction a flow of liquid is caused around the ball. Its speed decreases with increasing lateral distance from the ball.
From the NAVIER-STOKES *Equations*[1], which can be used to describe the motion of liquids, the friction force $\mathbf{F}_R$ can be calculated which the liquid exerts on a ball moving at the rate $\mathbf{v}$. Since the vectors $\mathbf{F}_R$ and $\mathbf{v}$ are oriented along the same axis, it is sufficient for the following to work with their magnitudes F_R and v. Following a complicated calculation, which will be explained in later semesters, we find that the *frictional force F_R* is proportional to the *velocity v* and to the *radius r of the ball*:

(1) $$F_R \sim v \qquad\qquad F_R \sim r$$

And it holds that:

(2) $$F_R = 6\pi\eta r v$$

The constant η is called *viscosity* (also *coefficient of the interior friction* or *dynamic viscosity*). Its SI-unit is $[\eta]$ = kg/(m·s) = Ns/m^2= Pa·s. The former CGS-unit, which is still used in many tabular works, is POISE[2] (1 POISE = 1 p = 1 g/(cm·s)).

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[1] CLAUDE LOUIS MARIE HENRI NAVIER (1785 – 1836); GEORGE GABRIEL STOKES (1819 – 1903).
[2] JEAN-LOUIS MARIE POISEUILLE (1799 – 1869).

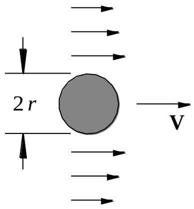

Fig. 1: Moving a ball through a liquid.

Equation (2) is called STOKES *law*. However, it only describes the motion of the ball correctly if the flow of the liquid is *laminar*. A laminar flow means that the different liquid layers glide over one another smoothly and do not mix. This means that smooth and connected streamlines are formed around the ball (Fig. 2). On the other hand, a *turbulent* flow means that the liquid layers mix. In that case, the streamlines are curled (Fig. 3; cf. also figures on the front page of this laboratory course script) and the force to be applied often becomes proportional to v^2:

(3) $$F_R \sim v^2$$

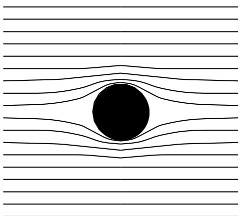

Fig. 2: Laminar flow around a ball.

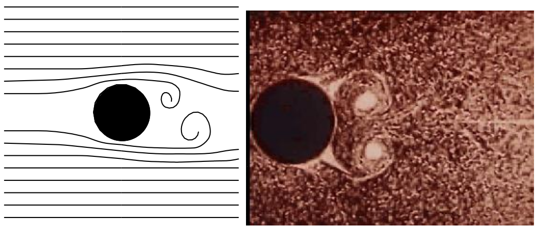

Fig. 3: Turbulent flow around a ball. Left: schematics; Right: original picture from LUDWIG PRANDTL (1875 – 1953)[3].

By means of the dimensionless REYNOLDS number[4] *Re*, whether a flow is laminar or turbulent can be assessed. It is given by:

(4) $$Re = \frac{\rho\, v\, l}{\eta}$$

ρ being the density of the liquid and l the characteristic length of the flow process being considered. In our case l corresponds to the diameter of the ball, in the case of a flow through a tube (cf. Eq.(36)), l would correspond to the diameter of the tube.

The REYNOLDS number has an illustrative physical significance: it is proportional to the quotient of the kinetic energy E_k of a volume particle with length l and the friction energy E_R „consumed“ when the particle is displaced by the distance l. For the example for a spherical liquid particle (mass m, velocity v, density ρ, diameter l) the kinetic energy is:

(5) $$E_k = \frac{1}{2}\, m\, v^2 = \frac{1}{12}\, \rho\, \pi\, l^3\, v^2$$

The friction energy results from the frictional force (Eq. (2) with $r = l/2$) and distance l:

(6) $$E_R = 3\, \pi\, \eta\, v\, l^2$$

Apart from the constant 1/36, the quotient of both quantities yields the REYNOLDS number from Eq. (4).

A flow is laminar for „small“ REYNOLDS numbers and turbulent for big ones[5], however, the terms „small“ and „big“ are to be understood as *relative* statements. It depends very much on the experiment as to what the terms „small“ and „big“ mean. Pipe flows, e.g., behave laminar for REYNOLDS numbers

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[3] Source: PHYSIK JOURNAL 3.10 (2004) 31-37.

[4] OSBORN REYNOLDS (1842 – 1912)

[5] After newer findings the conditions for laminar and turbulent flows are much more complicated then represented in this text and in common textbooks, see e.g. B. HOF et al: „Finite lifetime of turbulence in shear flows“, NATURE 443 (2006) 59-62. However, in the basic laboratory we cannot go into details.

Re < 2,000 - 2,500. For balls falling through fluids we need $Re < 0.2$ /3/ to prevent the flow from becoming turbulent and for STOKES law to remain valid.

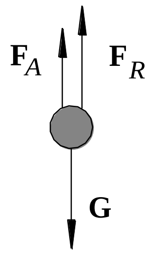

Fig. 4: Forces acting on a falling ball.

We now consider a little ball that is falling with a mass m, a radius r, and a volume V in an infinitely extended liquid with a density ρ_F and of the viscosity η. Three vertically oriented forces act on the ball (Fig. 4). Therefore, it is sufficient to consider their amounts. The forces are the gravity $G = mg$ (g: gravitational acceleration), pointing downwards, the buoyant force $F_A = \rho_F Vg$ pointing upwards and the force of friction, $F_R = 6\pi\eta rv$ also pointing upwards (eq. (2)). The resulting net force F is then:

(7) $$F = G - F_A - F_R$$

This force F accelerates the ball downwards with the increasing velocities v. Together with v, F_R increases as well, so that F decreases and finally becomes zero. From this moment on we have:

(8) $$F = G - F_A - F_R = 0$$

The ball is now falling with the constant velocity v_0.

Question 1:
- How does a gas bubble move that is released at the bottom of a water glass (e.g. a CO_2 bubble in a glass of mineral water)?

Inserting G, F_A and F_R with $v = v_0$ into Eq. (8), we obtain:

(9) $$mg - \rho_F Vg - 6\pi\eta rv_0 = 0$$

Inserting $m = \rho_K V$ (with ρ_K : density of the material) as well, and $V = \frac{4}{3}\pi r^3$, we obtain from Eq. (9):

(10) $$\frac{4}{3}\pi r^3 g\left(\rho_K - \rho_F\right) - 6\pi\eta rv_0 = 0$$

Rearranging this equation for η, we obtain

(11) $$\eta = \frac{2}{9}r^2 g\,\frac{\left(\rho_K - \rho_F\right)}{v_0}$$

A simple method for an indirect measurement of η follows from Eq. (11), provided that ρ_K and ρ_F are known: One drops balls of radius r in the liquid to be analysed and measures their falling velocity v_0 once the state $F = 0$ has been reached.
There is a problem with this method: In general we do not deal with infinitely extended liquids, but, e.g. a cylinder of radius R, in which the falling of the balls is observed. In these cases, the additional friction of the liquid swept away by the ball along the cylinder's wall must be taken into account. It leads to the fact that the measured velocity v_m is lower than the velocity v_0 in the case of an infinitely extended fluid. Since the deviation of v_m from v_0 depends mainly on the ratio of the cross sectional areas of the ball and the cylinder used, we can approximate v_m by:

(12) $$v_m \approx v_0 - k\left(\frac{r}{R}\right)^2$$

where k is an experimentally determined *correction factor*[6]. With this, it follows:

(13) $$v_0 \approx v_m + k\left(\frac{r}{R}\right)^2$$

2.2 Determining the Viscosity Using a Capillary Viscometer According to UBBELOHDE

A fluid flows through a vertical capillary of the radius r_0. The period Δt, which a liquid volume V takes to flow through the capillary is determined by the viscosity η of the liquid. The greater η, the greater Δt. Capillary viscometers function according to this simple principle. Fig. 5 shows such a capillary viscometer following UBBELOHDE, which is described in more detail in Chap. 3.2 and in Appendix 4.4.

The exact derivation of the quantitative relationship between η and Δt is time-consuming. The derivation is presented in the Appendix 4.4. Here, only the result is given:

(14) $$\eta = K\,\rho\,\Delta t$$

Here, ρ is the density of the liquid and K a meter constant of the viscometer employed, into which the passed volume V enters among others (Fig. 5).

For the *kinematic viscosity* $\upsilon = \eta/\rho$ with the unit $[\upsilon] = m^2/s$ we obtain:

(15) $$\upsilon = K\,\Delta t$$

In Eq. (14) and (15), a correction has to be made. Upon entering the narrow capillary from the large basin B (Fig. 5) of the capillary viscometer, the liquid must be accelerated according to BERNOULLI's law[7]. The required work leads to a small drop in pressure, which causes an increase in the efflux time Δt. Therefore, time correction terms t_k are to be subtracted from the measured times Δt (HAGENBACH*'s correction*), which are supplied by the manufacturers of the UBBELOHDE viscometers as *meter constants*. Hence, the final equation for determining the kinematic viscosity reads:

(16) $$\upsilon = K\left(\Delta t - t_k\right)$$

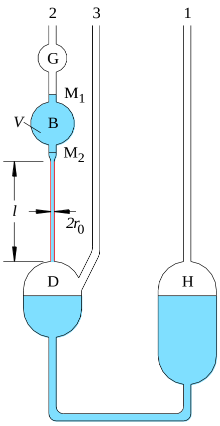

Fig. 5: Capillary viscometer according to Ubbelohde. During the time Δt the volume V flows through the capillary (red) of radius r_0 and length l. For further labels, refer to Chap. 3.2 and Appendix 4.4.

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[6] Eq. (13) is an empirically discovered law for the experimental setup used. The correction by LADENBURG (c.f. /2/, for example), which is used frequently, yields distinctly worse results for this experimental setup.

[7] DANIEL BERNOULLI (1700 – 1782).

2.3 Laminar and Turbulent Pipe Flows

Fig. 6 shows a setup that can be used to measure the transition of a laminar flow to a turbulent one in a cylindrical pipe[8]. A liquid, water in this case, flows through a long plexiglass pipe of bore diameter d. The water flows into the pipe from a reservoir. The reservoir is replenished though an inlet (a faucet). An overflow is used to maintain a constant water level in the reservoir, so that the pressure at the pipe's inlet remains constant. A sieve serves to calm the incoming water-flow. The velocity v of the flow can be adjusted by the faucet H_1 at the end of the pipe.

In addition to the water from the reservoir, a thin jet of coloured water is injected through a nozzle into the centre of the pipe. The strength of the current through the nozzle can be adjusted by a faucet H_2.The jet of coloured water is seen as a smooth *stream filament* for small flow velocities v. If the flow velocity is increased slowly by opening the faucet H_1, the current filament will begin to exhibit eddies starting at a certain velocity v_t, thus showing the transition from a laminar to a turbulent flow. By measuring the volume of flow per time in this setting of the faucet H_1, the flow velocity v_t can be determined and the corresponding REYNOLDS number Re may be calculated:

$$Re = \frac{\rho_w \, v_t \, d}{\eta_w} \qquad (17)$$

where ρ_w and η_w are the density and viscosity of the water.

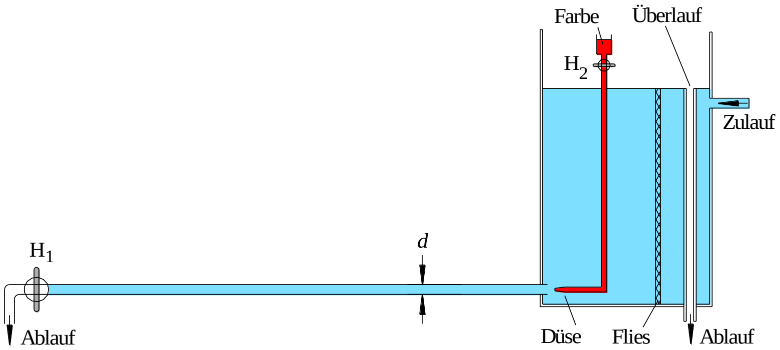

Fig. 6: Setup for measuring the transition of a laminar to a turbulent flow in a pipe having inner diameter d. For details, refer to the text.

Details for the quantitative description of the water flow through a pipe are given in Appendix 4.3.

3 Experimental Procedure

Equipment:

Six glass cylinders with different diameters in adjustable stand with water level, seals for the cylinders with a drill hole in their centres, 2 l vessel containing glycerine-water mixture, steel balls (about 100 balls with $d \approx 2$ mm), forceps, analytical balance (precision 0.001 g), laboratory balance (precision 0.01 g), micrometer gauge, slide measuring gauge, stop watch, thermometer (precision 0.1 C), magnet, UBBELOHDE viscometer ($K \approx 10^{-8}\,m^2/s^2$) in mounting and water bath, suction tube, ethanol, mounted flow tube ($d = (12.10 \pm 0.05)$ mm) with water reservoir, water with food color, measuring cylinder (100 ml and 1000 ml), bucket, floor cloth, kitchen paper roll.

3.1 Determining the Viscosity of a Glycerine-Water Mixture by Means of the Falling-Ball Method

Remarks:

- *The mixing ratios of the glycerine-water mixtures are not identical in all experiments. Since the viscosity strongly depends on the mixing ratio and the temperature (cf. Chap. 4.1 and 4.2), every working group must perform the whole experiment at constant room temperature using the mixture from one storage basin!*
- *The work place has to be tidied before leaving!*

Using a set-up as shown in Fig. 7 the falling movement of steel balls ($d = 2r \approx 2$ mm) in a glycerine-water mixture is to be investigated with the aim of determining the viscosity of the mixture according to Eq.(11)

[8] By recommendation of A. HEIDER, DEUTSCHES ZENTRUM FÜR LUFT- UND RAUMFAHRT (DLR), Göttingen.

. In order to quantify the influence of frictional effects at the wall of the basin (cf. Eq. (13)), various glass cylinders with different radii *R* are used.

The experiment is prepared by determining the following quantities:

(a) Density ρ_F of the mixture using a laboratory balance by weighing a volume determined with a measuring cylinder.
(b) Mean radius *r* of the balls by measuring the diameters of at least ten balls using the micrometer gauge and subsequently calculating the mean value.
(c) Density ρ_K of the ball material by weighing *n* balls ($n \geq 100$) by means of an analytical balance.
(d) Radii *R* of the glass cylinders used by measuring the inner diameter using a slide measuring gauge.
(e) Temperature of the mixture. Since the viscosity strongly depends on the temperature, it is only reasonable to record a result if the temperature of the mixture is simultaneously recorded. Keep room temperature as constant as possible during the experiment!

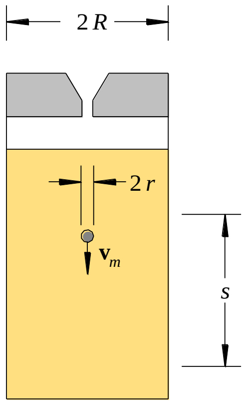

Fig. 7: Set-up of falling-ball experiment in liquids. The spheres of radius *r* fall through the upper drilled seal (grey). This is to ensure that they fall into the liquid (beige), contained in a cylinder with bore radius *R*, as close to the centre of the cylinder as possible.

Following the above preparations, the glycerine-water mixture is carefully filled into six glass cylinders with different radii *R* (avoid bubble formation!). The fluid level must be below the bottom of the seal[9]. Subsequently the cylinders are fixed in the mount (by careful tightening of the plastic screws) and the base plate of the mount is vertically adjusted using an integrated water level. The cylinders are now positioned vertically. Then ten balls are dropped centrally into each cylinder. For centring, a suitable seal with a drill-hole in its centre is used (Fig. 7). The time *t* it takes to fall a distance *s*, which is determined by the two horizontal mounting brackets on the cylinders (*s* measured with the caliper), is measured using a stop watch. The start of the distance *s* (upper mounting bracket) must be a few centimetres below the surface of the liquid.

Question 2:
- Why should the distance of the fall not begin at the surface of the liquid?

Question 3:
- Why is it important to drop the balls into the centre of the cylinder?

For every glass cylinder the sinking velocity

$$(18) \qquad v_m = \frac{s}{\bar{t}}$$

is determined, $\bar{t}$ being the mean value of the measured time for falling for each of the ten balls. Subsequently, v_m is plotted against $(r/R)^2$ (with error bars for v_m) and a regression line through the measured

[9] This work step has usually been prepared in advance by the technical assistance.

data is determined. The intersection of the line with the v_m-axis determines the velocity v_0 for an infinitely extended liquid ($R \rightarrow \infty$).
From the data obtained by this experiment for v_0, r, ρ_K, and ρ_F as well as from the gravitational acceleration for Oldenburg (g = 9.8133 m/s^2, error negligible[10]), the viscosity η of the glycerin-water mixture is determined according to Eq. (11) and the result is compared with the data from Table 1 (Chap. 4.1).

After completion of the measurement the liquid is carefully (again preventing the formation of bubbles!) poured from the glass cylinders back into the storage vessel. Towards the end, the balls are held back with a magnet. A sieve is used to catch the balls if necessary. The balls remaining in the glass cylinders are removed with a magnet. The balls are cleaned in water and dried **well** with crepe paper (otherwise there is a risk of rusting!).

3.2 Determining the Kinematic Viscosity Using a Capillary Viscometer

The kinematic viscosity of ethanol at room temperature is to be determined using a capillary viscometer according to UBBELOHDE. The viscometer is placed into a large water bath which provides a constant temperature (to be measured!) within the capillary throughout the experiment. Prior to the experiment the technical assistant adjusted the viscometer vertically and filled the storage basin H above tube 1 (cf. Fig. 5) with ethanol up to about three quarters.

Tube 3 is closed with a finger. By means of a suction tube connected with tube 2 the liquid is sucked up into tube 2 with a provided Peleus ball until the overhead flask G is completely filled. Then tubes 2 and 3 are opened and the time Δt during which the liquid level drops from mark M_1 to mark M_2 is determined. The measurement is repeated three times. The kinematic viscosity υ of ethanol for the predominant temperature of the water bath is determined from the measured values and the available meter constants K and t_k, and is then compared with values stated in the literature.

3.3 Determination of the REYNOLDS Number at the Transition from Laminar to Turbulent Pipe Flow

With a setup according to Fig. 6 the REYNOLDS number at the transition from a laminar to a turbulent pipe flow is to be determined. At first, the faucet controlling the flow to the reservoir is opened far enough, ensuring that the water level in the reservoir is just maintained at the level of the overflow during the whole experiment. The tube from the drain of the pipe is placed in the outlet. The faucet H_1 at the end of the pipe is opened slowly until water flows from the end of the pipe. At small flow velocities, the flow in the pipe is laminar. Next, the faucet H_2 is opened far enough, so that a thin, straight stream filament becomes visible. Now, the faucet H_1 is slowly opened further until the laminar flow changes to a turbulent one. This can be observed by the stream filament beginning to "jitter".

To measure the flow velocity v for the current position of the faucet H_1, a measuring cylinder is placed underneath the output tube of the pipe for a time interval Δt (measured with a stop watch) in order to capture the water. From the volume V, the (inner) diameter d of the pipe, the density[11] and the viscosity η of the water (c.f. Appendix 4.2), the flow velocity v, and thus, the REYNOLDS number R_e may be determined.

4 Appendix

4.1 Viscosity of Glycerine

Glycerine[12] ($C_3H_8O_3$) is *hygroscopic*, i.e. it adsorbs water. If it is exposed to the air for some time, it absorbs humidity from the ambient air. Thus a mixture with an increasing water content is formed over the course of time. The viscosity of such a mixture differs from that of pure glycerine. For your guidance some data for a temperature of 20 °C are stated in Table 1:

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[10] Value taken from http://www.ptb.de/cartoweb3/SISproject.php (15.10.18); the error of 2×10^{-5} m/s^2 is neglected.
[11] For the temperature-dependent density of water, see the experiment "*Surface tension…*".
[12] Further common names for glycerine are glycerol, propan-1,2,3-triol and others. The structure is described by $C_3H_5(OH)_3$.

Table 1: Viscosity of glycerine/water mixtures at 20 °C.[13]

$C_3H_8O_3$ weight-%	H_2O weight-%	η / kg $m^{-1}s^{-1}$
100	0	1,76
96	4	0,761
92	8	0,354
88	12	0,130
84	16	0,071
80	20	0,048

In addition, the viscosity is strongly dependent on temperature. For pure glycerine at T = 20 °C: η = 1,76 kg/(m s) (see above) and at T = 25 °C: η = 0,934 kg/(m s)[14]

4.2 Viscosity of Water

Fig. 8 shows the viscosity η of water as a function of temperature T. In the temperature interval between 10 °C and 35 °C, the data can be described in good approximation by a polynomial of degree 4 (T in °C)[13]:

(19) $$\eta \approx \begin{pmatrix} 1{,}77721 - 0{,}05798\{T\} + 0{,}00125\{T\}^2 \\ -1{,}66039 \cdot 10^{-5}\{T\}^3 + 9{,}814 \cdot 10^{-8}\{T\}^4 \end{pmatrix} 10^{-3}\ \frac{\text{kg}}{\text{m s}}$$

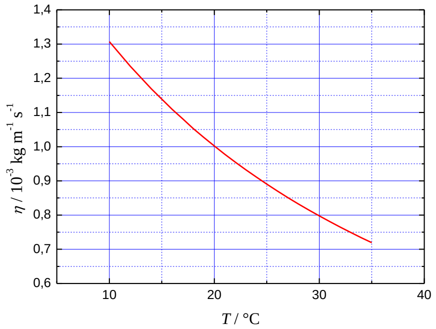

Fig. 8: Viscosity η of water as a function of temperature T.

4.3 Laminar Pipe Flow

This appendix details how the flow velocity v and its lateral profile $v(r)$ in a cylindrical pipe may be calculated quantitatively.[15]

An *ideal* liquid is *incompressible* and *free of internal friction forces*. We consider (as shown in Fig. 9) such a liquid flowing through a tapering horizontal tube. The *incompressibility* of the liquid means that the *rate of volume flow Q* (flowing volume per time) must be identical everywhere in the tube. A_1 is the cross-sectional area of the tube and v_1 the flow velocity on the left side of the tube, A_2 and v_2 are the corresponding quantities on the right side. This means:

(20) $$Q = \frac{\Delta V}{\Delta t} = A_1 v_1 = A_2 v_2 = \text{const.}$$

Eq. (20) is called the *continuity equation*.

To move a fluid volume ΔV from the left side of the tube to the right side by Δx_1, the work W_1 has to be performed by the *static pressure* p_1 on the left side:

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[13] Data after: WEAST, R. C. [Ed.]: „CRC Handbook of Chemistry and Physics", 56th Ed., CRC Press, Boca Raton, 1975 - 1976. All data without error statement.

[14] LIDE, D. R. [Ed.]: "CRC Handbook of Chemistry and Physics on CD-ROM", Taylor & Francis, Boca Raton, FL, 2006. All data without error statement.

[15] r is the lateral distance from the axis along the pipe.

(21) $$W_1 = F_1\,\Delta x_1 = p_1 A_1 \Delta x_1 = p_1 \Delta V$$

The required work W_2 for moving the same volume ΔV through the right side of the tube against the static pressure p_2 is given by:

(22) $$W_2 = F_2\,\Delta x_2 = p_2 A_2 \Delta x_2 = p_2 \Delta V$$

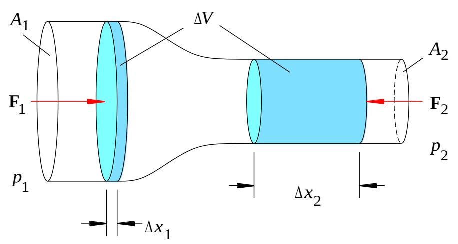

Fig. 9: Flow through a tapering horizontal tube. See text for labels.

From the law of conservation of energy it follows that the energy difference $W_1 - W_2$ must lead to an increase in the kinetic energy of the liquid (density ρ) on the right side of the tube. If m is the mass and v_1, v_2 are the velocities of the volumes ΔV, it follows:

(23) $$W_1 - W_2 = p_1\,\Delta V - p_2\,\Delta V = \frac{1}{2} m\, v_2^2 - \frac{1}{2} m\, v_1^2 = \frac{1}{2}\rho\,\Delta V\, v_2^2 - \frac{1}{2}\rho\,\Delta V\, v_1^2$$

Following division by ΔV and resorting the terms we finally obtain:

(24) $$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2 \;\coloneqq\; p + \frac{1}{2}\rho v^2 = \text{const.}$$

which is BERNOULLI*'s law*. It says that according to the stated assumptions, the sum of the *static pressure* p and *dynamic pressure* $\frac{1}{2}\rho v^2$ must be constant everywhere in the tube.

The *hydrostatic pressure* ρgh which depends on the height h has to be considered if we have a *vertical* tube instead of a horizontal one (g is the gravitational acceleration). Then BERNOULLI's law reads:

(25) $$p + \frac{1}{2}\rho v^2 + \rho g h = \text{const.}$$

Pressure and flow velocity are constant in the entire horizontal tube if the tube's diameter is constant and if an *ideal* liquid flows through it. In the case of a *real* fluid with the viscosity η friction forces appear between the fluid and the material of the tube and between the neighbouring fluid layers. These friction forces cause the pressure along the tube to decrease and the flow velocity to vary along the tube's cross-section, thus in the lateral direction. It must be zero on the boundary (because a boundary layer of the liquid adheres to the wall) and must assume its maximum value in the centre.
For a quantitative description of the transverse *velocity profile* of a *laminar* tube flow we consider (as in Fig. 10) a cylindrical tube with the length l and the radius r_0 in which a real fluid flows along the z-axis. Within this flow we observe a co-axial liquid cylinder with radius r and lateral surface area $A = 2\pi rl$. According to NEWTON*'s law of friction* the friction force F_R between this liquid cylinder and the adjacent layer of liquid is proportional to the lateral surface area A and to the velocity gradient dv/dr. The proportionality constant is the viscosity η:

(26) $$F_R = \eta\, A \frac{dv}{dr} = 2\pi\, r\, l\, \eta \frac{dv}{dr}$$

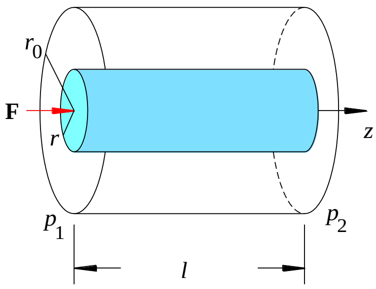

Fig. 10: Cylindrical tube with co-axial liquid cylinder of radius r. Pressure p_1 is on the left side; pressure p_2 is on the right side. For other labels, please refer to the text.

In the stationary case (temporally constant flow velocity) the friction force F_R for a fluid cylinder with the radius r must just equal the driving force F caused by the pressure gradient $\Delta p = p_1 - p_2$, thus:

(27) $$F = \pi r^2 \Delta p = 2\pi r l \eta \frac{dv}{dr}$$

From this we obtain:

(28) $$\frac{dv}{dr} = \frac{\Delta p}{2\eta l} r$$

and

(29) $$\mathrm{d}v = \frac{\Delta p}{2\eta l} r \, \mathrm{d}r$$, respectively,

and finally by the integration with the boundary condition $v(r_0) = 0$ the desired *velocity profile* $v(r)$:

(30) $$v(r) = \frac{\Delta p}{4\eta l}\left(r_0^2 - r^2\right); \quad 0 \le r \le r_0$$

Thus, the transverse velocity profile of a laminar flow through a tube is *parabolic* (cf. Fig. 11).

For calculating the volume V, which flows through the cylinder with a radius r_0 during the time Δt, we first look at the volume flow $\Delta V/\Delta t$ through a hollow cylinder with an inner radius r and an outer radius $r + \mathrm{d}r$ (cf. Fig. 12). This hollow cylinder has the cross sectional area A and length Δl. For a small wall thickness dr, the volume flow is given by:

(31) $$\frac{\mathrm{d}V}{\Delta t} = \frac{A \, \Delta l}{\Delta t} = 2\pi r \, \mathrm{d}r \, \frac{\Delta l}{\Delta t}$$

Since the liquid flows through the tube uniformly (i.e. without acceleration), it follows that the velocity is:

(32) $$v = \frac{\Delta l}{\Delta t}$$

Applying Eq. (30), Eq. (31) thus becomes:

(33) $$\frac{\mathrm{d}V}{\Delta t} = 2\pi r \, \mathrm{d}r \, v(r) = 2\pi r \frac{\Delta p}{4\eta l}\left(r_0^2 - r^2\right)\mathrm{d}r$$

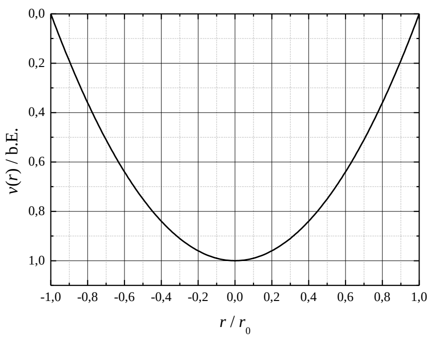

Fig. 11: Left: Calculated parabolic velocity profile of a laminar flow through a cylindrical pipe with the radius r_0. Right: Visualization of a parabolic velocity profile in a cylindrical plexiglass pipe (diameter about 1 cm) with the aid of coloured glue.[16]+

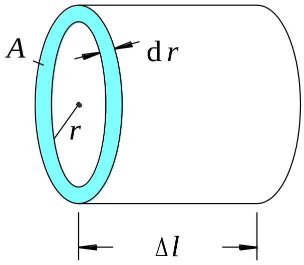

Fig. 12: Definition of the geometrical dimension of a hollow cylinder.

From this equation, by integration, we may calculate the *net volume V* flowing through the tube with the radius r_0 within the time Δt:

(34) $$\frac{V}{\Delta t} = \frac{\pi \Delta p}{2\eta l} \int_0^{r_0} \left(r_0^2 - r^2\right) r\, dr$$

and thus

(35) $$V = \frac{\pi \Delta p \Delta t}{8\eta l} r_0^4$$

This is the HAGEN-POISEUILLE[17] *law* for *laminar* flows. They appear when REYNOLDS *number Re*, here:

(36) $$Re = \frac{\rho \bar{v} 2 r_0}{\eta}$$ (ρ: density of the liquid; $\bar{v}$: mean value of the flow velocity according

to Eq. (30) is smaller than about 2,000 - 2,500.

4.4 Capillary Viscometer

In this appendix, the derivation of Eq. (14) is explained.

By means of the HAGEN-POISEUILLE law (Eq. (35)) we are able to determine the viscosity of fluids. For this we use a *capillary viscometer*. Fig. 5 shows a *capillary viscometer* according to UBBELOHDE. A defined volume of liquid *V* flows from a storage basin B with an overhead flask G through a capillary with the radius r_0 and the length *l*. The volume of liquid is defined by the volume between the marks M_1 und M_2. The viscosity η can be determined using Eq. (35) by measuring the time difference Δt in which the liquid mirror surface drops from M_1 to M_2 :

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[16] Source: T. GREVE: „Aufbau und physikalische Betrachtung eines Durchlaufreaktors zur Hydrothermalen Karbonisierung“, Diploma thesis, Carl von Ossietzky Universität Oldenburg, Institut für Physik, AG Turbulenz, Windenergie und Stochastik (TWiST), 2009.

[17] GOTTHILF HEINRICH LUDWIG HAGEN (1797 – 1884)

(37) $$\eta = \frac{\pi \, \Delta p \, r_0^4}{8 l V} \Delta t$$

In this case, the pressure difference Δp is given by the hydrostatic pressure:

(38) $$\Delta p(t) = \rho \, g \, h(t)$$ (ρ: density of liquid; g: gravitational acceleration)

$h(t)$ being the altitude difference between the momentary level of the liquid mirror surface in the basin and the lower end of the capillary. We use a trick so that the lower end of the capillary represents the reference level: by means of the ventilation tube 3 (cf. Fig. 5) there is air pressure in the overhead flask D. Consequently the liquid flows off in the form of a *thin film* along the *inner wall* of D.

Due to the time-dependence of the height $h(t)$ (dropping liquid mirror surface), $\Delta p(t)$ is also time-dependent. However, $h(t)$ may be replaced by a suitable mean value. This *mean height h* is given by:

(39) $$h = \frac{1}{\Delta t} \int_0^{\Delta t} h(t) \, dt$$

Hence it follows from Eq. (37):

(40) $$\eta = \frac{\pi \, \rho \, g \, h \, r_0^4}{8 l V} \Delta t$$

The quantity

(41) $$K = \frac{\pi \, g \, h \, r_0^4}{8 l V}$$

is a *meter constant* and is engraved into the viscosimeter ($[K] = m^2/s^2$; mostly given in mm^2/s^2). This yields the simple relationship for the viscosity following Eq.(14):

(42) $$\eta = K \, \rho \, \Delta t$$