Advanced Wind Energy
David Bastine

Fluid Dynamics

Lehrbrief

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Fluid Dynamics

by

David Bastine

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2 — Equations of Motion

Learning objectives

After processing this chapter and the corresponding exercise sheet you will be able to

- illustrate the basic ideas concerning the derivation of the fundamental fluid dynamic equations based on the conservation of energy, momentum and mass.
- enumerate the assumptions necessary to deduce the Navier-Stokes equations.
- interpret the different terms in the equations of motion particularly in the Navier-Stokes equations.
- deduce and apply the Bernoulli equation correctly.

Welcome to the second chapter! No boring warm up this time! Let us directly dive in and assume we have a fluid whose thermodynamic state is completely described by the state variables ρ, p, T, e which are related by known equations of state Eq. (1.7). If the fluid is in motion, its flow can be described by the velocity field $\mathbf{u}(\mathbf{x}, t)$ and the state variables also become field quantities $\rho(\mathbf{x}, t), p(\mathbf{x}, t), T(\mathbf{x}, t), e(\mathbf{x}, t)$, as described in the previous chapter. Our goal in this chapter is to find the dynamic equations whose solution yield the spatial dependence and temporal evolution of $\mathbf{u}, \rho, p, T, e$. These equations will be based on the fundamental conservation principles of physics: the conservation of mass, the conservation of momentum and the conservation of energy. It would be possible to deduce these equations only in the framework of our infinitesimal fluid particle Patty moving through the flow. However, we start our derivations considering a finite material volumes fixed in space and later moving with the fluid, since this leads to a more mathematically rigorous derivation. Furthermore, this way also yields the integral formulation of the equations which is important e.g. in numerical approaches such as the finite volume approach introduced in the CFD-module of your study programme.

Understanding the derivations might not be easy for some of you, especially if you have problems with the mathematical formulations using integrals andor tensors, but it can be pretty enlightening and in the end waits the rewarding feeling of not blindly believing every fundamental equation you get thrown at you. So I think, it is worth the effort. However, if you do not get every step, it is not the end of the world. In this case you should make sure that you can at least interpret the mathematical terms in the final equations and are aware of their limitations. In other words, you should know in which cases the equations should not be used since assumptions made for their derivations are violated.

2.1 Conservation Laws

2.1.1 Conservation of Mass

The conservation of mass states that no mass can be spontaneously created or destroyed. For an arbitrary fixed volume of fluid this means that the rate of change of mass in this volume equals the net flux of mass out and into this volume. Mathematically, we can express this by

$$\frac{d}{dt} \int\limits_V \rho(\mathbf{x}, t) dV = - \int\limits_{\partial V} \rho \mathbf{u} d\mathbf{A} \ , \tag{2.1}$$

where the left hand side is rate of change of mass in the volume V and the surface integral on the right hand side is the net flux of mass. Due to the linearity of the integral, we can move $\frac{d}{dt}$ into the integral. Furthermore, we can apply the Gauss theorem, which you know from you first exercise sheet, yielding

$$\int\limits_V \partial_t \rho(\mathbf{x}, t) dV = - \int\limits_V \nabla \cdot (\rho \mathbf{u}) dV \tag{2.2}$$

Since this integral equation holds for every volume V we can conclude that the integrands are equal[1]. This yields

$$\boxed{\partial_t \rho = -\nabla \cdot (\rho \mathbf{u})} \ . \tag{2.3}$$

[1] If we want to be more mathematically precise, we additionally have to assume that the integrands are at least piece-wise continuous.

With a little bit of algebra (you can do it!), we can also write down an alternative form of this equation given by

$$\boxed{\frac{1}{\rho}\frac{D\rho}{Dt} = -\nabla \cdot \mathbf{u}}\,. \tag{2.4}$$

Eq. (2.3) and (2.4) are are both called **continuity equation** and represent the differential form of the law of mass conservation. The continuity equation is one of the fundamental equations of fluid dynamics. The formulation in Eq. 2.4 can easily be interpreted in a Lagrangian way saying what happens to the density of little Patty. The rate of change of density of Patty moving through the flow equals the negative divergence of the velocity field times the density itself.

Additional knowledge 2.1 — Alternative Derivation of the continuity equation. Actually, we knew Eq. 2.4 already from Eq. (1.25). We know that a fluid particle has a constant infinitesimal mass δm and from Eq. (1.25) we know the change volume and thus we know the change of density. More precisely, when starting with the right hand side of Eq. (1.25) we obtain

$$\nabla \cdot \mathbf{u} = \frac{1}{\delta V}\frac{D}{Dt}\delta V = \frac{\delta m}{\delta V}\frac{D}{Dt}\frac{\delta V}{\delta m} = \rho\frac{D}{Dt}\left(\frac{1}{\rho}\right) = -\frac{1}{\rho}\frac{D}{Dt}\rho \tag{2.5}$$

where we used the chain rule of differentiation for the last equality.

2.1.2 Conservation of Momentum

In order to apply the conservation of momentum, it is much more convenient to use a moving instead of a fixed volume of fluid. The volume is a material volume as introduced in Sec. 1.5.3. If you like, you can imagine it as Patty's bigger sister Veronica, Veronica Volume. Remember that the mass of such a volume is constant by definition since the surface of the volume moves with the fluid. The conservation of momentum is given by Newton's second law. Thus, the volume simply moves according to this law changing its momentum $\mathbf{p}$ only through the application of external forces to the volume.[2] This is expressed by

$$\frac{d}{dt}\mathbf{p} = \mathbf{F}\,. \tag{2.6}$$

The momentum $\mathbf{p}$ of the volume can be obtained by integrating over the momentum per unit volume $\rho\mathbf{u}$. yielding

$$\frac{d}{dt}\int\limits_{V(t)} \rho\mathbf{u}\,dV = \mathbf{F}\,. \tag{2.7}$$

Reynolds Transport Theorem

Let us start with the left hand side (l.h.s.) of Eq. (2.7). In order to rewrite this term, we have to apply a general theorem concerning integrals over time-dependent volumes. Let f be an arbitrary quantity per unit volume, such as ρu_1 or the internal energy per unit volume ρe. Then the rate of

[2] Do not mistake the momentum with the scalar p we use for the pressure.

change of the corresponding integral quantity, for example the energy in the volume, is given by

$$\frac{d}{dt}\int\limits_{V(t)} f(\mathbf{x},t)dV = \int\limits_{V(t)} \partial_t f(\mathbf{x},t)\,dV + \int\limits_{\partial V(t)} f(\mathbf{x},t)\mathbf{u}\,d\mathbf{A}\;. \tag{2.8}$$

This the **Reynolds transport theorem**. The first term on the r.h.s. is the change caused by the implicit change of f in the volume. The second integral sums up the change caused by the motion of the surface ∂V with the flow $\mathbf{u}$ in the field f. It is thus simply the change caused by the motion of the volume in the field f, which would also occur when f itself is constant in time. Using the continuity Eq. (2.4), it is easy to deduce a special form of the Reynolds transport theorem given by

$$\frac{d}{dt}\int\limits_{V(t)} \rho(\mathbf{x},t)\tilde{f}(\mathbf{x},t)dV = \int\limits_{V(t)} \rho(\mathbf{x},t)\frac{D}{Dt}\tilde{f}(\mathbf{x},t)dV\;, \tag{2.9}$$

where $\tilde{f}$ is now any quantity per unit mass. The time derivative only acts on $\tilde{f}$ and not on ρ which has important consequences for the form of the fluid dynamical equations, as will be commented on in the following section.

Using Eq. (2.9) (component-wise), the l.h.s. of Eq. (2.7) can now be expressed as $\int\limits_{V(t)} \rho\frac{D\mathbf{u}}{Dt}dV$.

Types of Forces in a Fluid

Let us now have a closer on the r.h.s. of Eq. (2.7) an thus on the forces $\mathbf{F}$ acting on the volume. $\mathbf{F}$ can be divided into three different kinds of forces: *body forces*, *surface forces* and *line forces*. Thus, we can decompose $\mathbf{F}$ into

$$\mathbf{F} = \mathbf{F}_{\text{body}} + \mathbf{F}_{\text{surface}} + \mathbf{F}_{\text{line}}\;. \tag{2.10}$$

Body forces are forces, which act on the fluid without any physical contact arising somehow from "action at a distance". Usually they result from putting an object or in our case a volume of a fluid in a force field. This field can be gravitational, magnetic, electrostatic or electromagnetic. In this course, we consider only the gravitational force given by

$$\mathbf{F}_{\text{body}} = \mathbf{F}_G := -m_V\mathbf{g} = -\int\limits_{V(t)} \rho(\mathbf{x},t)\mathbf{g}dV\;, \tag{2.11}$$

where $\mathbf{g} := (0,0,g)^T$ and g is the acceleration caused by gravitational force field of the earth.

Surface forces are exerted on our volume through direct contact and thus through forces which act directly on the surface of our volume of fluid. They are proportional to the area of the surface and thus are usually given per unit area. The entire surface force on our volume can be obtained by integrating all surface force vectors $\mathbf{t}$ over the surface area yielding

$$\mathbf{F}_{\text{surface}} = \int\limits_{A(t)} \mathbf{t}dA\;. \tag{2.12}$$

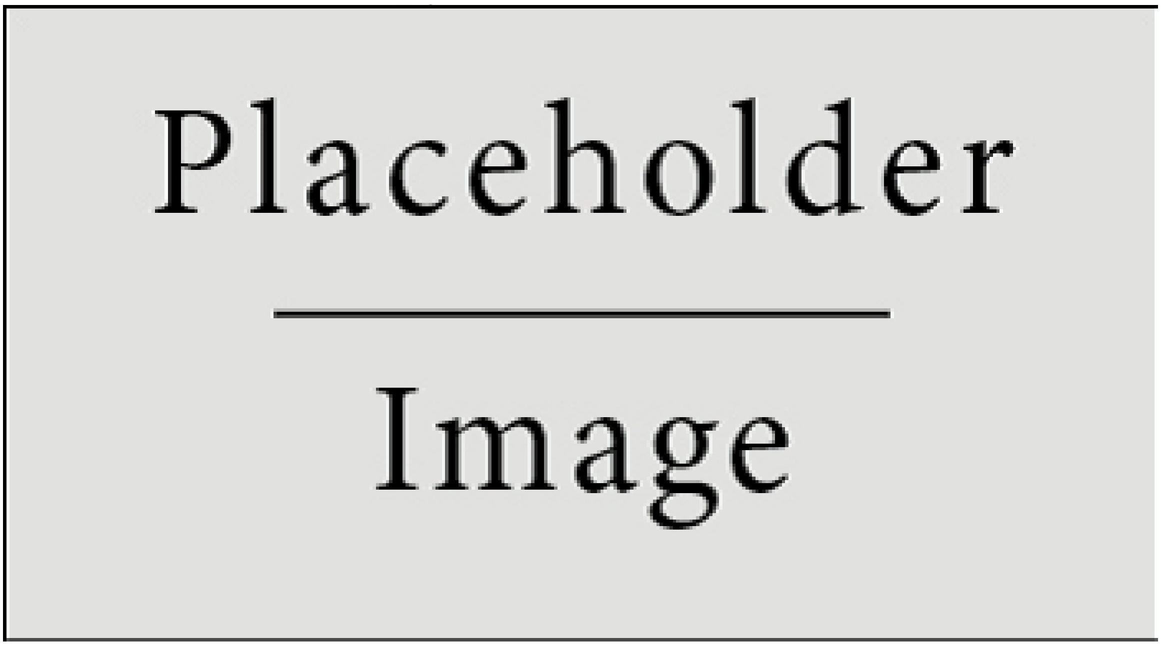

Figure 2.1: Illustration of the stress tensor taken from [**Kundu2008**].

The surface force vector $\mathbf{t}(\mathbf{x},\mathbf{n})$, also called stress vector, depends on the position $\mathbf{x}$ in the flow and the orientation of the surface, it is acting on. The orientation of the surface is given by its local normal vector $\mathbf{n}$. Let us now consider the stress $\mathbf{t}$ for an infinitesimal cubic volume at a point $\mathbf{x}$. Based on the principle of stress equilibrium, following e.g. [1], one can show that it is enough to know $\mathbf{t}(\mathbf{x},\mathbf{n})$ for $\mathbf{n} = \mathbf{e}_i\ ,\ i \in \{1,2,3\}$, in order to calculate t for every surface orientation. We can express this necessary information in a matrix

$$\boldsymbol{\tau}(\mathbf{x}) = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} := (\mathbf{t}(\mathbf{x},\mathbf{e}_1),\mathbf{t}(\mathbf{x},\mathbf{e}_2),\mathbf{t}(\mathbf{x},\mathbf{e}_3))^T \text{ or } \tau_{ij} := t_j(\mathbf{x},\mathbf{e}_i)\ . \tag{2.13}$$

Thus, $\tau_{23}(\mathbf{x})$, for example, represents the third component of the surface force vector on a surface with the normal vector $\mathbf{e}_2$, as illustrated in Fig. 2.1. For an arbitrary surface orientation, $\mathbf{t}(\mathbf{x},\mathbf{n}$ can then be obtained by

$$\mathbf{t}(\mathbf{x},\mathbf{n}) = (\mathbf{n}^T\boldsymbol{\tau}(\mathbf{x}))^T \tag{2.14}$$

following [1]. You can easily check for yourself that inserting e.g. $\mathbf{e}_1$ for $\mathbf{n}$ yields the transpose of the first row of $\boldsymbol{\tau}$ and thus $\mathbf{t}(\mathbf{x},\mathbf{e}_1)$, which is a consistent result. $\boldsymbol{\tau}$ is a second order tensor which will be called stress tensor in the following. Applying the principle of conservation of angular momentum, one can show that $\boldsymbol{\tau}$ has to be a symmetric tensor fulfilling $\boldsymbol{\tau}^T = \boldsymbol{\tau} \Leftrightarrow \tau_{ij} = \tau_{ji}$. [3](see e.g. [1] or [2]). Inserting Eq.(2.14) in Eq. (2.12) and using $\boldsymbol{\tau}^T = \boldsymbol{\tau}$ leads to

$$\mathbf{F}_{\text{surface}} = \int\limits_{A(t)} (\mathbf{n}^T\boldsymbol{\tau})^T dA = \int\limits_{A(t)} \boldsymbol{\tau} d\mathbf{A} = \int\limits_{V(t)} \nabla\cdot\boldsymbol{\tau} dV\ , \tag{2.15}$$

[3]This only true in the absence of so called *body couples*. In a *Ferrofluid*, for example, magnetic dipoles exist, which contribute to the balance equation of angular momentum[1].

where $V(t)$ is the volume enclosed by $A(t)$. To obtain the last equality, you have to apply the Gaussian theorem (component-wise) and use the definition $(\nabla \cdot \boldsymbol{\tau})_i := \partial_j \tau_{ij}$. Try for yourself!

Line forces act along a line and in fluid flows are given by surface tension forces. They appear at the interface of immiscible fluids and enter the fluid dynamical equations only in the boundary conditions. Since this course does not cover fluid interfaces, line forces will not be discussed any further and neglected in the deduced equations.

Cauchy's Equation of motion

Using the deduced expressions for body (Eq. (2.11)) and surface force (Eq. (2.15)), Newton's second law $\frac{d}{dt} \int_{V(t)} \rho \mathbf{u}\, dV = \mathbf{F}$ becomes

$$\frac{d}{dt} \int_{V(t)} \rho \mathbf{u}\, dV = \int_{V(t)} \rho(\mathbf{x}, t) \mathbf{g} dV + \int_{A(t)} \nabla \cdot \boldsymbol{\tau} dV \, . \tag{2.16}$$

If we know apply the variation of Reynold's transport theorem Eq. (2.9) to the l.h.s. and combine the integrals on the r.h.s., we obtain

$$\int_{V(t)} \rho \frac{D}{Dt} \mathbf{u}\, dV = \int_{V(t)} \rho(\mathbf{x}, t) \mathbf{g} + \nabla \cdot \boldsymbol{\tau} dV \, . \tag{2.17}$$

Since this equation is valid for every Volume $V(t)$, the integrands must be equal and we obtain

$$\boxed{\rho \frac{D}{Dt} \mathbf{u} = \rho \mathbf{g} + \nabla \cdot \boldsymbol{\tau}} \, , \tag{2.18}$$

which is called **Cauchy's Equation of motion**. It is basically Newtons second law formulated for a continuous medium. Coming back to Patty, we can say that her velocity (momentum per unit mass) changes with the rate $\frac{D\mathbf{u}}{Dt} = -\mathbf{g} + \frac{1}{\rho} \nabla \cdot \boldsymbol{\tau}$.

Further interpretation of the term $\nabla \cdot \boldsymbol{\tau}$ will be first given in Sec. 2.1.4.

2.1.3 Conservation of Energy

Let us now reformulate the first law of thermodynamics (Eq. (1.1)) for our moving material volume (Veronica). In addition to the internal (thermodynamical) energy of the volume, we now have to include its kinetic energy since it is moving through the flow. Thus, energy conservation can be expressed as

$$\frac{D}{Dt} \int_{V(t)} \frac{1}{2} \rho \mathbf{u} \cdot \mathbf{u} + \rho e \, dV = \begin{pmatrix} \text{rate of work} \\ \text{done on the volume} \\ \text{by external forces} \end{pmatrix} + \begin{pmatrix} \text{flux of internal energy} \\ \text{through the surface} \\ \text{of the volume} \end{pmatrix} . \tag{2.19}$$

This can be written as

$$\frac{D}{Dt} \int_{V(t)} \frac{1}{2} \rho u^2 + \rho e \, dV = \int_{V(t)} \rho \mathbf{g} \cdot \mathbf{u} \, dV + \int_{A(t)} \mathbf{t} \cdot \mathbf{u} \, dA - \int_{V(t)} \mathbf{q} \, d\mathbf{A} \, . \tag{2.20}$$

The first term on the r.h.s. represents the work done by the gravitational force while the second term is the work done by surface forces. The last term represents the heat transported in and out of the volume, where $\mathbf{q}$ is the heat flux. Using Eq. (2.9) , Eq. (2.14) and the Gauss theorem, we obtain

$$\int\limits_{V(t)} \rho \frac{D}{Dt}(\frac{u^2}{2} + e) dV = \int\limits_{V(t)} \rho \mathbf{g}\mathbf{u} + \nabla \cdot (\boldsymbol{\tau}\mathbf{u}) - \nabla \cdot \mathbf{q}\, dV \,. \tag{2.21}$$

Since this is again valid for every volume $V(t)$, this yields

$$\rho \frac{D}{Dt}(\frac{u^2}{2} + e) = \rho \mathbf{g}\mathbf{u} + \nabla \cdot (\boldsymbol{\tau}\mathbf{u}) - \nabla \cdot \mathbf{q} \,. \tag{2.22}$$

This is the differential form of the conservation of energy in a moving fluid. It is easily possible to deduce two separate equations for the internal energy $\rho \frac{De}{Dt}$ and for the term representing the kinetic energy $\rho \frac{D}{Dt}(\frac{u^2}{2})$. The latter can be obtained by simply taking the dot-product of Eq. (2.18) leading to

$$\frac{\rho}{2} \frac{Du^2}{Dt} = \rho \mathbf{g} \cdot \mathbf{u} + \mathbf{u}(\nabla \cdot \boldsymbol{\tau}) \,. \tag{2.23}$$

This equation is called the *mechanical energy balance* and is a direct consequence of Newton's second law. Subtracting this equation from Eq. (2.22), we obtain after some algebra (see exercise) the *thermal energy balance*

$$\frac{De}{Dt} = \frac{1}{\rho} \boldsymbol{\tau} : \mathbf{E} - \frac{1}{\rho} \nabla \cdot \mathbf{q} \,, \tag{2.24}$$

where $\boldsymbol{\tau} : \mathbf{E} := \tau_{ij} E_{ij}$ (summation implied), which is a scalar quantity. In a Lagrangian interpretation the internal energy (per unit mass) e of Patty changes due to the transport of heat $\frac{1}{\rho}\nabla \cdot \mathbf{q}$ and a more mysterious term $\frac{1}{\rho}\boldsymbol{\tau} : \mathbf{E}$ related to the work of the surface forces A further interpretation of this term will be done in the next sections.

2.1.4 A Complete Set of Equations?

Our goal, formulated in the first paragraph of this chapter, was to obtain evolutionary equations for the quantities $\mathbf{u}, \rho, p, e, T$. One possible way to summarize the deduced equations is the following set of equations

$$\begin{aligned}
\frac{D\mathbf{u}}{Dt} &= \mathbf{g} + \frac{1}{\rho} \nabla \cdot \boldsymbol{\tau} \\
\frac{D\rho}{Dt} &= -\rho \nabla \cdot \mathbf{u} \\
\frac{De}{Dt} &= \frac{1}{\rho} \boldsymbol{\tau} : \mathbf{E} - \frac{1}{\rho} \nabla \cdot \mathbf{q} \\
e &= e(T, \rho) \quad \text{caloric equation of state} \\
p &= p(T, \rho) \quad \text{thermal equation of state}
\end{aligned} \tag{2.25}$$

plus initial and boundary conditions. Thus, we now have seven equations for the desired seven variables $\mathbf{u}, \rho, p, e, T$. Yeah!!! But wait... Mhmm.. We actually have twelve additional unknowns. The tensor $\boldsymbol{\tau}$ and the heat flux $\mathbf{q}$. Shhhhhhhhhhhocolate... How did this happen? The reason for this problem is that fluids are not really a continuum, as implied in the derivation (see Sec. 1.4) and that the molecular structure of fluids cannot be completely ignored. Even though the molecular structure is not directly evident at the relevant scales of the flow, it still plays an important role. The considered macroscopic velocity $\mathbf{u}$ leaves out the stochastic motion and collisions of individual molecules which play an important role for the transport of heat and momentum in the flow. These transports are therefore represented by the additional unknowns $\boldsymbol{\tau}$ and $\mathbf{q}$.

In order to obtain a closed set of equations, we have to introduce additional equations which model $\boldsymbol{\tau}$ and $\mathbf{q}$ as functions of the variables $\mathbf{u}, \rho, p, e, T$. These equations are called *constitutive equations* and a special kind of such equations for fluids called *Newtonian fluids* will be introduced in the next section.

Together with constitutive equations for $\boldsymbol{\tau}$ and $\mathbf{q}$ the set of equations Eq. (2.26) provide a closed set of equations. With appropriate boundary and initial conditions they completely describe the spatio-temporal behavior of fluids in motion (as long as the continuum approximation is valid). Obviously, the equations of state and constitutive equations will differ for different kinds of fluids. In a more complex case, the equations of state might also depend on additional quantities such as salinity of seawater or moisture of air. In this case an additional transport equation for these quantities has to be included.

2.2 Equations of Motion for an Incompressible Newtonian Fluid

Starting from the set of equations given by Eq. (2.26), we will now deduce a closed set of equations for a special kind of fluids, namely incompressible Newtonian fluids. These are the only fluids considered in this course. Except in Chap. 8, we will also approximate the flow as isothermal leading to a constant density of the fluid.

2.2.1 Continuity Equation for an Incompressible Fluid

Let us return to the continuity equation given by Eq. (2.4): $\frac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{u}$. We know from the thermal equation of state $p = p(T,\rho)$, that the density of the fluid depends on the temperature and the pressure ($\rho = \rho(T,p)$). A fluid is called incompressible when the dependence on pressure can be neglected. It can be shown, that this is a useful approximation when the typical order of velocities in the flow U is much smaller than the speed of sound c_S in the fluid. This is commonly expressed as Ma $<<$ 1 where Ma is the so called Mach number defined as Ma $:= \frac{U}{c_{sound}}$. In typical flows in the atmosphere or the ocean this is a valid assumption.

In a lot of relevant cases, such as the flow through a wind turbine, the temperature is almost constant in the flow region and thus we can additionally assume that the flow is isothermal. Thus, the density ρ is approximately constant in the flow and hence $\frac{D\rho}{Dt} \approx 0$ and the continuity equation becomes

$$\nabla\cdot\mathbf{u} = 0\,. \tag{2.26}$$

Even for non-isothermal fluids, this equation is often still a good approximation since $\frac{1}{\rho}\frac{D\rho}{Dt} << \nabla\cdot\mathbf{u}$ still holds, which will be discussed further in Chap. 8.

2.2.2 Constitutive Equation for the Stress Tensor

As noted in Sec. 2.1.4, we need to express the stress tensor $\boldsymbol{\tau}$ in terms of the other field variables $\mathbf{u}, \rho, p, e, T$, in order to obtain a closed set of equations. $\boldsymbol{\tau}$ has to model the surface stress caused by the underlying molecular structure of the fluid.

The Static Case

Let us start with the simple case of a resting fluid. In this case, we already know from thermodynamics or your fluid dynamics primer that the only surface force is the thermodynamic pressure. It is directed inwardly normal to any surface in the flow and does not itself depend on the orientation of the surface. Thus, we obtain $\mathbf{t}(\mathbf{x}, \mathbf{n}) = -p(\mathbf{x})\mathbf{n}$ and therefore $\boldsymbol{\tau} = -p\mathbf{I}$. Note that through this equation, we have now taken into account molecular effects, since the pressure arises from the collisions of molecules to a surface.

The Dynamic Case

For a fluid in motion, things get more complicated. In this case, it is useful, to express $\boldsymbol{\tau}$ as

$$\boldsymbol{\tau} = -p\mathbf{I} + \boldsymbol{\sigma} \,, \tag{2.27}$$

where $\boldsymbol{\sigma}$ represents the stress stemming from all surface forces except the thermodynamic pressure. It is sometimes called the *deviatoric stress tensor*. $\boldsymbol{\sigma}$ is caused by internal friction in the fluid flow. For illustrative purposes let us first consider a simple shear flow given by $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} cx_2 \\ 0 \end{pmatrix}$ with the constant $c \in \mathbb{R}$, as sketched in Fig **??**. Obviously, the layers on top of each other have different velocities. Since they are in direct contact there has to be some kind of friction or in other words exchange of momentum. We can also imagine two trains traveling parallel to each other with different velocities, where coal is shoveled from one train to the other (in both directions). This way momentum is exchanged and the trains will eventually end up with the same speed. In the case of a fluid flow, momentum will additionally transported by collisions of molecules. Obviously, the exchange of momentum will be higher when there is a stronger velocity difference between the different layers or trains. In analogy to Fick's law for mass diffusion, Newton suggested therefore that in a simple shear flow the transport of momentum and hence the deviatoric stress σ_{12} depends linearly on the velocity gradient:

$$\sigma_{12} \propto \frac{\partial u_1}{\partial x_2}, \tag{2.28}$$

which has been experimentally confirmed for many fluids.

We will follow a similar approach for a general fluid flow and assume that $\boldsymbol{\sigma}$ depends in a linear form on the velocity gradients $\partial_i u_j$ and thus the corresponding velocity gradient tensor $\boldsymbol{\Lambda}$, defined in Eq. (1.17). As we learned in Sec. 1.5.3, $\boldsymbol{\Lambda}$ can be decomposed into a symmetric tensor $\mathbf{E}$ and an antisymmetric tensor $\boldsymbol{\Omega}$. The antisymmetric tensor simply represents rigid body rotation. In such a rotation, obviously no internal friction can occur. Thus, we assume that $\boldsymbol{\sigma}$ depends only on $\mathbf{E}$ and not on $\boldsymbol{\Omega}$. This can be expressed as:

$$\boldsymbol{\sigma} = \mathbf{A} : \mathbf{E} \;\; \text{with} \;\; \sigma_{kl} = A_{ijkl} E_{kl} \,, \tag{2.29}$$

where $\mathbf{A}$ is an arbitrary tensor of fourth order.

We will additionally assume that the fluid is isotropic meaning it does not show a different structure in different directions. Taking into account that $\boldsymbol{\tau}$ and thus $\boldsymbol{\sigma}$ is symmetric, the following equation can be shown:

$$\boldsymbol{\tau} = -p\mathbf{I} + \lambda \text{tr}(\mathbf{E})\mathbf{I} + 2\mu\mathbf{E} \text{ with } \lambda \in \mathbb{R}\,, \;\; \mu \in \mathbb{R}\,. \tag{2.30}$$

In the case of an incompressible fluid this can be further simplified to

$$\boldsymbol{\tau} = -p\mathbf{I} + 2\mu\mathbf{E} \tag{2.31}$$

by neglecting $\text{tr}(\mathbf{E}) = \nabla \cdot \mathbf{u}$. Note that the former complex dependence on a fourth order tensor has been reduced to a single scalar μ. This scalar is material property of the fluid called the *viscosity*. It can be shown using the second law or thermodynamics that it has to be positive (see e.g. [1]). Its role will be discussed later (e.g. in Sec. 4.1.3) but you probably already have an intuitive understanding of μ. Honey, for example, is a fluid with a relatively high viscosity in comparison to fluids such as water.

Fluids which can be modeled adequately by Eq. (2.30) or Eq. (2.31) are called *Newtonian Fluids*. The reason behind this name is the central assumption of a linear dependence of the stress on the rate of strain (Eq. (2.29)), as originally done by Newton for a simple shear flow. We have implicitly neglected derivatives of higher order, which might be a good idea since on very small molecular scales, which we try to model, the velocity changes might be well described by first order derivatives. Furthermore, we have assumed that the dependence on $\mathbf{E}$ is instantaneous and does not depend on past or future values of $\mathbf{E}$.

Even though all these assumptions are plausible, we have not said anything about their validity in the physical reality. Fortunately, experiments show, that the deduced relatively simple constitutive equations (Eq. (2.30) or Eq. (2.31)) work well for a lot of fluids, namely almost all gases and liquids with low to moderate molecular weights. Hence, we have found a good model for the description of air and water as long as the rate of strain does not become huge which leads to a violation of the linear dependence in $\boldsymbol{\sigma} = \mathbf{A} : \mathbf{E}$.

Non-Newtonian Fluids

However, they are also a lot of *complex liquids* which do not follow the assumptions we made in this section. These fluids are called *non-Newtonian fluids*. Their deviatoric stress tensor is a nonlinear function of the strain rate. Consequently, no constant scalar material property μ exists. Their “apparent viscosity” depends on the rate of strain. Thus, such fluids can appear more or less viscous when the rate of strain is increased. Furthermore, the stress often depends not only on the instantaneous strain but on its history. The fluids, in some sense, “possess a memory” leading to elastic behavior. They are often called *viscoelastic*. Examples for non-Newtonian fluids are solutions containing complex polymer molecules, emulsions and slurries containing suspended particles such as water with clay or blood. A very impressive example is a suspension of corn starch, which will be part of the exercise sheet. Non-Newtonian fluids are exciting but their quantitative treatment is really complicated, since it is much more difficult to find adequate constitutive equations. Fortunately, in renewable energies we only deal with Newtonian fluids such as water and air. A first look into non-Newtonian fluids can e.g. be found in [1].

2.2.3 Navier-Stokes Equations (NSE)

Since we have now found a constitutive equation for the stress tensor, we can substitute the tensor $\boldsymbol{\tau}$ in the differential equation describing the momentum (Eq. (2.18)). This leads to

$$\boxed{\rho\frac{D}{Dt}\mathbf{u} = -\rho\mathbf{g} - \nabla p + \nabla\cdot(2\mu\mathbf{E})}\,. \tag{2.32}$$

which is the *Navier-Stokes-Equation* for an incompressible fluid. The dependence of the viscosity μ on the pressure is almost always negligible. If we additionally assume that our fluid is isothermal, μ and ρ can be considered as constants and we obtain the NSE in its most famous form

$$\boxed{\rho\frac{D}{Dt}\mathbf{u} = -\rho\mathbf{g} - \nabla p + \mu\Delta\mathbf{u}}\,. \tag{2.33}$$

Slightly rewriting this equation and combining it with the continuity equation leads to

$$\begin{aligned} \partial_t\mathbf{u} &= -(\mathbf{u}\cdot\nabla)\mathbf{u} - \mathbf{g} - \frac{1}{\rho}\nabla p + \nu\Delta\mathbf{u} \\ \nabla\cdot\mathbf{u} &= 0\,, \end{aligned} \tag{2.34}$$

where $\nu := \frac{\mu}{\rho}$ is called the kinematic viscosity. Combined with appropriate initial and boundary conditions, this is now a closed set of equations. Assuming we know the constant density and viscosity of the fluid, we have four equations with 4 unknowns. Yuhuuuuuu! Finally:) The first term on the r.h.s. of the momentum equation $-(\mathbf{u}\cdot\nabla)\mathbf{u}$ is simply the advection of the velocity (momentum per unit mass) by the velocity field itself. This is a nonlinear term. Due to this term, sums of solutions of the NSE are not again solutions of the equation. The term plays an important role for the development of instabilities in fluid flows and hence for the generation of turbulent flows. The second term simply represent acceleration through gravitation. The third is a consequence of the internal friction in the fluid leading to a diffusion of velocity with the kinematic viscosity ν as its diffusive constant.

Pressure and Gravitation

The role of the thermodynamics in this set is reduced to a constraint on the velocity field ($\nabla\cdot\mathbf{u} = 0$). The pressure p appears in the equations only as a gradient. Therefore, the absolute value of the pressure is not relevant for the motion of the fluid and cannot be determined from the equations of motion.

In an incompressible and isothermal with constant density we can simplify the Navier-Stokes equations even further. We define a new kind of pressure $\tilde{p}$ by

$$\tilde{p} = \rho g x_3 + p\,. \tag{2.35}$$

with

$$-\frac{1}{\rho}\nabla\tilde{p} = -\begin{pmatrix}0\\0\\g\end{pmatrix} - \frac{1}{\rho}\nabla p = -\mathbf{g} - \frac{1}{\rho}\nabla p\,.$$

Thus, we can simplify the momentum equation to

$$\boxed{\partial_t \mathbf{u} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\rho}\nabla \tilde{p} + \nu \Delta \mathbf{u}} \tag{2.36}$$

and the gravitation is simply lumped into the new pressure term. Consequently, the gravitation term does not have any direct influence on an incompressible isothermal fluid flow.[4] $\tilde{p}$ is sometimes called *dynamic pressure* but this can be a misleading term since the term $\frac{\rho}{2}u^2$ sometimes gets the same name.

2.2.4 Constitutive Equation for the Heat Flux

So now we have a closed set of equations for velocity and pressure. Why should we care about the energy now? First of all it is obviously relevant when the flow is compressible or non-isothermal, since in this case the set Eq. (2.34) is not valid. Secondly, understanding the energetic behavior of a fluid flow can sometimes offer important additional insight.

In the energy equation of the set, an additional vector $\mathbf{q}$ occurred describing the heat flux in the fluid. As discussed in Sec. 2.1.4, the heat flux is caused by the molecular motions. More precisely, it is the transport of mean molecular kinetic energy stored in the stochastic motion of the particles. The transport occurs due to random exchange of particles and collisions between them. The temperature T is a measure for the mean molecular kinetic energy. The "random" heat transport is therefore obviously higher when there are strong temperature gradients. Therefore, in analogy to Fick's law for mass diffusion a linear dependence of $\mathbf{q}$ on ∇T is a reasonable guess leading to

$$\mathbf{q} = \mathbf{K}\nabla T \, , \tag{2.37}$$

where $\mathbf{K}$ is an arbitrary second order tensor. Under the assumption of an isotropic fluid, the second-order tensor has to take the form $\mathbf{K} = k\mathbf{I}$ with the *thermal conductivity* k. This leads to Fourier's law of heat conduction:

$$\mathbf{q} = -k\nabla T \, . \tag{2.38}$$

2.2.5 Energy equation

Substituting $\boldsymbol{\tau}$ and $\mathbf{q}$ in the energy equation of the set Eq. (2.26) leads to

$$\rho\frac{De}{Dt} = -p\nabla \cdot \mathbf{u} + 2\mu\mathbf{E} : \mathbf{E} - \nabla \cdot (k\nabla T) \tag{2.39}$$

and assuming a constant k to

$$\boxed{\frac{De}{Dt} = -\frac{1}{\rho}p\nabla \cdot \mathbf{u} + 2\nu\mathbf{E} : \mathbf{E} - \frac{k}{\rho}\Delta T} \, . \tag{2.40}$$

This equation now shows that the rate of change of internal energy (per unit mass) of our moving fluid particle Patty is caused by three different terms. The first term on the r.h.s. is work done

[4] An exception occurs in the presence of interfaces of different fluids where the pressure has to be taken into account in the boundary conditions.

by the pressure, the third term is the diffusion of energy caused by transport of heat. The second term

$$\varepsilon := \frac{1}{\rho}\boldsymbol{\sigma} : \mathbf{E} = 2\nu\mathbf{E} : \mathbf{E} = \frac{\nu}{2}(\partial_i u_j + \partial_j u_i)(\partial_i u_j + \partial_j u_i) \tag{2.41}$$

is called the dissipation. It plays a fundamental role in the theory of turbulence discussed in Chap. 7. It describes the increase of internal energy due to the conversion of macroscopic kinetic energy into heat. It depends on the square of velocity gradients and is therefore much higher in regions of strong shear. Also note that it is proportional to the kinematic viscosity μ. A higher viscosity leads to a higher dissipation which obviously makes sense if we interpret viscosity as a parameter responsible for internal friction in the fluid.

Using the second law of thermodynamics it can be shown that μ and ν have to be positive [1, 2]. This makes a lot of sense, In simplified terms the second law states that intensive quantities such as the temperature or the momentum per unit mass (the velocity) tend to a uniform distribution. Otherwise, we would have a negative diffusion of e.g. velocity $\nu\Delta\mathbf{u}$. This would mean that the velocity gets more and more concentrated in a smaller and smaller region. Finally ending with an infinite velocity in an infinitely small region. This does not sound realistic.

Last but not least we can interpret the equation in an Eulerian way $\frac{De}{Dt} = \partial_t e + \mathbf{u}\nabla e$, where the second term describes the transport of energy caused by the motion of the fluid. This process is called *convection*.

Note that in Eq. (2.40), we have not simply left out $-p\nabla\cdot\mathbf{u}$ even though we consider nearly incompressible fluids. We did not do so since $\nabla\cdot\mathbf{u} << \frac{1}{\rho}\frac{D\rho}{Dt}$ in the continuity equation does not necessarily mean that we can neglect it in Eq. (2.40).

Since they are many thermodynamic relation between different quantities, there many ways to express thermodynamic laws. In a non-isothermal fluid where T is not constant it makes a lot of sense to use e.g. $e = e(\rho, T)$ to deduce an alternative equation in terms of the temperature given by

$$\rho c_V \frac{DT}{Dt} = 2\mu\mathbf{E} : \mathbf{E} - k\Delta T - \frac{T}{\rho}\left(\frac{\partial p}{\partial T}\right)_\rho \frac{D\rho}{Dt} \tag{2.42}$$

or another equivalent form derived in [1]:

$$\rho c_p \frac{DT}{Dt} = 2\mu\mathbf{E} : \mathbf{E} - k\Delta T - \frac{T}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p \frac{Dp}{Dt}\,. \tag{2.43}$$

2.2.6 Boundary and Initial Conditions

here or with the NSE? In order to solve the set Eq. (2.34) including the Navier-Stokes Equations or any other closed set of hydrodynamic equations derived from Eq. (2.26), we need appropriate initial and boundary conditions.

The Role of the Pressure

In the case of the set (2.34), it turns out that we only need initial and boundary conditions for the velocity field and not for the pressure p. In a strict sense, we aim to solve a set of partial

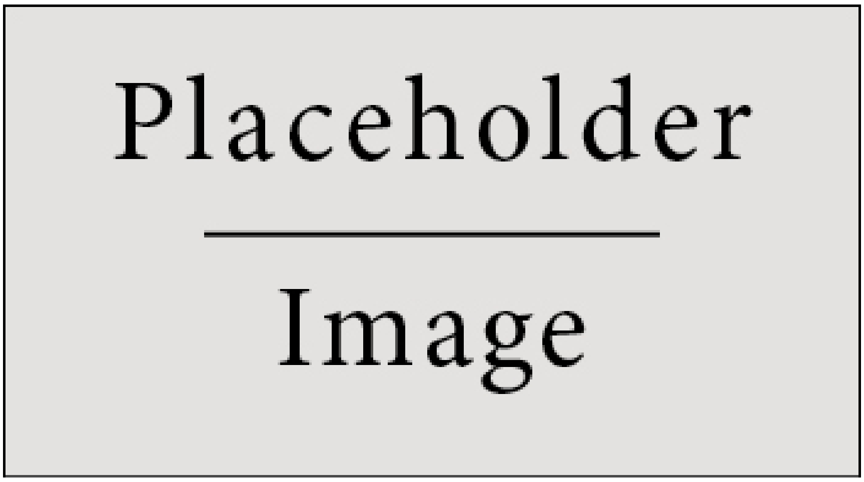

Figure 2.2: Cylinder Flow

differential equations for the variables u_1, u_2, u_3 under the constraint $\nabla \cdot \mathbf{u} = \partial_i u_i = 0$. Mathematically, the pressure p is not an independent variable but a Lagrange-multiplier corresponding to the constraint $\nabla \cdot \mathbf{u} = 0$. Only the gradient of p can be uniquely determined, since its absolute value does not play any role in the set Eq. (2.34). One way to determine the gradient is through the Poisson equation (Eq. (**??**)). The boundary condition for this equation can be determined from the boundary conditions on $\mathbf{u}$.

Initial Conditions

Setting up initial conditions is mostly relatively trivial. We simply choose the velocity field $\mathbf{u}(\mathbf{x}, t = 0) = u_0(\mathbf{x})$ for the whole domain we consider in our problem. You might run into some problems though when choosing an unrealistic discontinuous velocity field as initial condition.

Boundary Conditions

Boundary Conditions are a complex topic. There are many different kinds of boundaries in fluid flows, such as the boundaries between two different fluids or different phases of a fluid, between the fluid and a solid wall or conditions describing the flow out of a domain. However, the boundary conditions between different fluids or phases will not be treated in this reader.

As a first example, you can think of a fixed cylinder C, which has been placed in an otherwise uniform flow field defined by $(u_1, u_2, u_3)^T = (u_\infty, 0, 0)^T$, where u_∞ is a constant. This situation is sketched in Fig. 2.2. We can assume that far away from the flow will not be influenced by the presence of the cylinder. Thus, if we consider a domain which is large enough we can simply choose $(u_1, u_2, u_3)^T = (u_\infty, 0, 0)^T$ at the boundaries. Nice!

However, we still need boundary conditions on the wall of the cylinder. What happens on the wall? Using mass conservation, it is relatively easy to show that the velocity component normal to a solid surface has to equal the normal component of the velocity of the solid and thus in the case of a fixed cylinder we obtain

$$(\mathbf{u} \cdot \mathbf{n})|_{\partial C} = 0 \; , \tag{2.44}$$

which is sometimes called the *kinematic boundary condition*[2, 1]. If this condition was not true, the boundary would act as a sink of mass.

Eq. (2.44) is only a condition for one direction of the velocity but what happens in the tangential directions? There is no obvious way to derive corresponding conditions. On a molecular level this condition is described by the collisions of individual fluid molecules with the solid wall or in more general terms by their interaction with the atomic structure of the wall. But which consequences does this have for the macroscopic velocity $\mathbf{u}$. Experimentally it has been shown, that in most cases (not always) it works well to choose the tangential velocities on a fixed solid surface as zero. This is called the *no-slip condition*. Thus, in combination with the kinematic condition we can conclude

$$\boxed{\mathbf{u}|_{\partial C} = \mathbf{0}} \, . \tag{2.45}$$

Analogously, for a moving cylinder or surface, the velocities of fluid and solid are the same on the surface or in other words are continuous across the surface, which can be expressed by

$$\mathbf{u}^{(fluid)}|_{\partial C} = \mathbf{u}^{(solid)}|_{\partial C} \, . \tag{2.46}$$

In cases where the a inhomogeneous temperature of the flow is relevant, the temperatures across the surface is also assumed to be continuous

$$T^{(fluid)}|_{\partial C} = T^{(solid)}|_{\partial C} \, . \tag{2.47}$$

This is based on the assumption that the system of fluid and solid is always in a local thermodynamic equilibrium.

2.2.7 Solving the Navier-Stokes Equation

Now we finally can formulate well-posed fluid dynamical equations to specific problems using the NSE (Eq. (2.34)) and the boundary and initial conditions corresponding to the specific problem. However, analytical solutions can be found only in a very few cases under strong assumptions. Even proving the existence of a solution can be very difficult. Not without reason one of the so called millennium problems deals with such a proof.

Therefore, computational methods, as introduced in Chap. 9 are a very important tool in the area of fluid dynamics. However, even these methods have some principle limitations since fluid flows are often very complex and show dynamics on a wide range of scales, which will be discussed further in the Chap. 7, which deals with turbulence.

2.3 Bernoulli Equation

In this section, we will deduce the famous Bernoulli equation. It is not a separate equation but simple an alternative formulation of the momentum equation for *inviscid* flows. A flow is called inviscid if viscosity effects can be neglected. In this case, the Navier-Stokes Equation becomes the Euler equation given by:

$$\partial_t \mathbf{u} = -(\mathbf{u} \cdot \nabla)\mathbf{u} - \mathbf{g} - \frac{1}{\rho}\nabla p \, . \tag{2.48}$$

You will show in the exercise sheet that the advective term can be substituted according to

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \frac{1}{2}\nabla q^2 + \mathbf{u}\times\boldsymbol{\omega}\ , \tag{2.49}$$

where $\boldsymbol{\omega}$ is the vorticity from Eq. (4.1) and $q^2 := u_j u_j$ is simply twice the kinetic energy. Additionally, using in the Euler equation $\mathbf{g} = -\nabla(gx_3)$ leads to

$$\partial_t\mathbf{u} + \frac{1}{2}\nabla q^2 + \nabla(gx_3) - \frac{1}{\rho}\nabla p = \mathbf{u}\times\boldsymbol{\omega}\ . \tag{2.50}$$

This equation is the basis for many forms of the Bernoulli equation, as shown e.g. in [2]. Here, we will derive the Bernoulli equation for a steady flow with a constant density yielding:

$$\nabla(\frac{1}{2}q^2 + gx_3 - \frac{p}{\rho}) = \mathbf{u}\times\boldsymbol{\omega}\ . \tag{2.51}$$

The function in the brackets of the l.h.s. is sometimes called the Bernoulli function. Its gradient is therefore perpendicular to $\mathbf{u}$ and $\boldsymbol{\omega}$ and hence:

$$\boxed{\frac{1}{2}q^2 + gx_3 - \frac{p}{\rho} = \text{constant along stream and vortex lines}}\ . \tag{2.52}$$

In the case of an *irrotational* flow with $\boldsymbol{\omega} = \mathbf{0}$ we obtain

$$\boxed{\frac{1}{2}q^2 + gx_3 - \frac{p}{\rho} = \text{constant}}\ , \tag{2.53}$$

which is the most famous form of the Bernoulli equation.

⚠ Eq. (4.24) is only valid for irrotational flows. If vorticity cannot be neglected we have to stick to Eq. (2.52) for a precise application of Bernoulli's law.

The application of the Bernoulli equation is illustrated in the following example.

Example 2.1 — Pitot tube.
One important example for the use of the Bernoulli equation is the principle of a pitot tube, a simple device to measure a velocity in a fluid. This is illustrated in Fig. 2.3. By measuring the two heights of the water columns in the figure it is possible to determine the fluid's velocity at point 1. How does this work? We assume that friction is negligible and that point 1 and point 2 lie on the same streamline. In this case we can apply Bernoulli's equation yielding

$$\frac{p_1}{\rho} + \frac{u_1^2}{2} = \frac{p_2}{\rho} + \frac{u_2^2}{2}\ , \tag{2.54}$$

where the gravitation term has been subtracted since it is the same on both sides. On point 2, immediately in front of the tube the velocity u_2 is zero and we obtain

$$u_1 = \sqrt{2\frac{p_2 - p_1}{\rho}} \tag{2.55}$$

The pressures at both points can be found from hydrostatic balance leading to $p_1 = \rho g h_1$ and $p_2 = \rho g h_2$, where we have implicitly assumed that at the top of the columns the atmospheric

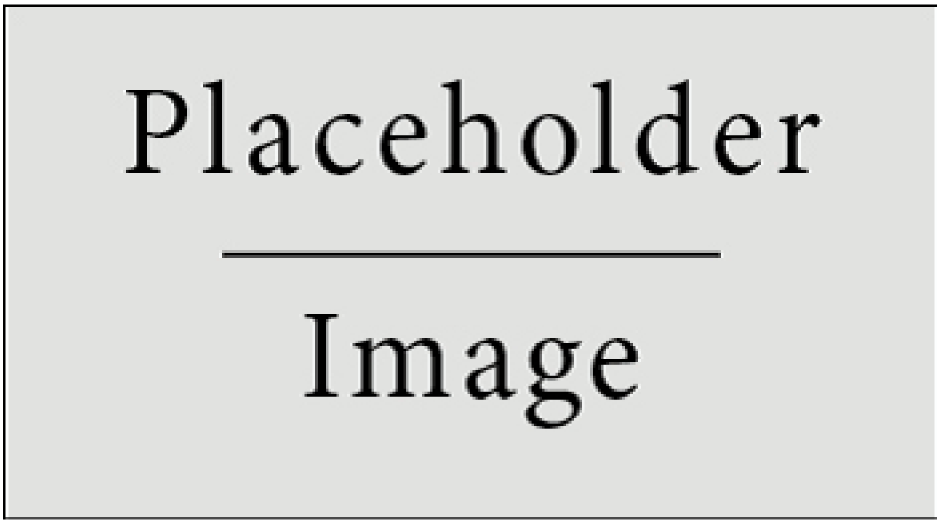

Figure 2.3: Pitottube

pressure is approximately the same. This leads us to

$$u_1 = \sqrt{2g(h_2 - h_1)} \,, \tag{2.56}$$

which can now be used to determine the velocity. Even when pitot tube is present the terms $\frac{\varrho}{2}u^2$ and $p + \frac{\varrho}{2}u^2$ are sometimes called *dynamic pressure* and *stagnation pressure*, respectively.

2.4 Summary of Chapter Two

Puuuhhhh. A lot of equations in this chapter but you made it! As said in the beginning, no need to feel frustrated if you did not get every step but you have two make sure that you at least have a solid idea about the different terms in the deduced equations. Simply go through the chapter again and have a look at all boxed equations. What can you say about the different terms? Furthermore, look again at the learning objectives in the beginning. Can you put a check mark behind every item?

We started this chapter by applying the fundamental physical conservation principles to continuous media, such as fluids, and by formulating them in a differential form. We end up with a set of equations (Eq. (2.26)) which represent the conservation of momentum, energy, and mass plus the thermodynamic equations of state. For the formulation of these equations we introduced two new quantities the stress tensor $\boldsymbol{\tau}$ and the heat flux $\mathbf{q}$ which have to be modeled to take into account the neglected molecular structure of the fluid and to obtain a closed set of equations.

Modeling $\boldsymbol{\tau}$ in a Newtonian manner and additionally assuming incompressibility and and isothermal flow, we were able to deduce a much simpler closed set of equations called the Navier-Stokes equations (Eq. (2.34)) which will be the basis for a large part of this textbook. Additionally, we could deduce an evolution equation for the internal energy in the flow (Eq. (2.40)) by modeling $\mathbf{q}$ to be linearly dependent on the temperature gradients and assuming isotropic flow.

3 — Solving, Analyzing and Approxin

Learning objectives

After processing this chapter you will be able to

- solve the NSE for simple unidirectional flows and interpret the results
- describe the concept of dynamic similarity
- describe the meaning of some important non-dimensional numbers in fluid mechanics
- analyze fluid flows and other physical problems from a purely dimensional perspective
- explain the idea of a self-similar flow solution

Welcome back fluid experts :) I hope you could relax a little bit from the excessive and sometimes pretty complex and difficult mathematical derivations in the last chapter. Time for something much easier, at least mathematically, and also a little bit more concrete.

After all this work put in to end up with the Navier-Stokes equation, we obviously would like to use and solve it to understand different hydrodynamic flows. Thus, in the following section we start this chapter with a very special class of flows, called *unidirectional flows*, since they often allow an analytical treatment of the NSE. Subsequently, in order to see what we can do when no analytical solution can be found, we take a closer look at the NSE in terms of physical dimensions or units (Sec. 3.2). This way a lot can be learned about the nature of a fluid dynamical problem and in some cases the underlying equations can be systematically approximated. In the last section (Sec. 3.3) a unidirectional flow problem is solved using some of the tricks learned from our dimensional analysis.

3.1 Unidirectional Flows

A unidirectional flow is a hydrodynamic flow with the fluid flowing only in one direction and thus the velocity vector becomes

$$\mathbf{u} = u\mathbf{e}_1 \ , \ u \in \mathbb{R} \ . \tag{3.1}$$

The components u_2, u_3 and the corresponding derivatives are all assumed to be zero. Consequently, we only need to solve an equation for the first component. This leads to

$$\partial_t u = -u\partial_{x_1} u - \frac{1}{\rho}\partial_{x_1} p + \nu\Delta u \ . \tag{3.2}$$

We have lumped the gravitational term into the pressure term following 2.2.3 but kept the p symbol instead of $\tilde{p}$. The continuity equation becomes

$$\partial_{x_1} u = 0 \tag{3.3}$$

and thus the nonlinear term in Eq. (3.2) $u\partial_{x_1} u$ also vanishes and u does not depend on x_1 yielding

$$\partial_t u = -\frac{1}{\rho}\partial_{x_1} p + \nu(\partial^2_{x_2} + \partial^2_{x_3})u \ , \text{ with } u = u(x_2, x_3, t) \ . \tag{3.4}$$

Nice! This already seems a lot simpler than treating the original NSE. The main reason why an analytical treatment is possible for a lot of unidirectional flows is that the nonlinear term has vanished. Does it get even simpler? What can we say about the pressure term? Any Idea?

We know that the pressure gradients $\partial_{x_2} p$ and $\partial_{x_3} p$ have to be zero. Otherwise, these gradients would cause a change of u_2 and u_3 which are required to stay zero. This leads to $p = p(x_1, t)$. But we can say even more when examining Eq. (3.4) again. Due to $u = u(x_2, x_3, t)$ the term on the l.h.s. and the second term on the right hand side cannot depend on x_1 and therefore neither can the pressure term, leading to $p = p(t)$. It can be convenient to denote the pressure term as

$$\partial_{x_1} p = G(t) \ . \tag{3.5}$$

This leads to the final form of the Navier-Stokes equation for unidirectional flows

$$\boxed{\partial_t u = -\frac{1}{\rho}G(t) + \nu(\partial^2_{x_2} + \partial^2_{x_3})u} \ , \tag{3.6}$$

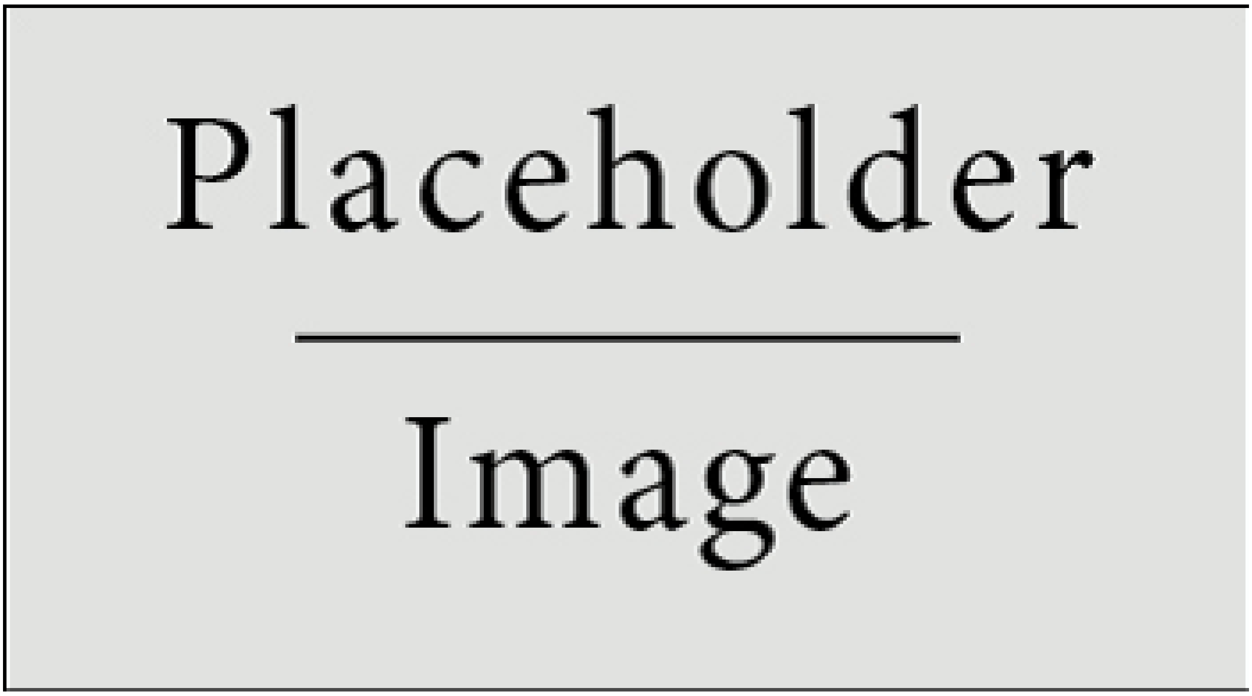

Figure 3.1: Steady flow between two paralell plates taken from [1].

where $u = u(x_2, x_3, t)$.

3.1.1 Steady Flow between Parallel Plates

The easiest example of a unidirectional flow is a steady flow between two infinitely extended parallel plates, represented by two x_1x_2-planes with a distance of $2b$, as sketched in Fig. 3.1. The upper plate moves with a velocity $U\mathbf{e}_1$, which results in a unidirectional flow in in the x_1-direction.

Obviously, the infinite extension is an idealization. It is used to approximately describe flows where neither the boundaries in x_2-direction nor the in- and outflow regions in the direction of the flow play a significant role. Concerning the direction of the flow, one could say that the flow is assumed to be "fully developed" since we are always infinitely far away from the inflow and outflow regions. We are also only interested in the "fully developed" flow with respect to time. Thus, we are only considering a flow, which has reached its steady state.

Let's start! Due to the infinite extension in x_1- and x_2-direction and the assumed steady state, there cannot be any dependence of the flow on x_1, x_2 and time t.[1] Consequently, u in Eq. (3.6) depends only on x_3 and furthermore the pressure gradient G is simply a constant parameter. This yields

$$\frac{d^2u}{d{x_3}^2} = \frac{G}{\nu\rho} \; , \text{ subject to } u(0) = 0 \; , \; u(2b) = U \; , \tag{3.7}$$

where $u = u(x_3)$. Here, we examine only two special cases of this flow, called *Couette flow* and

[1] This should be intuitively clear. If not, it might become after working on the exercise sheet.

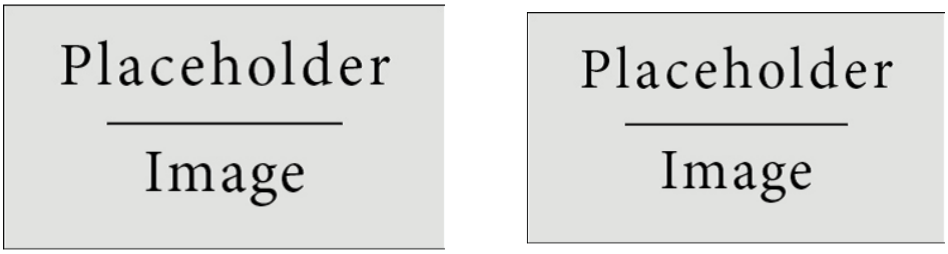

Figure 3.2: Velocity profiles of Pouseille flow (left) and Couette flow (right) taken from [1]. The Pouseille flow yields a constant shear while a linear shear profile can be found in the Pouseille flow, as sketched in the left part of the right figure.

Pouseille flow.

Couette Flow

In the case of the Couette Flow the pressure gradient G is zero and the flow is only driven by the upper moving plate. Integrating the resulting equation $\frac{d^2u}{dx_3{}^2} = 0$ twice and using the boundary conditions leads to

$$\boxed{u = \frac{U}{2b}x_3}\ , \tag{3.8}$$

as sketched in Fig. 3.2 (left). Thus, the motion of the upper plate induces a simple linear shear flow, which has already been investigated in Sec. 1.5.3 and 2.2.2.

The shear stress in the flow is given by

$$\tau = \mu\frac{\partial u}{\partial x_3} = \frac{\mu U}{2b}\ , \tag{3.9}$$

which is constant throughout the channel.

Pouseille Flow

For the Pouseille flow, U is chosen to be zero and consequently the flow is only driven by a constant pressure gradient $\partial_{x_1}p = G$. Integrating Eq. (3.6) twice and using the boundary conditions easily leads to

$$\boxed{u = -\frac{x_3 G}{\mu}(b - \frac{x_3}{2})}\ , \tag{3.10}$$

which is a well-known parabolic velocity profile, as sketched in Fig. 3.1. Such a profile can, for example, be found in a river in the non-vertical direction, if we look not too close to the ground.

The shear stress in the flow is given by

$$\tau = \mu\frac{\partial u}{\partial x_3} = (b - x_3)G\ , \tag{3.11}$$

which is also sketched in Fig. 3.1. It should be noted that this linear shear stress distribution as well as the constancy of the pressure gradient are general results which hold even in the case of a fully developed turbulent flow, which is not completely uni-directional.

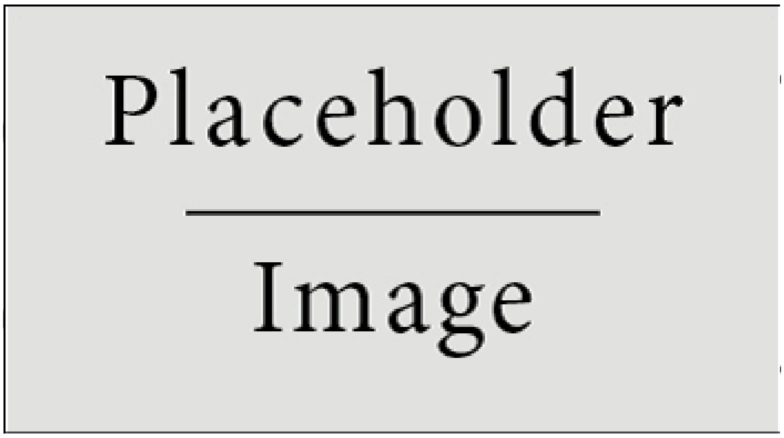

Figure 3.3: Flow in a pipe taken from [1].

3.1.2 Pipe Flow

Let's go back again to Eq. (3.6) and $u = u(x_2, x_3)$ and try to draw conclusion on the flow in an infinitely long pipe. Due to the symmetry of the problem, the solution of this equation is much easier in cylinder coordinates $u = u(r, \theta, x_1)$.

Analogoue to the line of arguments for the flow between plates, there cannot be any dependence of u_1 on x_1 and θ.

> **Self-reflection 3.1**
> What is this line of arguments here? Can you explain this in your own words? Why can't there any dependence on x_1 and θ?

Following [2], the x_1-momentum equation is given by

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{G}{\mu} \, , \tag{3.12}$$

which can be solved by

$$\boxed{u = \frac{r^2 - a^2}{4\mu} G \, .} \, , \tag{3.13}$$

The profile is also sketched shown in Fig. 3.3. Hence, as in the case of the Pouseille flow we obtain a parabolic profile but now in the radial instead of the vertical direction.

Integrating over one cross-section of the pipe leads to the volume flow rate

$$Q = -\frac{\pi a^4}{8\mu} G \, . \tag{3.14}$$

The pipe flow is a fundamental example, since it occurs in many applications such as the flow through water pipes or blood vessels. Being able to calculate analytical formulas for such flows is thus of an immense value in many areas of engineering and science.

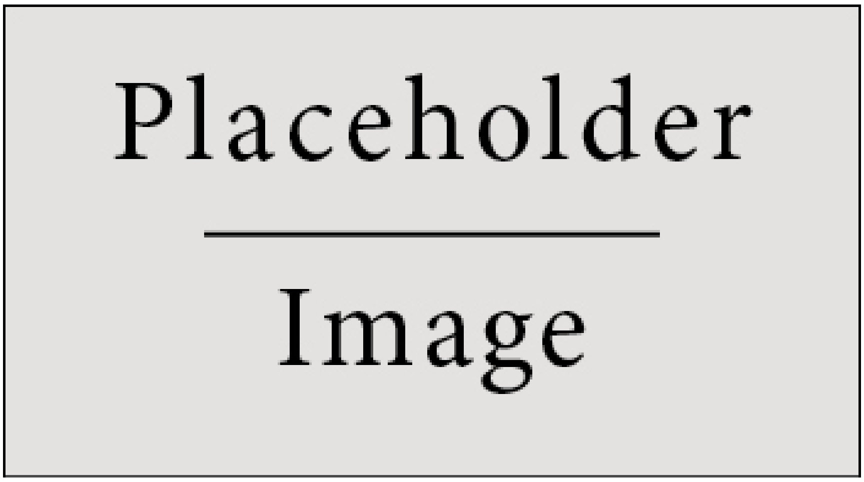

Figure 3.4: Sphere in a uniform flow.

3.2 Dimensional Analysis

As discussed before, cases where the general Navier-Stokes equation can be solved analytically are rare. Hence, in this second part of the chapter, we take a step back and look again closely at the NSE. What can we learn from this equation and its boundary conditions in the case of a concrete physical problem, when an analytical solution cannot be found? Is the only way to get an idea of a flow's behavior, to run experiments and/or simulations in the complete parameter space of a problem? Are there ways to approximate the NSE in some limit cases?

In order to find some answers to these questions, we will introduce the tool of dimensional analysis. It is a very powerful and simple tool and sometimes it is enough to squint our eyes together and look sharpely at a physical problem to draw an important conclusion. You might even learn to make impressive off the top of your head remarks like "No, this is not possible from a dimensional point of view." in scientific discussions. If that is no motivation, I don't know what is ;)

3.2.1 Dynamic Similarity

In order to introduce the concept of *dynamic similarity*, we choose the steady flow around a solid sphere, as an illustrative example. This is sketched in Fig. 3.4. The parameters determining the flow are given by the velocity U_∞, the diameter of the sphere l, the viscosity μ and density ρ of the fluid.

This idealized flow is relevant for any kind of ball flying through the air or a spherical raindrop falling from the sky. Let us start with the steady NSE, given by

$$-(\mathbf{u}\nabla)\mathbf{u} - \nabla p + \nu\Delta\mathbf{u} = 0 \tag{3.15}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{3.16}$$

with the boundary conditions being U_∞ in the undisturbed ambient flow and $\mathbf{u} = 0$ on the surface of the sphere.

We now introduce the following non-dimensional quantities

$$\tilde{\mathbf{u}} = \frac{\mathbf{u}}{U_\infty} \ , \ \ \tilde{P} = \frac{P - P_\infty}{\rho U_\infty^2} \ , \ \ \tilde{\mathbf{x}} = \frac{\mathbf{x}}{d} \ . \tag{3.17}$$

Substituting u_i by $U_\infty \tilde{u}_i$ in the NSE etc. leads to

$$-(\tilde{\mathbf{u}}\nabla)\tilde{\mathbf{u}} - \nabla\tilde{P} + \frac{1}{\mathrm{Re}}\nabla^2\tilde{\mathbf{u}} = 0 \tag{3.18}$$

$$\nabla \cdot \tilde{\mathbf{u}} = 0 \ , \tag{3.19}$$

where $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}(\tilde{x})$ and

$$\mathrm{Re} := \frac{U_\infty l}{\nu} \tag{3.20}$$

is called the *Reynolds number*. It also has to be noted that the operator ∇ has to be understood as a differential operator with respect to $\tilde{\mathbf{x}}$ in this normalized form of the NSE. Obviously, the normalized boundary conditions do not depend on any parameter since at infinity we obtain $\tilde{\mathbf{u}} = 1$ and at the boundary of the sphere $\tilde{\mathbf{u}} = 0$.

What did we gain from this alternative formulation of the NSE? It is important to note that the non-dimensional NSE including boundary conditions, only depends on one non-dimensional parameter called Re. This might not seem impressive at first, but it actually makes things a lot easier. It means that the behavior of the flow is completely determined by the single parameter Re. Every flow with the same Re essentially looks the same. In other words, flows with the same Reynolds number are said to be *dynamically similar*.

If you know the solution of the NSE for one specific Re, you know the solution for every flow with the same Re, regardless of the specific values of $U_\infty,\ l,\ \mu,\ \rho$. For example, instead of investigating the flow around a real size wind turbine with a diameter of a $100\,m$ at a wind speed of $5\,\frac{m}{s}$, we can use a small wind turbine in the lab with a diameter of $1\,m$ at a wind speed of $50\frac{m}{s}$, since both cases yield the same Reynolds number. If $\mathbf{u}$ is a solution of the NSE for some specific values $U_\infty,\ l,\ \mu,\ \rho$, you can simply use Eq. (3.27) to obtain the non-dimensional solution $\tilde{\mathbf{u}}$. This solution is the same for every flow with the same Re and you can now insert other values of $U_\infty,\ l,\ \mu,\ \rho$ with the same Re in Eq. (3.27) to obtain the alternative dimensionful solution.

Now suppose you perform an experiment, where you measure the drag force[2] D on the sphere and you want to know its dependence on the flow parameters $U_\infty,\ l,\ \mu,\ \rho$. A first naive way would be to measure D simply for a lot of different values of these parameters trying to cover as much of this four dimensional parameter space as possible. This is a lot of work!!!

But there is an alternative much simpler way to do this. Let us define the non-dimensional drag coefficient[3]

$$c_D := \frac{D}{\frac{\rho}{2}U_\infty^2 A_l} \ , \tag{3.21}$$

where A is the area of the cross section of the sphere given by $A_l := \pi(\frac{l}{2})^2$. Except for normalization the flow is completely determined by Re. Consequently, this is true for every non-dimensional property of the flow. The normalized drag given by c_D can thus be expressed as:

[2] See 4.2.5 for an exact definition of the drag force.

[3] The factor $\frac{1}{2}$ in the denominator is simply conventional and not really necessary!

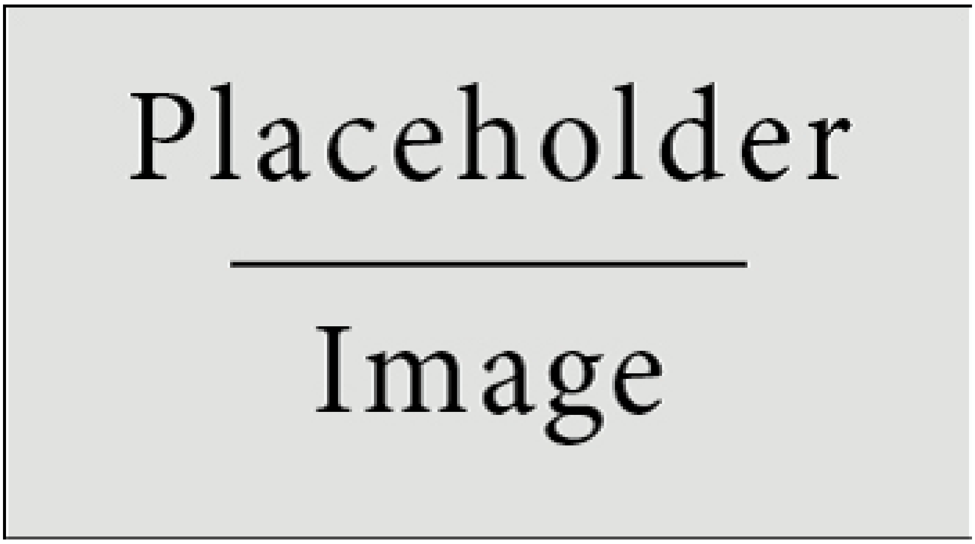

Figure 3.5: Drag coefficient of a sphere

$$c_D = f(\mathrm{Re}) \;, \tag{3.22}$$

which will become even clearer in Sec. 3.2.4. We now measure c_D for a lot of different Reynolds numbers to approximately determine the functional dependence, given by $f(\mathrm{Re})$. A possible result is illustrated in Fig. 3.5.

From this experimental result, we can now calculate the drag force D for all values of $U_\infty,\ l,\ \mu,\ \rho$, which lead to a Reynolds number in the measurement range. We simply need to insert $U_\infty,\ l,\ \mu,\ \rho$ into

$$D = c_D \frac{\rho}{2} U_\infty^2 A_l = f(\mathrm{Re}) \frac{\rho}{2} U_\infty^2 A_l \;. \tag{3.23}$$

This is a huge simplification leading to an enourmous reduction of the number of necessary measurements. We only need to cover a simple one-dimensional parameter space instead of the former four dimensional one.

Even though the concept of dynamically similar flows can make experiments a lot easier, the dependence on the Reynolds number can still be huge challenge. In wind energy research, for example, it is often difficult to reach realistically high Reynolds numbers in the laboratory, as will be discussed further in your exercise sheet. Similar problems arise when solving the NSE numerically since the computational effort is usually proportional to Re^3, as discussed further in Chap. 7.

Furthermore, it should be noted that other flows can depend on different and also on more non-dimensional parameters. For example, if the inflow in the problem above is given by $U(t) = U_\infty \sin(\frac{2\pi}{T} t)$, the flow becomes unsteady and a new characteristic time scale T is present in the flow. We would then additionally normalize the time by $\tilde{t} = \frac{t}{T}$. and the time-dependent NSE becomes

$$-(\tilde{\mathbf{u}}\nabla)\tilde{\mathbf{u}} - \nabla\tilde{P} + \frac{1}{\mathrm{Re}}\Delta\tilde{\mathbf{u}} = \mathrm{St}\,\partial_{\tilde{t}}\tilde{\mathbf{u}} \;, \tag{3.24}$$

where $\mathrm{St} = \frac{l}{U_\infty T}$ is an additional non-dimensional parameter called Strouhal number. Flows in this physical setting are dynamically similar, if both Reynolds and Strouhal number are the same.

It should be noted that the set of non-dimensional parameters, also called non-dimensional groups, is not unique and depends on your choice of normalization. However, if you normalize every variable involved and end up with non-dimensional equations and boundary conditions every set of non-dimensional parameters, obtained in this way, is correct. Hence, your choice of normalization cannot really lead to any mistakes. This is slightly different when the normalized equations are used in the context of approximations, which will become clear in the following section.

3.2.2 Approximations of the NSE

Another application of dimensional analysis is to use it to draw conclusions on the magnitude of the different terms in the equations. In some special cases, comparably small terms might then be neglected leading to a simplified set of equations. In order to compare different terms, the concrete choice of normalized variaables plays a more important role than in the context of dynamical similarity. We aim for a normalization which yields non-dimensional variables whose order order of magnitude is approximately 1.

Let us now go back to the example of the sphere and assume that the velocities occurring in the region of the flow are all of the same order as the inflow velocity and thus $\frac{\tilde{u}_i}{U_\infty}$ is of order 1 for all $i \in \{1, 2, 3\}$. In other terms, we can write $O(\frac{\tilde{u}_i}{U_\infty}) = O(1)$. The relevant length scales of the flow are assumed to be of order l. This means that significant changes of the velocity or pressure occur on this length scale and hence the gradient $\frac{\partial u_i}{\partial x_j}$ is of order $\frac{U_\infty}{l}$. Furthermore, we obtain $O(\frac{\partial^2 u_i}{\partial x_j^2}) = O(\frac{U_\infty}{l^2})$ for the second derivative.

Consequently, we find for the ratio of inertia force and viscous force

$$O(\frac{(\mathbf{u}\nabla)\mathbf{u}}{\nu\Delta\mathbf{u}}) = \frac{U_\infty l}{\nu} = \mathrm{Re}\ . \tag{3.25}$$

This helps to understand the important role of the Reynolds number. For relatively low Reynolds numbers viscous forces play an important role in the flow. Hence, perturbations in the flow get dissipated relatively fast, leading to very regular flow behavior such as the unidirectional flow in a pipe. For higher Re, the inertia forces and thus the nonlinear term in the NSE becomes more and more dominant. Perturbations cannot be completely dissipated but are amplified by the nonlinear inertia. This leads to instabilities which result in irregular and chaotic flow behavior, which is called *turbulent*. The flow in a pipe becomes fully turbulent for Re > 2600. Relevant flows in wind energy applications are almost all in the turbulent regime.

In the following, we will investigate whether we can approximate the NSE in the limit cases of very high and low Reynolds numbers.

Equations for Very High Reynolds Numbers

We keep the example of the sphere with the uniform inflow in our mind but the following analysis applies to all flows, which allow a similar normalization, such as a cylinder or an airfoil in the flow.

We use the normalization $\tilde{\mathbf{u}} = \frac{\mathbf{u}}{U_\infty}$, $\tilde{\mathbf{x}} = \frac{\mathbf{x}}{l}$, as introduced above, but reconsider the normalization of the pressure. The pressure is often the most tricky part in the normalization procedure.

Different choices are possible, such as $\tilde{P} = \frac{P-P_\infty}{\rho U_\infty^2}$ or $\tilde{P} = \frac{P-P_\infty}{\mu \frac{U_\infty}{L}}$. The pressure p here is not the thermodynamic pressure and its scaling is usually not directly determined by external parameters. It is therefore in some sense a passive variable and has to be normalized by the dominant effects in the flow. In the case of a very high Re, the inertial force is much higher than the viscous force (see Eq. (3.25)). Hence, the flow is dominated by the balance between pressure and inertia force. Consequently, we assume the pressure to scale with ρU_∞^2, which is related to the inertia the flow. As shown in Sec. 3.2.1, this scaling leads to the non-dimensional Eq. (3.18), given by

$$-(\tilde{\mathbf{u}}\nabla)\tilde{\mathbf{u}} - \nabla\tilde{P} + \frac{1}{\text{Re}}\Delta\tilde{\mathbf{u}} = 0$$

Neglecting the last term on the l.h.s. for high Re leads to

$$-(\tilde{\mathbf{u}}\nabla)\tilde{\mathbf{u}} - \nabla\tilde{P} = 0 \; , \tag{3.26}$$

which is basically a form of the steady Euler-equation. Even though the concept of inviscid flows can be very helpful (see Sec. 4.2), there are some fundamental problems with Eq. (3.26) and with the neglect of the viscous term in general. This neglect can actually not be valid near the surface of the sphere, where strong shear leads to stronger viscous forces. Near the surface, the velocity is not of order U_∞ but goes to zero when approaching it. Hence, close to the surface $\frac{1}{\text{Re}}$ is actually not a small parameter. The corresponding perturbation problem is therefore called singular, as further discussed in 5.7. Furthermore, it can also be shown that Eq. (3.26) cannot fulfill the no-slip condition on the sphere's surface, since derivative of the highest order (second order) has been neglected. Inviscid flows will be further discussed in Sec. 4.2.

Equations For Low Reynolds numbers

What happens if the Reynolds number is low? If, for example, we go swimming in a pot of honey or if the relevant length scales are very small as for the motion of a swimming bacteria (see FIG.). In this case, the viscous forces dominate over the inertial forces. Consequently, the pressure as a passive variable scales with $\mu\frac{U_\infty}{L}$, which is related to the viscous forces. Normalizing the NSE with

$$\tilde{\mathbf{u}} = \frac{\mathbf{u}}{U_\infty} \;\; , \;\; \tilde{P} = \frac{P-P_\infty}{\mu\frac{U_\infty}{L}} \;\; , \;\; \tilde{\mathbf{x}} = \frac{\mathbf{x}}{d} \; . \tag{3.27}$$

leads to

$$-\text{Re}(\tilde{\mathbf{u}}\nabla_{\tilde{\mathbf{x}}})\tilde{\mathbf{u}} - \nabla_{\tilde{\mathbf{x}}}\tilde{P} + \Delta_{\tilde{\mathbf{x}}}\tilde{\mathbf{u}} = 0 \; . \tag{3.28}$$

In the limit of small Re we obtain

$$\nabla\tilde{P} = \Delta\tilde{\mathbf{u}} \tag{3.29}$$

or

$$\boxed{\nabla P = \mu\Delta\mathbf{u}} \tag{3.30}$$

for the dimensionful variables (Check yourself). Eq. (3.30) is called the *Stokes equation* or the *creeping flow equation*. In contrast to the high Reynolds number-case, no fundamental problems arise when Re goes to zero. In other words, treating Re as a small parameter leads to a non-singular perturbation problem. Combined with the $\nabla \cdot \mathbf{u} = 0$, it provides a complete description of flows with very low Reynolds numbers. Since these equations are linear a lot of analytical theory exists to draw conclusions from these equations see e.g. leal.

The resulting flow at low Reynoldsnumbers can exhibit a lot of counterintuive effects, such as their reversible character, shown in

VIDEO

Such effects are counterintuitive, since humans are relatively large animals and mostly experience high Reynoldsnumber flows. However, a lot of the life on this planet is based on the motion on very small scales and hence Physics at low Re is sometimes even called the "Physics of Life", as in the following video.

VIDEO

If you are further interested in such kind of flows we refer you to... In the rest of this textbook we will only deal with intermediate and high Reynolds numbers since these are most relevant in the context of renewable energies.

3.2.3 Typical Non-dimensional Numbers

For the sake of completeness we will list a few typical non-dimensional numbers in the following.

- Reynolds number
- Froude number
- Mach Number
- shortly mention internal Froude and Richardson number but refer to chapter 8.

3.2.4 General Dimensional Analysis

Dimensional Analysis is a very powerful tool and not only useful when the underlying fundamental equations of a physical problem are known. It can be helpful in many areas of science and engineering and not only in the context of fluids. Let us start again with the steady flow around a sphere, but now assume we do not know the exact form of the underlying hydrodynamic equations. We are again interested in the drag force on the sphere. Since the only occurring parameters are $l,\ U_\infty,\ \rho,\ \mu$, we know

$$D = f(l, U_\infty, \rho, \mu)\ . \tag{3.31}$$

Obviously, not every dependence on these parameters is possible, since the function f must lead to the dimension of a force, given by $\frac{kg\,m}{s^2}$.

We could also reformulate our problem and say we search for a different function f fulfilling

$$\frac{D}{\rho A U_\infty^2} = f(l, U_\infty, \rho, \mu)\ . \tag{3.32}$$

In this case, f has to lead to a non-dimensional combination of $l,\ U_\infty,\ \rho,\ \mu$. We should therefore be able to choose a different function $\tilde{f}$, which only depends on non-dimensional parameters. For example, every term in a sum has to be dimensionless to yield a dimensionless result in the end. How can such dimensionless parameters be found? We could simply play around to find one or follow a systematic way by expressing such a parameter as a non-dimensional product given by

$$\Pi = U_\infty^a l^b \rho^c \mu^d\ , \tag{3.33}$$

where a, b, c, d are integers. Denoting the dimensions of mass, length and time as M, L, T respectively, we obtain

$$(LT^{-1})^a L^b (ML^{-3})^c (ML^{-1}T^{-1})^d = L^0 M^0 T^0 = 1 \,, \tag{3.34}$$

where we used that Π is dimensionless. Simple algebra shows that $a = 1, b = 1, c = 1, d = -1$ is a possible choice, as you will show in the exercises. Thus we obtain $\Pi = U_\infty l \rho \mu^{-1}$, which is the Reynolds number Re. It can be shown, as discussed later in this section, that no other relevant non-dimensional paramater exists in this case. We can therefore conclude

$$\frac{D}{\rho A U_\infty^2} = \tilde{f}(Re) \,, \tag{3.35}$$

as we also found from the normalized NSE. It is important to note that we found this result without any knowledge of the fundamental equations behind this problem! Awesome!

A similar analysis is possible for almost all physical problems but the dependence on non-dimensional parameters can be more complicated. An important role in this analysis is played by the Π-Theorem, presented in the following.

Π-Theorem

Let $\{q_1, q_2, ..., q_n\}$ be a set of dimensional variables fulfilling

$$F(q_1, q_2, ..., q_n) = 0 \,. \tag{3.36}$$

Furthermore, let r be the number of different basis units which occurr in the dimensions of $\{q_1, q_2, ..., q_n\}$.

Then the Π-Theorem states $n - r$ *independent* non-dimensional products $\{\pi_1, \pi_2, ...\pi_{n-r}\}$of the variables $\{q_1, q_2, ..., q_n\}$ exist which fulfill

$$\tilde{F}(\pi_1, \pi_2, ...\pi_{n-r}) = 0 \,. \tag{3.37}$$

A proof and a more precise formulation of this theorem can be found e.g. in Bridgeman1963...

Are you as confused as I am? Probably even more, if you heard of this for the first time. But what does this mean? Let us put this in very simple terms. In the relation in Eq. (3.31), five dimensionful variables occur.[4] We have three occurring basis units kg, m, s and thus the problem can be reduced to a dependence two non-dimensional products. As it turns out, one possible choice is given by $\frac{D}{\rho A U_\infty^2}$ and Re $= \frac{U_\infty l \rho}{\mu}$.

By normalizing the drag force in Eq. (3.32), we already did some of the work. In this case, c_D only depends 4 dimensionful variables $l,\ U_\infty,\ \rho,\ \mu$ with three basis units and thus the problem can be reduced to one non-dimensional product, which can be chosen as Re $= \frac{U_\infty l \rho}{\mu}$.

Not so difficult, is it? The real challenge lies in finding the non-dimensional products. This can be a lot of work, particularly if more than one non-dimensional product has to be found. However, it is not really difficult since a lot of systematic approaches exist to this problem as for

[4]Note that we could rewrite the relation as $\tilde{f}(l, U_\infty, \rho, \mu, D) = 0$ with $\tilde{f} = f - D$.

example illustrated in Kundu284 Durst216... for example, or several youtube videos discussing dimensional analysis. In

It is important to note that the non-dimensional products are not unique and often different ones can be chosen. If the problem is reduced to the dependence on only one non-dimensional product, as done above, this product is in some sense unique, since possible products can only differ by an exponent. For example, if Re is a possible non-dimensional product, so is Re^2, obviously.

Further Conclusions on the Drag on the Sphere

For very high Reynolds numbers, we know that the viscous forces in a fluid dominate over the inertial forces. Hence, we might suspect that we can drop the dependence on viscosity in $D = f(l, U_\infty, \rho, \mu)$ yielding

$$D = f(l, U_\infty, \rho) \ . \tag{3.38}$$

Now we have four occurring dimensionful variables l, U_∞, ρ, D and again three occurring basis units and thus the relation can be reduced to the dependence on non-dimensional product, which can be chosen as $\frac{D}{\rho U_\infty^2 l^2}$. It is easy to show that in this case we can conclude

$$D \propto \rho U_\infty^2 l^2 \ . \tag{3.39}$$

Experiments have shown that this easily obtained result actually leads to the the correct functional dependence in high Reynolds number flows.

For very low Reynolds numbers inertial forces play a minor role and we might reduce $D = f(l, U_\infty, \rho, \mu)$ to

$$D = f(l, U_\infty, \mu) \ . \tag{3.40}$$

Analogously to the argumentation above we can easily conclude that

$$D \propto \mu U_\infty l \ , \tag{3.41}$$

which has also been experimentally verified. Note that this result could have also been derived from the creeping flow equations . The non-dimensional form of these equations does not depend on any parameter. Hence, a non-dimensional quantity such as $\frac{D}{\mu U_\infty l}$ does neither and has to be given by a constant, which proofs Eq. (3.41).

The missing constant could be found by really solving the creeping flow equations in the case of the sphere with the uniform inflow, as done e.g. in .... This leads to

$$D = 6\pi\mu U_\infty \frac{l}{2} \ , \tag{3.42}$$

which is also called *Stokes law of resistance*.

3.3 Impulsively Started Plate: Diffusion of Momentum

So far, we have discussed uni-directional flows and dimensional analysis in this chapter. Even though there is no direct connection between these topics, we will now bring both topics together and investigate a unidirectional flow using a little bit of dimensional analysis. The difference to the flows discussed in Sec. 3.1 is that we now consider a time-dependent flow. Our goal is to investigate the diffusion of momentum. When you start a steady motion of sth., say a spoon, in perfectly still fluid. How fast does the momentum travel? When does the motion affect the fluid parcels far away. This is a very difficult problem and thus we begin again by looking at a very simple case.

We consider an infinite plate in the in the x_1x_2-plane and a fluid above it. The plane and the fluid are at rest for $t < 0$. At $t = 0$, the plane suddenly begins to move with a velocity U. We assume that there is no process of acceleration.

We assume that the flow is unidirectional and thus we can use Eq. (3.6), given by

$$\partial_t u = -\frac{1}{\rho}\partial_{x_1} p + \nu(\partial^2_{x_2} + \partial^2_{x_3})u \ .$$

Since the region of the flow is infinite in x_1- and x_2 direction, there cannot be any dependence on x_1 and x_2. This leads to

$$\partial_t u = \nu\partial^2_{x_2} u \ \text{ with } \ u = u(x_3, t) \tag{3.43}$$

subject to

$$u(x_3, 0) = 0 \ , \ u(0, t) = U, \ u(\infty, t) = 0 \ , \tag{3.44}$$

where we have assumed the motion of the plate does not affect fluid infinitely far away.[5] One possible way to solve this equation is using tools from dimensional analysis. Let us start by normalizing the velocity using $\tilde{u} = \frac{u}{U}$. This leads to

$$\partial_t \tilde{u} = \nu\partial^2_{x_2}\tilde{u} \ . \tag{3.45}$$

subject to

$$u_(x_3, 0) = 0 \ , \ u(0, t) = 1, u(\infty, t) = 0 \ . \tag{3.46}$$

Note that we have not normalized x_3 and t yet, since there is no obvious scaling at hand. The solution of the normalized equation can be expressed as

$$\tilde{u} = \frac{u}{U} = f(x_3, t, \nu) \ . \tag{3.47}$$

We know from dimensional analysis that this can be expressed by the dependence on only one non-dimensional product, since we have three dimensionful variables with two occurring basis dimensions (check for yourself). This product can be easily found to be $\frac{x_3}{\sqrt{\nu t}}$. Hence, our solution, in principle, depends only on one variable and thus it should be able to express Eq. (3.45) as an ordinary instead of a partial differential equation. Let us define $\eta := \frac{x_3}{\sqrt{\nu t}}$ fulfilling

$$\frac{u}{U} = F(\eta) \ . \tag{3.48}$$

Using this in Eq. (3.45), we can in fact obtain after some algebra (see e.g. Kundu309) an ordinary differential equation given by

$$-2\eta F' = F'' \ \text{ with } \ F(\infty) = 0, F(0) = 1 \ . \tag{3.49}$$

[5]This obviously makes a lot of sense for $t < \infty$.

This equation can be solved (see e.g. kundu) by

$$F = 1 - \text{erf}(\eta) \ , \tag{3.50}$$

where $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0 x e^{-x^2} dx$ is called error function. This finally leads to

$$\frac{u}{U_\infty} = 1 - \text{erf}(\frac{x_2}{2\sqrt{\nu t}}) \ . \tag{3.51}$$

This solution is sketched in Fig. 3.6 for two different points in time. What do we learn from this result?

First, we have definitely learned something about the diffusive nature of momentum. FIG illustrates how momentum is transported away from the moving plate setting more and more fluid in motion. What determines the time-scales of this transport? Let us define the thickness of the diffusive layer as the distance at which the velocity falls to 5%. You can look up that the error function $\text{erf}(x)$ reaches 0.95 at approximately 1.38 and thus we obtain $\frac{\delta}{\sqrt{\nu t}} = 1.38$ and consequently

$$\delta \propto \sqrt{\nu t} \ . \tag{3.52}$$

The thickness of the layer scales with $\sqrt{\nu t}$. Such a scaling is typical for all kinds of diffusive processes. Unsurprisingly, the parameter determining the rate of diffusion in the case of momentum is the viscosity. A higher viscosity therefore leads to a faster diffusion of momentum. This makes sense. Imagine a fluid with a huge viscosity, which almost acts like a solid. Doesn't it it seem natural that the momentum is transported really fast when one portion of the fluid begins to move?

A second lesson we have learned, is that Eq. (3.51) represents a *self-similar* solution. When normalizing u by U and the x_3 by the width of the diffusive layer $\sqrt{\nu t}$, all solutions collapse onto a single curve, as shown in Fig. 3.7. Thus, the functional dependence on x_3 stays the same for all times t and so does the dependence on t for all x_3. The reason for this behavior is that there is no characteristic length scale or time scale determined by the boundary or initial conditions of the problem. Thus, a similarity variable η could be defined reducing the original pde to a relatively simple ode. Self-similar flows and similarity solution appear quite often in fluid dynamical problems.

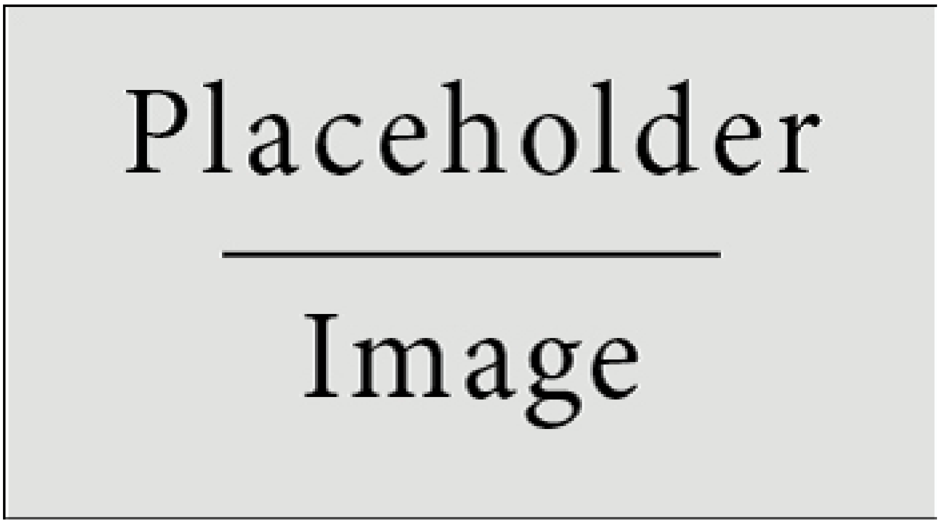

Figure 3.6: Impulsively started plate: Solution for two consecutive points in time. The figure is taken from [1].

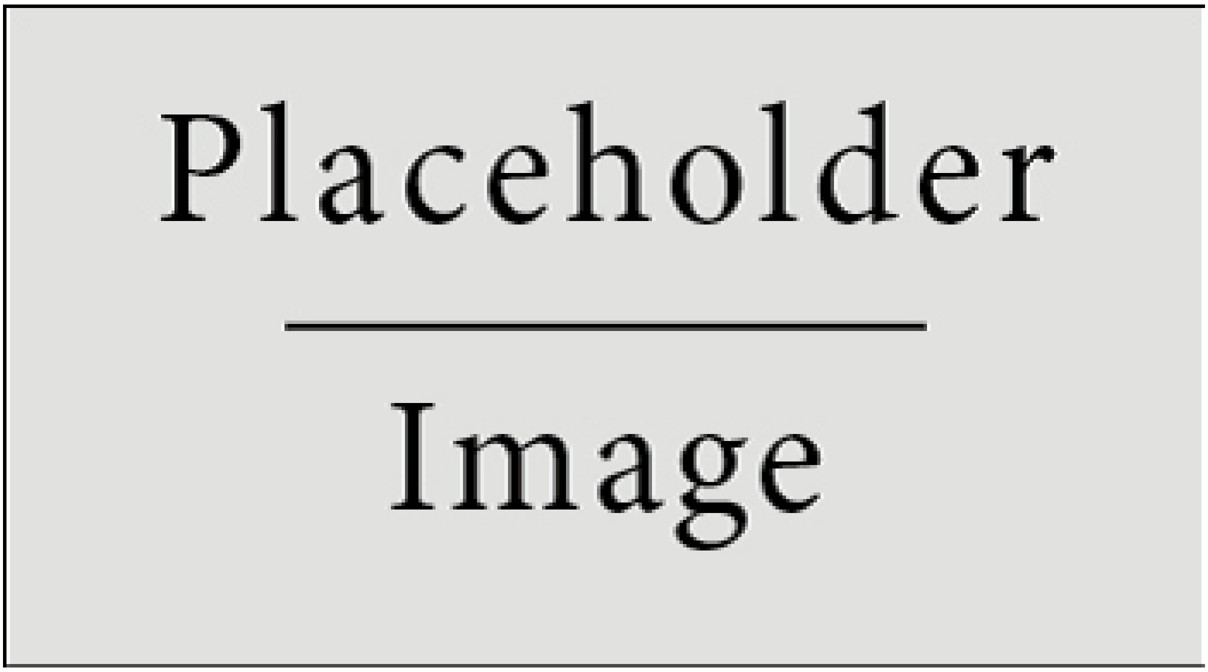

Figure 3.7: Impulsively started plate: Illustration of the similarity solution given by Eq. (3.51). The figure is taken from [1].

3.4 Summary of Chapter Three

- unidirectional flows, no nonlinear term, solutions of nse possible
- important solutions such as pipe flow occurring in a lot of problems...
- dimensional analysis teaches a lot, reduces number of relevant parameter to a minimal number of important non-dimensional products... Note an Re number.
- better understanding of the problem but also much easier experiments.
- Also how such non-dimensional numbers can help to a systematic approximations of the fundamental fluid dynamic equations.
- diffusive processes and momentum and self-similar which often occurs