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A Two-Stage Diffusion Modeling Approach to the Compelled-Response Task

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The issue of how perception and motor planning interact to generate a given choice between actions is a fundamental question in both psychology and neuroscience. Salinas and colleagues have developed a behavioral paradigm, the compelled-response task, where the signal that instructs the subject to make an eye movement is given before the cue that indicates which of two possible target choices is the correct one. When the cue is given rather late, the participant must guess and make an uninformed random choice. Perceptual performance can be tracked as a function of the amount of time during which sensory information is available. In Salinas' accelerated race-to-threshold model, two variables race against each other to a threshold, at which a saccade is initiated. The source of random variability is in the initial state of information buildup across trials. This implies that incorrect decisions are due to the inertia of the racing variables that have, at the start, sampled a constant buildup in the "wrong" direction. Here we suggest an alternative, non-time-homogeneous two-stage-diffusion model that is able to predict both response time distributions and choice probabilities with a few easy-to-interpret parameters and without assuming cross-trial parameter variability. It is falsifiable already at the level of qualitative features, for example, predicting bimodal reaction time (RT) distributions for particular gap times. It connects the compelled-response paradigm with an approach to decision making that has been uniquely successful in describing both behavioral and neural data in a variety of experimental settings for the last 40 years.

Keywords: compelled-response task, race-to-threshold model, two-stage diffusion model, non-time-homogeneity

Riding your bike in a busy city street, you constantly have to predict events and prepare for quick decisions. Will the pedestrian entering the road actually stop to let you ride past? Will the door of the car you are just about to pass open suddenly? In both cases, you must decide either to go straight on or swerve. Your response will be the result of a perceptual decision, but how long did it take you prepare the movement and to make the decision before executing your motor response? Disentangling these different components of the reaction time (*RT*) has proven to be a challenging task even under highly controlled laboratory conditions. Within a psychophysical paradigm only the total amount of *RT* is directly observable, but even when underlying neural circuits are accessible, the task is not straightforward because neurons that encode perceptual decisions are often also involved in motor planning (e.g., Horwitz & Newsome, 1999). Over the last 150 years, experimental psychology has developed a host of paradigms to unravel the elemental processes involved in generating a response in such tasks within a certain amount of time (see Luce, 1986, for an overview).

More recently Salinas and colleagues have developed a paradigm that seems particularly suited in separating the process of perceptual decision making from motor planning and execution (Salinas et al., 2010, 2014; Shankar et al., 2011; Stanford et al., 2010). In this *compelled-saccade* (CS) *task* participants start fixating a spot displayed in the middle of the screen, and the color of the spot, red or green, indicates the target color. Then, two yellow spots —potential targets—appear equidistant to the left and right of the fixation point. Next, the fixation point disappears telling the participant to initiate a saccade (go signal). At this time, the positions of target and distractor are not yet revealed. After a delay, the peripheral spots change color: One turns red and the other green. The delay between the offset of the fixation point (go screen) and the onset of the cue screen is called *gap time* and typically varies between 50 and 250 ms in steps of 25 ms. Thus, crucially, the signal that instructs the subject to start an eye movement, that is, the disappearance of the fixation point, is given prior to the cue (color identity) and indicates which of two possible choices is the correct one, the target. When the cue is given rather late (or, say, never), the participant must guess and make an uninformed random choice. In contrast, the earlier the cue is given the more informed the participant will be in making a choice. The idea is that the motor process is initiated early on and that perceptual information, once presented, influences a motor plan that is already evolving. Probing the paradigm on monkeys, it has been suggested that perceptual performance can be tracked as a function of the amount of time during which sensory information is available (called *tachometric curve*) independently of motor demands (Costello et al., 2013; Shankar

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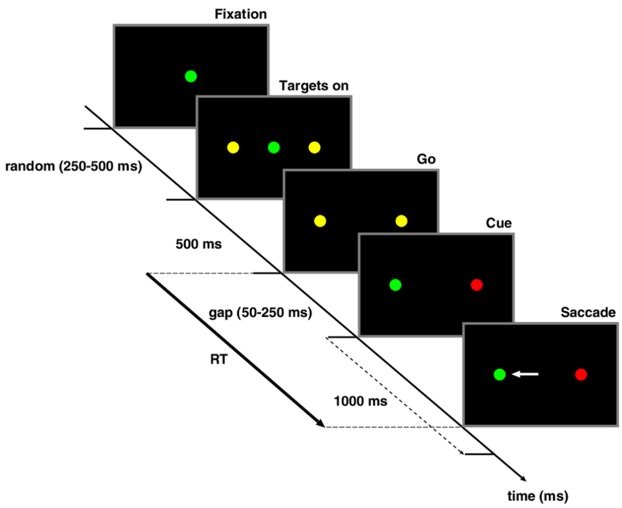
Figure 1
Compelled-Saccade Task



Note. Timeline of events in the CS task. The fixation point indicates the color of the target (green, in this example). The participant is instructed to gaze left or right as soon as the fixation point disappears (go screen). Target and distractor colors and positions are only revealed after a gap of 50–250 ms (onset of cue screen). A trial is correct if the participant makes an eye movement to the peripheral location that matches the color of the fixation point (green, in this example). Response time (RT) is defined from offset of the fixation point to saccade initiation. See the online article for the color version of this figure.

et al., 2011; Stanford et al., 2010;). Figure 1 shows the timeline of one trial (cf. Stanford et al., 2010, Figure 1).

Salinas and Stanford also proposed a computational model of perceptual decision making for the compelled-response task. Specifically, in their *accelerated race-to-threshold model* two variables race against each other to a threshold, at which a saccade is initiated. Each variable represents the motor plan to perform a saccade to one of the two spots, and the variable that first crosses the threshold determines the spot to which the saccade goes. Importantly, once the target position is signaled, the incoming perceptual information differentially modulates their trajectories toward it: The variable corresponding to the correct target accelerates and the one corresponding to the incorrect distractor decelerates. This model was fit successfully to data from two monkeys trained in this task. The authors showed that it captured both the full response time distributions and the dependence of decision accuracy on the effective sensory processing time. Specifically, they concluded that stimulus information (color identification) needs to be processed for just 25–50 ms to have an impact on the developing oculomotor plan.

Nevertheless, as observed by Drugowitsch and Pouget (Drugowitsch & Pouget, 2010) in a comment on the (Stanford et al., 2010) article, the accelerated race-to threshold model has an unusual feature: "*all stochasticity in the subject's responses is attributed to the random choice of the initial race speeds.* [. . .] *No extra noise is added at a later time, even after the delay period, when the color of the target is revealed. As a result, incorrect choices occur only because of the inertia of the racing variables.*" They also state that this is in contrast to standard models of decision making, such as diffusion or race models where response time variability is due to sensory noise and uncertainty in the stimulus itself, and suggest that *"it would be interesting to see whether the drift diffusion model and its neural counterparts would fit the data of these experiments as well as the deterministic race model . . . "* (see Drugowitsch & Pouget, 2010, p. 280).

Taking up the proposal by Drugowitsch and Pouget, here we propose an alternative to the race-to-threshold model, a *two-stage diffusion model* of behavior in the compelled response task. Note that we do not claim this alternative modeling approach—limited as it is here to behavioral data—to be uniquely preferable to the accelerated race-to threshold model. Rather, as a proof of principle, we show that stimulus processing in this task can be accounted for by an approach that (a) has been uniquely successful in describing the time-course of behavioral decision processes (RT and accuracy; e.g., Diederich, 1992; Ratcliff et al., 2016; Ricciardi, 1977) and that (b) has been shown to match the pattern of activity growth recorded in SC buildup neurons (Ratcliff et al., 2003; Smith & Ratcliff, 2004).

Note, that in psychology diffusion models with changing drift rates within one trial have been proposed for different applications. For instance, in the domain of the psychophysics of visual perception, Philip L. Smith and colleagues proposed a model, called time-changed diffusion model, in which the drift and diffusion coefficient grow in proportion to one another continuously over time. (e.g., Smith et al., 2010; Smith, Ratcliff, & Sewell, 2014; Smith & Lilburn, 2020). The change of drift/diffusion coefficients is closely related to the visual stimulus properties itself. Ratcliff (1980) proposed to divide—theoretically—the diffusion process into two parts but did not provide any solution to the problem. Diederich and colleagues have developed diffusion models with step changes in drift rate—to model shifts in attention to attributes of choice options and to formalize dual process models (e.g., Diederich, 1997, 2008; Diederich & Busemeyer, 2006; Diederich & Oswald, 2016; Diederich & Trueblood, 2018) Furthermore, they introduced these models to account for experimental situations in which stimuli from different modalities were presented at different points in time (stimulus onset asynchronies; e.g., Diederich, 1992, 1995).

It is to note, however, that neither the accelerated race-to threshold model nor simple diffusion models take into account that motor preparation processes are more complex and dynamic than represented in either modeling approach; that is, they currently ignore the possibility that the brain may exploit information that is in "the processing pipeline when the initial decision is made to subsequently either reverse or reaffirm the initial decision." (Resulaj et al., 2009, p. 263; see also Lepora & Pezzulo, 2015).

This note is organized as follows. After adding some details about the race-to-threshold model, we present the two-stage diffusion model in a non-technical manner and consider different scenarios for the compelled-response task. Various predictions of the two-stage diffusion model, hereafter abbreviated as 2SD model, are explicated in the two subsequent sections. Then we show how the 2SD model can account for the eye movement data from two monkeys presented in Stanford et al. (2010). We will point out in the final section what our demonstration contributes to the current discussion in the general area of race and diffusion modeling in psychology. Further technical details for both models are given in the appendix.

Accelerated Race-to-Threshold Model

In the *accelerated race-to-threshold model* (Shankar et al., 2011; Stanford et al., 2010), two competing variables, x_L and x_R, denote the developing motor plan to perform a saccade to one of the two targets (in neurophysiological terms, the activity of neurons that trigger eye movements to the left or right), and the variable reaching a certain fixed threshold first (the winner of the race) determines the direction of the saccade that occurs a short efferent delay later. Two different stages must be distinguished: One in which no cue information is yet available, and another, in which the target cue boosts one of the motor plans and suppresses the other. This information is incorporated into the model by assuming build-up rates r_R and r_L of x_L and x_R, respectively, drawn from a (truncated) bivariate Gaussian distribution; if one of the variables reaches a threshold during this stage, the outcome is a coin toss because the buildup rates were sampled randomly from a symmetric distribution. Otherwise, the two oculomotor plans keep changing until the cue information arrives. Once the cue arrives showing, for example, that the target is on the right-hand side, the build-up rate of x_R increases and that of x_L decreases (again at constant rates) until a threshold is reached. Figure 2 depicts the trajectories of six developing motor plans (hypothetical trials) for the race-to-threshold model. The figure legend includes details on specific assumptions.

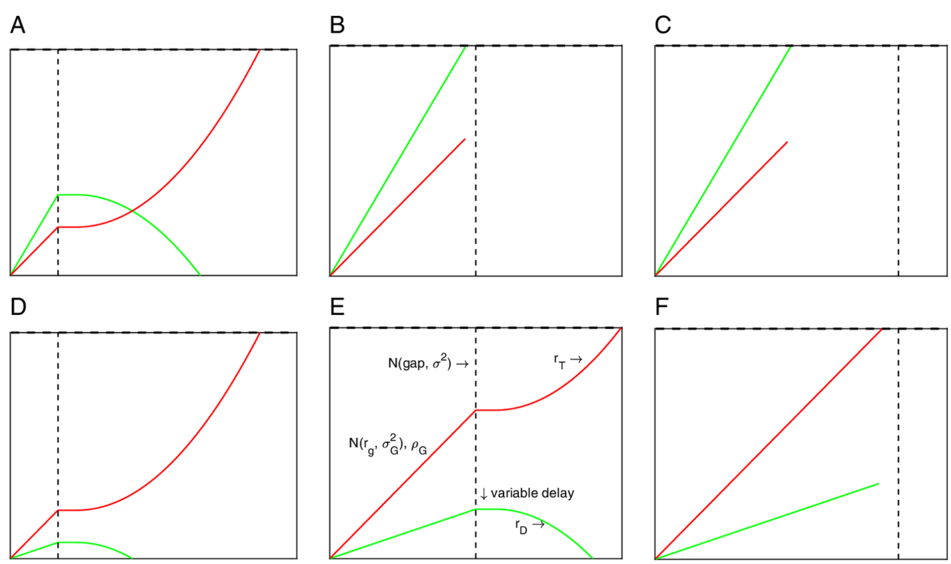
The Two-Stage Diffusion Model

The two-stage diffusion model, like the accelerated race-to-threshold model, assumes that the decision is driven by two sub-processes, the first stage feeding directly into the second. The diffusion model is conceived as a stochastic process, $X(t)$, representing the numerical value of the accumulated "evidence" at time t with drift rate, $\mu_i(x, t)$ ($i = 1, 2$), changing from the first to the second stage. Note that, here, "evidence" is conceived of as an abstract concept and may have different interpretations (see below).

The first stage represents the process from onset of the go signal to the cue signal (the *gap time*). During the gap time, the direction of the trajectory may randomly change from right to left and *vice versa*. Because no target evidence is yet provided, the probability to make an eye movement to the right or left is the same, that is, $\mu_1(x, t) = 0$. With increasing gap time, the probability to reach one of the criteria for making a right or left movement increases (Figure 3). Thus, while the drift rate is zero, the latter fact allows interpreting $X(t)$ in the first stage as representing an (increasing) propensity-to-respond signal because noise is accumulating.

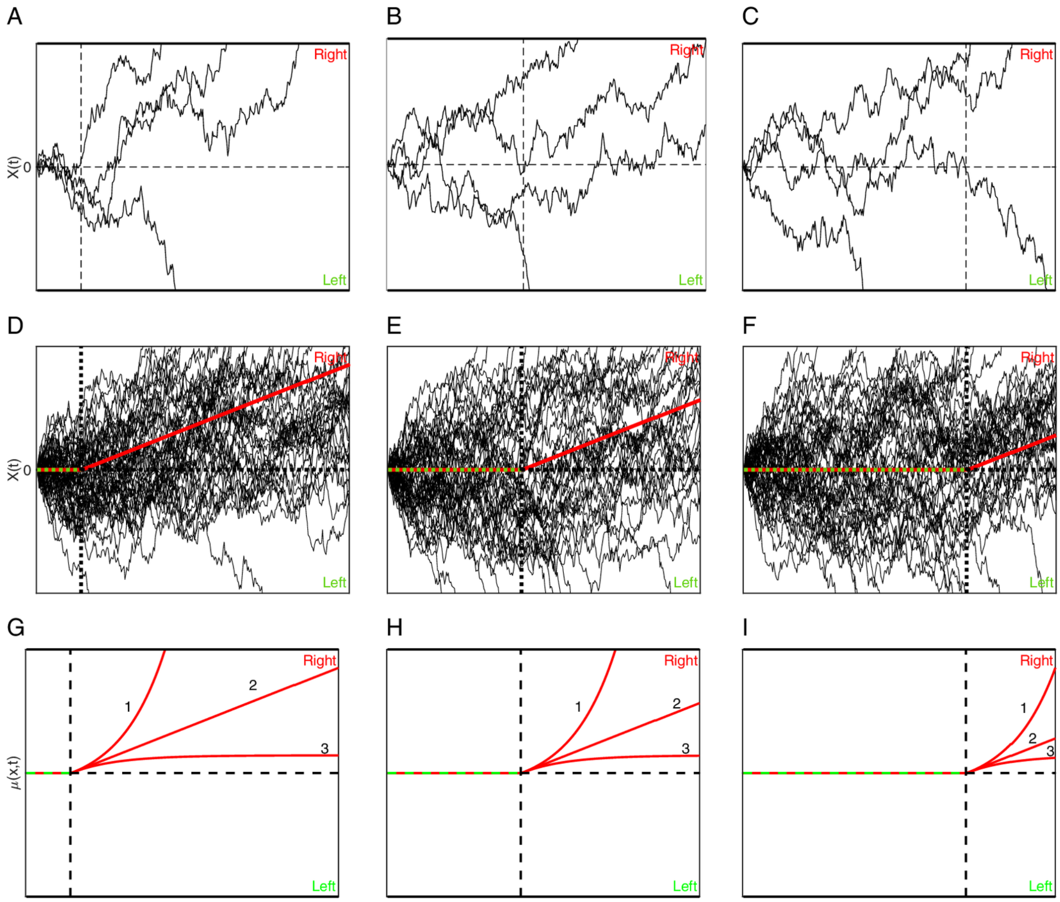
The second stage encompasses the process from cue onset to saccade initiation. For a given trial, the accumulation process of the second stage starts at the level where the process of the first stage

Figure 2
Accelerated Race-to-Threshold Model



Note. Hypothetical trajectories of six developing motor plans for the race-to-threshold model with gaps of 50 ms (A and D), 150 ms (B and E) and 250 ms (C and F), indicated by the dotted vertical lines. The target here is always red and located to the right of fixation. The top line of each panel is the threshold. Trials in which the red trajectory crosses the threshold first are correct, and trials in which the green trajectory crosses the threshold first are errors. In A–C, the rate is larger for the incorrect response and it wins the race for larger gaps. In D–F the rate for the correct response is larger and it always wins the race. E shows some details on the parameters. In the first stage (no cue information up to the gap time, vertical dotted line), buildup rates are drawn from a (truncated) bivariate Gaussian distribution and run in parallel. The winner determines the response. The gap varies according to a normal distribution with mean gap and variance identical to the variance of the non-decision time. Before and after the gap there is a variable time of inactivity or delay (slope = 0). Onset and offset of this time are free parameters. In the second stage, the buildup accelerate for targets (r_T) and decelerate for distractors (r_D). The remaining parameters are described in the appendix. See the online article for the color version of this figure.

Figure 3
Two-Stage-Diffusion Model



Note. For these examples, the target is red, located on the right-hand side. Each trial is presented as a trajectory from a stochastic process $X(t)$. A–C show four trials with three different gap times [ms]: 50 (A), 150 (B), and 250 (C), indicated by vertical dashed lines. Once a trajectory crosses a threshold (upper and lower horizontal bold face lines), a response is initiated. The upper boundary is associated with the right response, the lower one with a left response. In A, three out of four trajectories cross the upper threshold and, therefore, three correct responses are initiated. In B, two trajectories cross the thresholds and a correct resp. incorrect response are initiated. From the figure it is not clear yet where the two remaining trajectories would end up. In C, one response occurs during the gap (trajectory hits the lower threshold) and in this example, an incorrect response is initiated. Of the remaining three trials, two are correct responses (hitting the upper threshold) and one is incorrect (hitting the lower threshold). D–F show the same scenario with 50 trials each. The straight lines indicate the expected (mean) values of all trajectories. From the onset to the cue, the expected value is the same for making a movement to the right or left side, that is, the drift rate of the process is zero (red-green line). After the target color has been revealed, the chances for making a correct response increase and the drift rate in this example is positive (red line). G–I show the mean drift (expected value) for three different model assumptions after the target appeared. Case 1: OUP with $\gamma < 0$; Case 2: Wiener process and identical to the one in D–F respectively; and Case 3 with $\gamma > 0$. See the online article for the color version of this figure.

ended, continuing now with a drift rate $\mu_2(x, t)$, different from zero. Depending on gap time length, the first process may be closer or farther away from the decision criterion (threshold), and the direction of the trajectory in the second stage may be reversed with more or less success. Once the cumulative evidence exceeds a threshold criterion, the process stops and a response is initiated. Let us consider the behavior of this stochastic process in more detail.

Different Scenarios for the Trajectories in the Two-Stage Diffusion Model

In this example, the target is a red dot presented on the right, a correct response is initiated as soon as the upper threshold is crossed. Thus, the upper threshold is associated with response *R* (looking to the right), the lower one with response *L* (looking to the left).

Figure 3 shows the basic ideas of the model with hypothetical trajectories. The decision process for each trial is presented as a trajectory generated from the stochastic process $X(t)$. Assuming that the target is red and presented on the right, a correct response is initiated a soon as the upper threshold, $\theta > 0$, is crossed: $X(t) = \theta$. Alternatively, as soon as the accumulated evidence reaches the lower criterion value [here, $X(t) = \theta < 0$], the incorrect response L is initiated.

A–C of Figure 3 show four trials with three different gap times [ms]. A: 50, B: 100, and C: 250, indicated by the vertical dashed lines. At that time point, the process switches from stage 1 to stage 2. Once a trajectory crosses a threshold (upper or lower horizontal bold face lines) a response is initiated. In A, three out of four trajectories cross the upper threshold and, therefore, three correct responses are initiated. Note that even if the evidence is overwhelmingly strong (when the cue is presented rather soon after the go signal), there is a small chance for making an incorrect response, which is an integral part of the stochastic processing mechanism. In B, one trajectory crosses the upper, another crosses the lower threshold and a correct and incorrect response is initiated, respectively. From the figure it is not visible where the two remaining trajectories would end up, that is, initiating an R or L response. In C, one response occurs during the gap time (trajectory hits the lower threshold) and, in this case, an incorrect response is initiated. Of the remaining three trials, two are correct responses (hitting the upper threshold) and one is incorrect (hitting the lower threshold).

D–F of Figure 3 illustrate the same scenario with 50 trials each. The simulated trajectories show that with an increasing number of trials, the overall process converges toward specific directions. In D, with a short gap time, the vast majority of trajectories are absorbed at the upper threshold and most of the initiated responses are correct. The reason is that the information about the target occurs almost immediately after the go signal, so an incorrect direction can more easily be reversed. In F, with a long gap time, about half of the trajectories are absorbed at the upper or lower threshold. That is, with long gap times, the chances to make a correct or an incorrect response are about the same. A redirection of the movement is almost impossible. The bold lines in D–F indicate the expected (mean) values of all trajectories. From the onset of the go signal up to the cue, the expected value is the same for making a movement to the right or left side, that is, the drift rate of the process is zero (red-green line). After the target color has been revealed, the chances for making a correct response increase and the drift rate, for this example, becomes positive (red line).

G–I show the mean drifts (expected value) for three different model versions after the target appearance. Stage 1 is the same for all versions and described by a Wiener process with drift rate $\mu_1(x, t) = 0$. After the target color has been revealed (in the example, red)—stage 2—evidence accumulation for making an R response is directed toward that threshold. First, consider the mean drift case 2 that assumes a Wiener process with drift rate $\mu_2(x, t) = \delta > 0$, which is identical to the one in D–F: the expected (mean) evidence increases linearly as a function of time.

A widely accepted assumption in diffusion modeling is that the drift rate mirrors the quality of the stimuli (e.g., Ratcliff et al., 2016): The better the quality, the larger is the drift rate (slope). For instance, when the target and distractor colors are difficult to discriminate (e.g., yellow and orange instead of red and green), then the drift rate is assumed to be smaller and the probability for a correct response decreases for all gap times.

Stanford et al. (2010) argue that, after the target color has been revealed, the process for making a correct response accelerates and the one for an incorrect slows down. The 2SD model incorporates acceleration/slowing down in processing in stage 2 depending on the current state by modifying the drift rates to $\mu_2(x, t) = \delta - (-\gamma)x$ and $\mu_2(x, t) = \delta - \gamma x$, respectively, and shown in Figure 3 G–I, case 1 (acceleration) and case 3 (slowing down). The processes are instances of the Ornstein–Uhlenbeck process (OUP; for details see e. g., Diederich, 1992; Ricciardi, 1977; Smith, 2000).

Predictions

The two-stage diffusion model allows us to derive several qualitative and quantitative predictions from the above definitions and assumptions.

1. The probability for making a correct response decreases as the gap time increases.

As shown in Figure 3 for very short gap times, information about the target is revealed almost immediately after the go signal leading to a high probability of making a correct response. The longer the gap the more trials have either been finished during the gap time, that is, guessing about the correct side, or the process is too close to one threshold so that reverting incorrect direction is no longer possible, that is, also leading to chance level. Figure 4 shows the predictions of a Wiener process in stage 2 for a large range of parameter values μ_2, gaps, and three different threshold that is, A: low, B: medium, and C: high. The same pattern occurs when assuming an OUP in stage 2 (not shown).

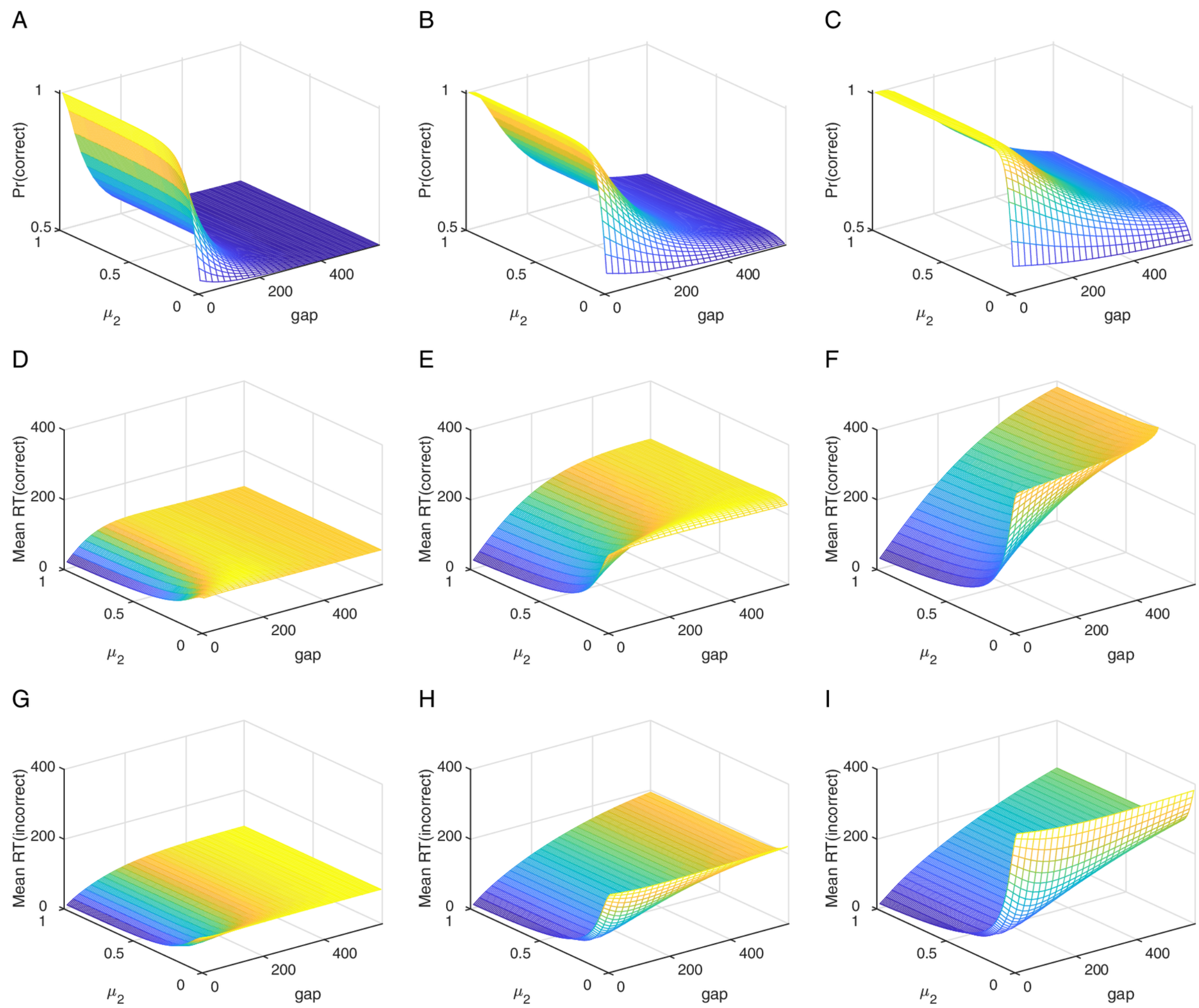
2. The mean choice response time for incorrect responses is shorter than for correct responses (fast errors).

During the gap time no information about the target is yet available. Correct and incorrect responses during this time are random with probability .5. After the target is presented information for a correct response is available. That is, when an incorrect response occurs it needs to be fast before new evidence for a correct response is provided. When responses are made only during gap time, that is, for long ones, the mean choice response times for correct and incorrect responses are identical. Figure 4 (D–I) shows the predictions assuming a Wiener process in stage 2 for a large range of parameter values μ_2, gaps, and three different thresholds. (D–F) refer to mean choice times for correct response; (G–I) for incorrect responses. The same pattern occurs when assuming an OUP in stage 2 (not shown).

3. The probability distributions are right skewed and bimodal. Furthermore, the overall pattern of the probability distributions is the same regardless of the specific process (Wiener, OUP) of stage 2.

Figure 5 shows predicted distributions for the three versions of the model introduced above as a function of gap time. To make them comparable [and in accordance with Stanford et al. (2010)] they are unconditioned (sum of both areas adds up to one) and normalized (all values are divided by the largest value). Assuming a slowing down depending on the current state in stage 2 obviously predicts slower RT (left column), while an acceleration in stage 2 predicts faster RTs (right column) as compared to a linear increase in

Figure 4
Predicted Choice Probabilities and Mean Choice RT



Note. A–C: Predicted choice probabilities for a correct response as a function of μ_2 and gap times and three different thresholds. A: low; B: medium; C: high. D–F: Predicted mean choice response times for correct responses as a function of μ_2 and gap times for the respective three different thresholds. G–I: Predicted mean choice response times for incorrect responses as a function of μ_2 and gap times for the respective three different thresholds. See the online article for the color version of this figure.

evidence (middle column). When all responses are made within the gap time (last row), the process is driven only by stage 1, for which we assumed a standard Wiener process. Thus, the distributions for correct and incorrect response are identical, thus mean RTs are also identical and the probability to choose *R* or L is .5.

Some Extensions

The basic decision model with two or three parameters (Wiener: drift rate δ_2 for stage 2 and threshold θ; δ, γ, and θ for OUP) can be extended to account for experimentally induced effects and other theoretically motivated variations.

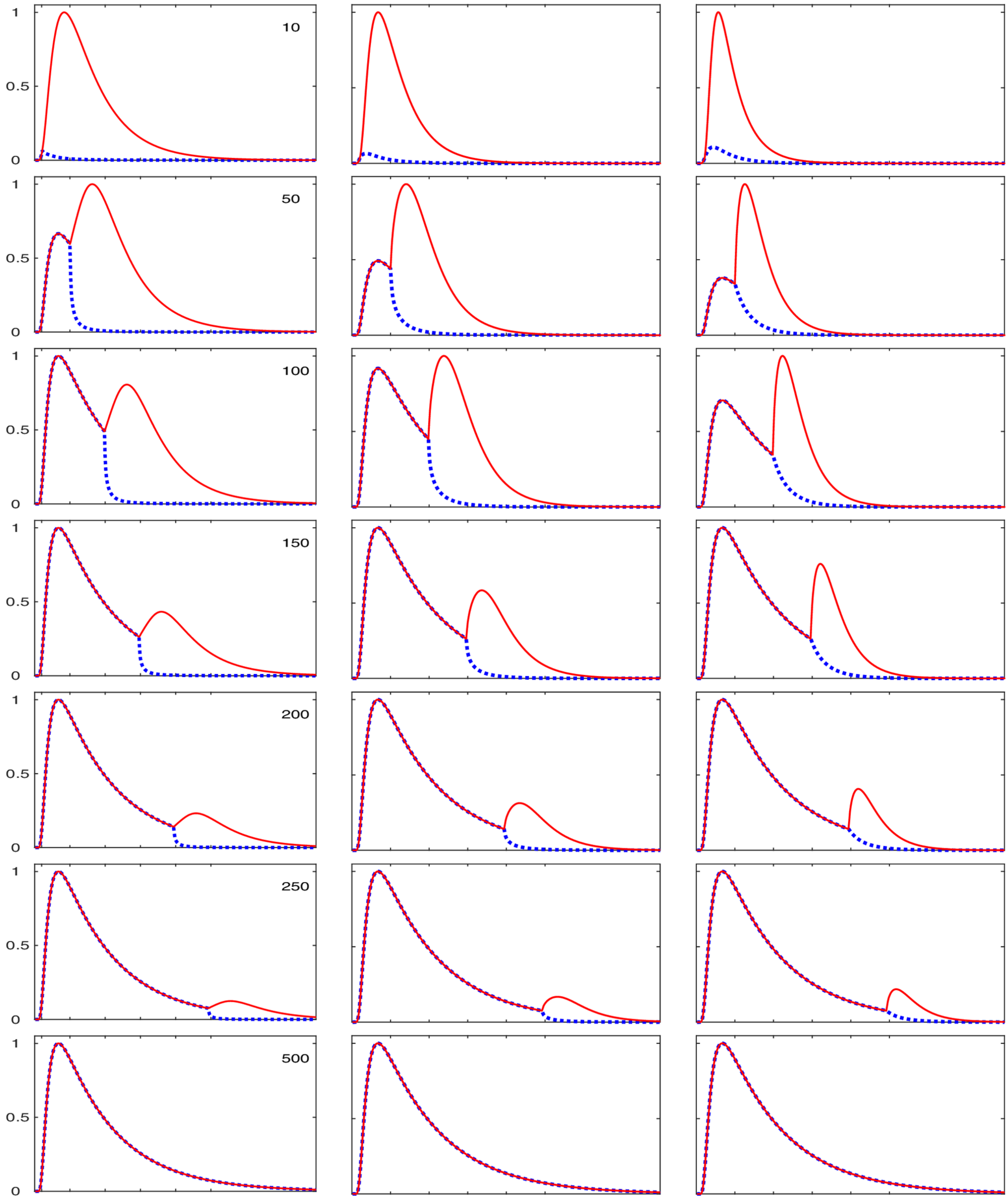
1. Biases toward the left or right may be reflected in the starting point of the process or in the drift rate of stage 1.

Instructions or rewards may influence the accumulation process. A tendency to look to the right, for instance, may be incorporated by shifting the starting position $X(0)$ closer to the right-hand threshold [in the current example, $X(0) > 0$]. Similarly, a tendency to look to the left is reflected by assuming $X(0) < 0$. In this case, the bias may wear off when the target appears late (long gaps). Assuming that the bias is incorporated into the drift rate, that is, $\mu_1(x, t) \neq 0$ the bias becomes stronger with longer gaps (Diederich, 2008; Diederich & Busemeyer, 2006).

2. The gap time may not be fixed but may follow a distribution.

For the basic model, we assume that the switch from stage 1 to stage 2 occurs exactly at the respective gap time point. That is, the switch occurs according to the time course of the experimental set-up. However, it is conceivable that the switch is delayed and/or follows a

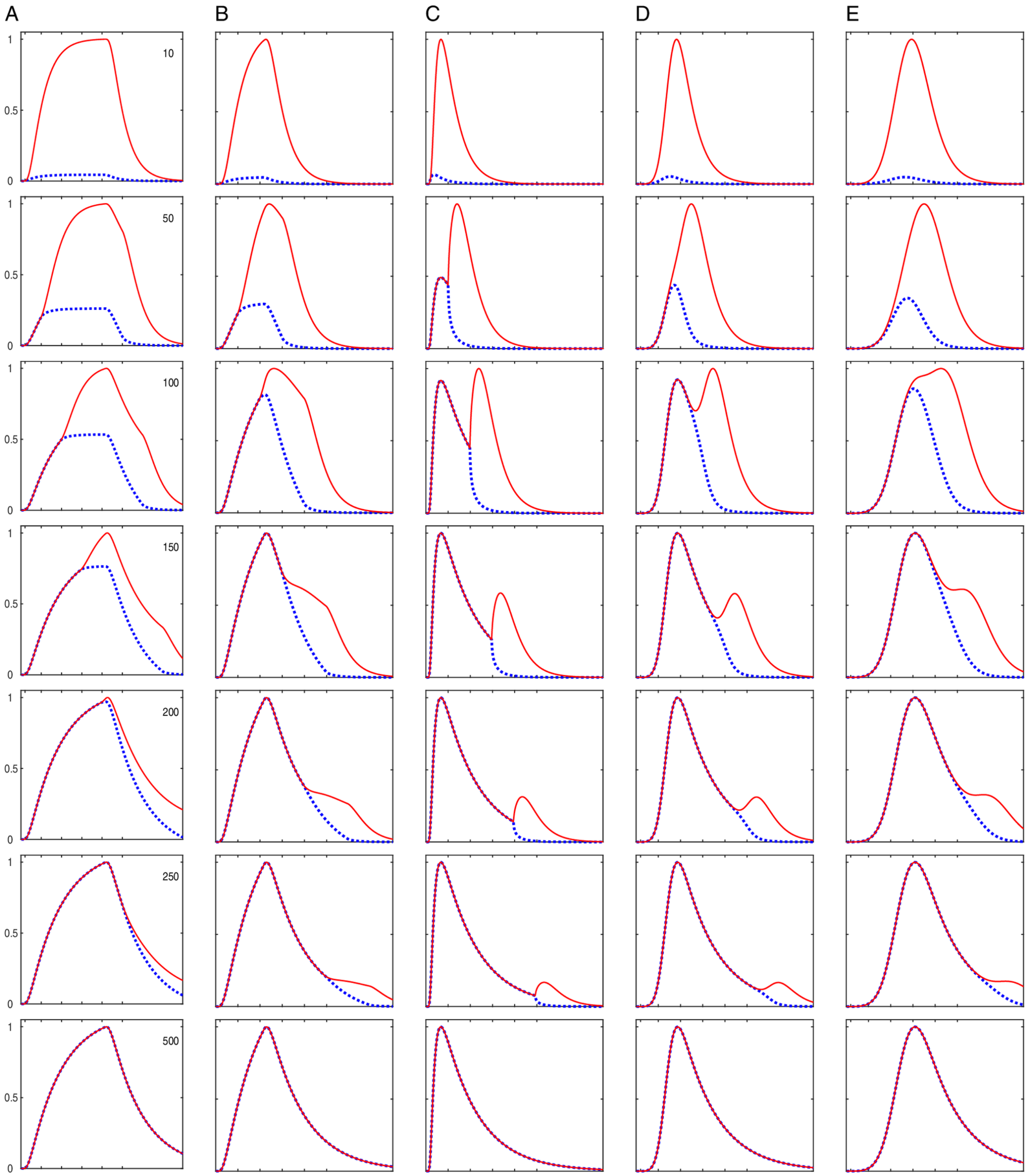
Figure 5
Predicted Probability Distributions



Note. Each column shows the predicted RT probability distribution (correct: red, incorrect: blue) for one model version for different gap times. Left : OUP with $\mu_2(x, t) = .2 - .02$; center: Wiener with $\mu_2(x, t) = .2$; right: OUP with $\mu_2(x, t) = .2 + .02$. As in Shankar et al. (2011), distributions are unconditioned (with respect to correct/incorrect) and normalized (division by maximum value). Time (arbitrary time units, labels omitted) is on the x axis. Gap times (50, 100, 150 200, 250, 500 ms) are shown in the upper right corner of the first column panels. See the online article for the color version of this figure.

Figure 6

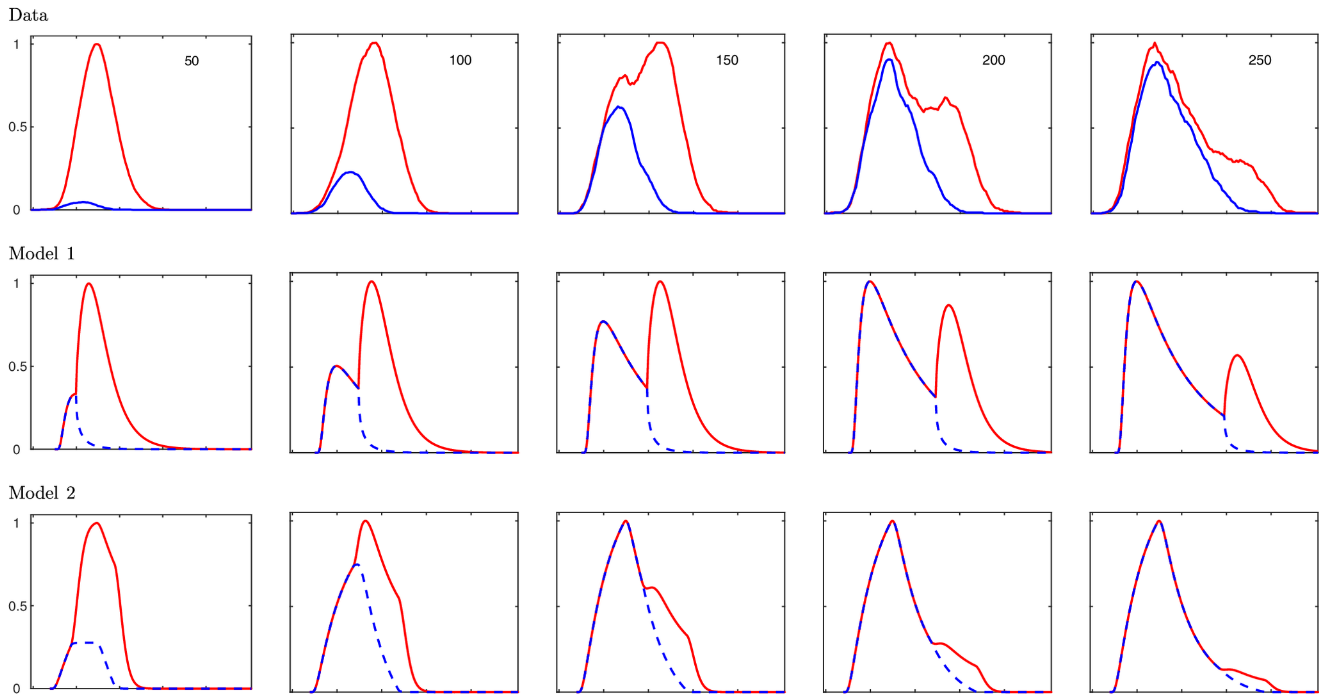
Predicted Probability Distributions With Different Non-Decision Times



Note. Each column shows the predicted RT probability distribution (correct: red, incorrect: blue) for $R_T = T_D + T_{ND}$. T_D follows a Wiener process with drift as in five. A and B: T_{ND} is uniformly distributed across the range of 50 to 250 time units (A) and of 100 to 200 time units (B). C: T_{ND} is fixed, here to zero. Increasing it shifts the distribution to the right. D and E: T_{ND} follow a normal distribution with mean 150 and standard deviation 15 (D) and 30 (E). See the online article for the color version of this figure.

Figure 7

Data and Model Account for Monkey S: Estimation



Note. The upper row shows the smoothed empirical distributions for correct (red) and incorrect (blue) response times for gap conditions 50, 100, 150, 200 and 250 ms. Model parameters were estimated *only* from these data. Model 1 (second row) depicts the corresponding predictions from the two-stage diffusion model assuming a fixed residual time. The estimated parameters values are $\mu_2 = 0.2$, $\theta = 12$, and T_{ND} =150 ms. Model 2 (third row) assumes the same underlying two-stage decision model but with a uniformly distributed residual time. The estimated parameters values here are $\mu_2 = 0.427$, $\theta = 9$, and $E(T_{ND})$ =139 ms with a fixed time range of 100 ms. See the online article for the color version of this figure.

probability distribution. In this case, the first stage exceeds the actual gap time and a (variable) time step is added. Assuming a fixed delay is similar to prolonging the actual gap time and decreases the second peak of the distribution, as seen from Figure 5. Assuming distributions on the delay requires a specification of it. (Diederich & Oswald, 2014) investigated several distributions for the switching time, among them geometric, Poisson, uniform. The exact shape of the distributions changes whereas, however, the overall pattern (bimodal, fast error for the present parameters) does not. Note that in all these cases, the drift rate is still $\mu_1 = 0$ (i.e., no information). Rather than assuming the drift rate and diffusion coefficient to be a smooth, time-varying function already from the beginning of a trial (e.g., Smith & Ratcliff, 2009), here it is the change of color carrying the discriminative information that initiates modulation of the drift.

3. The threshold may not be constant but change as a function of gap time.

The basic model assumes a constant threshold. However, it is possible that with increasing gap time the threshold converges (or diverges). For the two-stage diffusion model, (Diederich & Oswald, 2016) derived prediction assuming various functional forms for the thresholds and, most importantly and different from all other models, with converging boundaries, separately for stages 1 and 2. Also note that the threshold may be related to the available time to make a decision. That is, the threshold is assumed to be a function of deadlines with smaller thresholds for time constraints (Diederich & Oswald, 2016).

At the moment, we do not consider these model extensions any further, but they could be incorporated if physiological evidence require it and if corresponding data to test those assumptions were available.

Two-Stage Diffusion Model Account of Data From Stanford et al., 2010

In the following, we show how the basic two-stage diffusion model accounts for the data from (Stanford et al., 2010; data for monkeys S and G, provided by courtesy of Emilio Salinas).

Typically, response time, RT, is assumed to be the sum of two random variables, the decision time T_D, here modeled by the diffusion model, and a non-decision time T_{ND} (e.g., Luce, 1986):

$$RT = T_D + T_{ND}. \tag{1}$$

Various assumptions have been made about the non-decision time, such as a normal distribution with a small variance, as done in (Stanford et al., 2010); for references see also (Luce, 1986). Often, a uniform distribution is postulated since it is mathematically very tractable (e.g., Ratcliff & Tuerlinckx, 2002), or it is simply set to a

Table 1
Estimates Parameters for Two Monkey and the Two Models

	Monkey S		Monkey G	
	Model 1	Model 2	Model 1	Model 2
μ_2	.200	.427	.200	.228
θ	12	9	12	10
$E(T_{ND})$	150	139	160	139

constant because behavioral data do not allow to identify exactly the separate parts in Equation 1, (but see Verdonck & Tuerlinckx, 2016, for special cases). Figure 6 in the appendix shows the prediction of the model when uniformly distributed non-decision times are assumed (panel A) and when a truncated normal distribution is assumed (panel B). Judging from the figure, the effect on the distributional shape is more pronounced when a uniform distribution is postulated, as compared to a normal distribution. In the following, model 1 includes T_{ND} as a constant, and model 2 T_{ND} as a uniformly distributed random variable. Thus, the T_D part of the two-stage diffusion model has only two parameters to be estimated from the data: Drift rate μ_2 and threshold θ (note that $\mu_1 = 0$ a priori). Setting T_{ND} to a constant adds another parameter; and assuming a distribution for T_{ND}, the number of additional parameters depends on the specific distribution. Here we limited it also to one additional parameter by fixing the variance (see, e.g., Smith & Ratcliff, 2009).

The model is implemented as a continuous Markov process approximated by a discrete Markov chain (for the matrix approach, see Diederich, 1997; Diederich & Busemeyer, 2003; Diederich & Oswald, 2014, 2016). The diffusion coefficient was set to $\sigma^2 = 1$; the time unit to $\tau = .1$ (i.e., 10 sample points per 1 ms). The resulting step size of the process is $\Delta = \sqrt{\tau}\sigma$.[1]. The step size reflects the amount of evidence sampled per time unit (for details, see appendix).

Importantly, in testing the model we first estimate the parameters from a subset of the (smoothed) data for some of the gap conditions and then predict the remaining gap conditions as well as mean choice response times and choice probabilities. In particular, we estimate the parameters using maximum likelihood estimation and utilize the Matlab routine *fminsearchbnd*, which allows parameters restrictions. It is similar to *fminsearch* that uses the Nelder–Mead simplex search method. The estimated parameter values are in Table 1, in the appendix.

Figure 7 to Figure 10 show the data and the models‘ accounts for monkeys *S* and *G*. To generate the empirical distributions in Figures 7 to 10, upper rows and labeled Data, the observed response times were arranged in bins of 4 ms and then smoothed with a moving average filter with filter length $M = 20$ (see Stanford et al., 2010). Parameters were estimated from gap conditions 50, 100, 150, 200, and 250 ms (Figure 7 and Figure 9, upper rows) and the respective accounts of Model 1 (constant T_{ND}) and Model 2 (uniformly distributed T_{ND}) are shown in the second and third rows of Figure 7 and Figure 9, respectively. With these parameter estimates, the models predict the patterns (Figure 8 and Figure 10, second and third rows) for the gap conditions 75, 125, 175, and 225 ms for data (Figure 8 and Figure 10, first rows). Furthermore, with the same parameters, the models predict choice probabilities and mean choice response times for correct and incorrect choices.

Overall, the different models (based on T_{ND}) give a good qualitative account of the data. As predicted, mean choice response times are faster for incorrect responses ("fast errors") for all gap times. The bimodality observed in the data, in particular for medium gap times, could be accounted for assuming a constant T_{ND}. The bimodality is less pronounced when a uniform distribution of the non-decision time is assumed. Quantitatively, the models overestimate/underestimate for some gap conditions. Given that only three parameters were estimated from just one half of the data (two parameters for T_D and one parameter for T_{ND}, the model provides a reasonable account of the data.

Discussion

The CS task, introduced in Stanford et al. (2010) and further analyzed in a number of eye movement studies (e.g., Costello et al., 2013; Salinas et al., 2014; Salinas & Stanford, 2013, 2018; Shankar et al., 2011), has been very successful in tracking perceptual performance. Salinas and colleagues have developed a simple, heuristic model that is able to reproduce most of the observed behavior while separating the perceptual from motor processing components. As observed by Drugowitsch and Pouget (2010), however, this accelerated-race-to-threshold model does not square with the most successful theoretical approach for decision making under time pressure, that is, stochastic diffusion models (e.g., Ratcliff et al., 2016). Specifically, in the former, the primary source of random variability is in the initial state of information buildup across trials whereas, in diffusion-type models, variable sensory information is accumulated randomly fluctuating over the entire duration of the trial. Thus, the accelerated-race-to-threshold model implies that incorrect decisions are due to the inertia of the racing variables that have started with an across-trials randomly sampled constant buildup in the "wrong" direction[2].

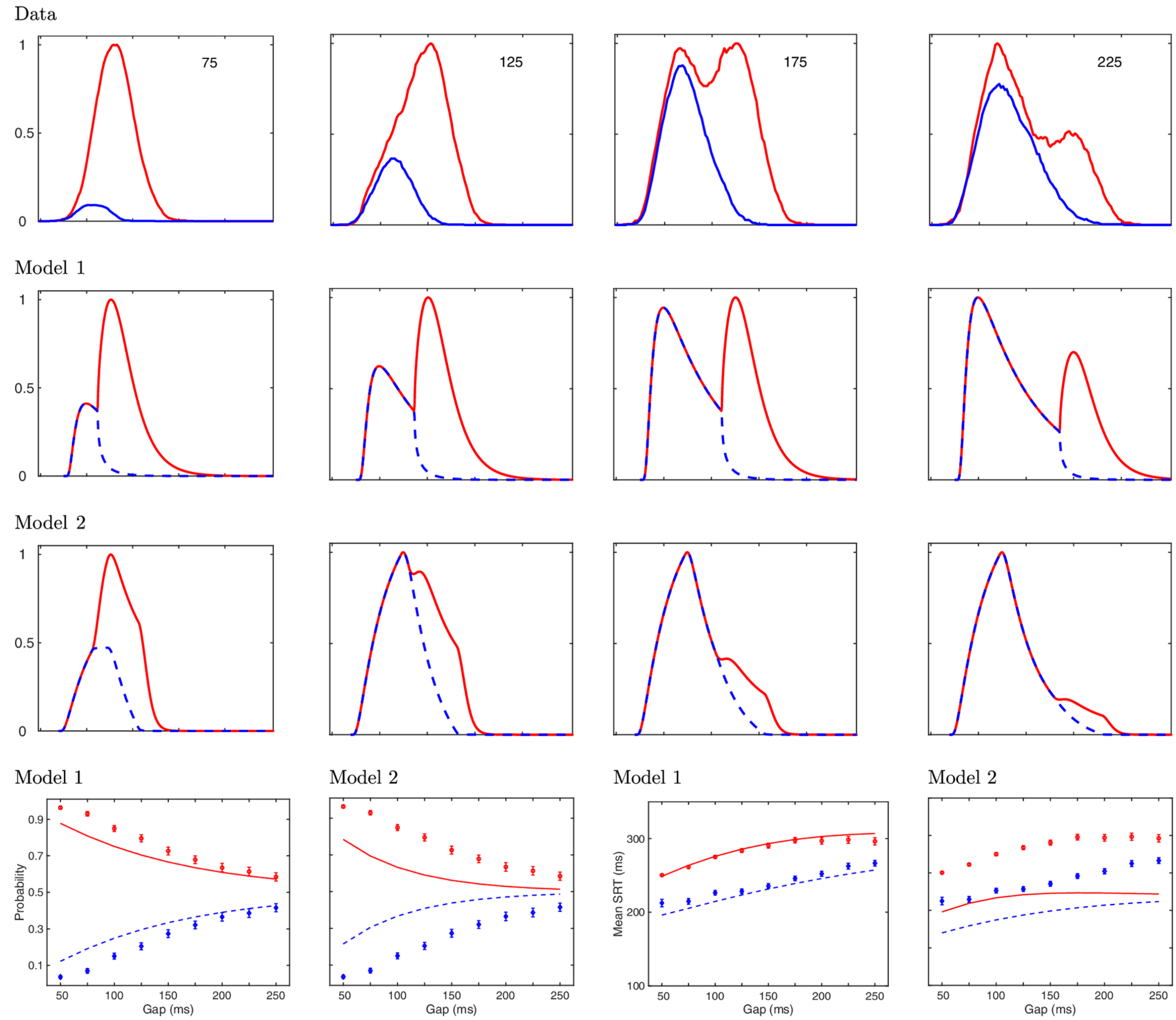
Here, we have shown that a two-stage-diffusion model is able to account for the rather complex pattern of data generated in the compelled response task: With only three easy-to-interpret parameters, the model is able to predict both response time distributions and choice probabilities for arbitrary gap time conditions in the compelled-response task. It is important to realize, however, that an important feature of the two-stage model proposed here is that it is based on a non-time-homogeneous stochastic process, in contrast to the "common" diffusion model (e.g., Ratcliff & Tuerlinckx, 2002). This non-homogeneity property is essential

[1] Philip L. Smith, one of the reviewers, pointed out, that the time step may be much larger (even few milliseconds) to approach a continuous process when smoothly interpolating between adjacent δ points. The reason for choosing τ very small is that the state space is discretized in units proportional to the square root of the time step, which is much coarser, and this granularity can create problems for fitting routines.

[2] Similar notions of a linear rise-to-threshold have been proposed earlier for typical eye movement tasks with highly detectable stimuli, e.g., by Carpenter and colleagues (Carpenter et al., 2009; Carpenter & Williams, 1995). Moreover, the linear-ballistic-accumulator model, introduced in Brown and Heathcote (2008) as a simplified alternative to diffusion models that also proposes a constant buildup of random slope, has gained some popularity for psychological choice RT paradigms.

Figure 8

Data and Model Account for Monkey S: Predictions



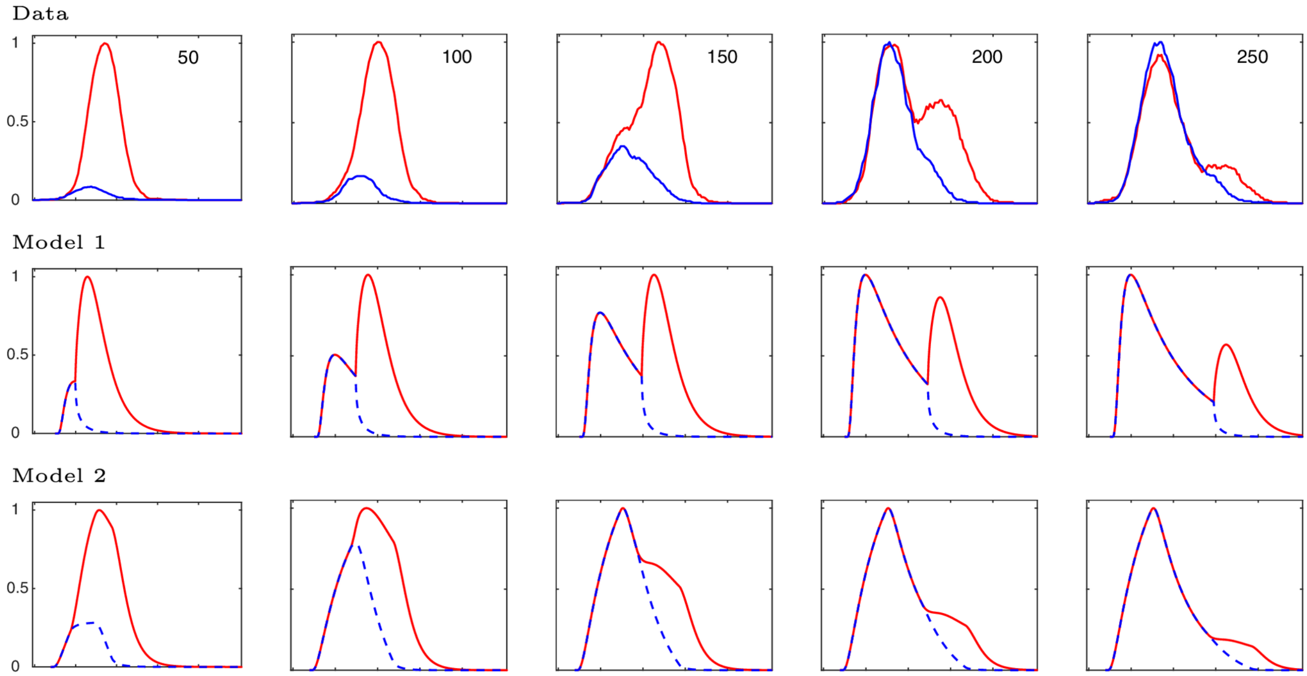
Note. The upper row shows the smoothed empirical distributions for correct (red) and incorrect (blue) response times for gap conditions 75, 125, 175 and 225 ms. Model 1 and Model 2 depict the corresponding predictions of the two-stage diffusion model with fixed and uniformly distributed residual time, respectively. Note that the parameters are those estimated from the conditions described in Figure 7. The last row shows the probabilities and mean RT data (points including 95% confidence intervals) and model predictions (lines; red correct, blue: incorrect responses), again with the same parameter as before: No new estimations are performed here. See the online article for the color version of this figure.

in representing the effect of the cue on the on-going saccade trajectory. Although the model is mathematically complex, it is analytically tractable using the discrete Markov chain approach developed by Diederich and colleagues (Diederich, 1992, 1997; Diederich & Busemeyer, 2006; Diederich & Oswald, 2016) that permits us to derive predictions without having to rely on model simulations.

Another feature that distinguishes the two-stage model from both the race-to-threshold and the diffusion model as proposed early on by Ratcliff (Ratcliff, 1978) is that there is no need to assume between-trial variability to match the pattern of data across conditions, that is, all parameters remain constant from one trial to the next rather than being sampled from some additionally-assumed probability distribution. This is an important property in light of the recent criticism by Jones and Dzhafarov (Jones & Dzhafarov, 2014a, 2014b) arguing that across-trial variability makes models overly flexible (for discussion see Heathcote et al., 2014; Smith, Ratcliff, & McKoon, 2014). The two-stage diffusion model is more

Figure 9

Data and Model Account for Monkey G: Estimation



Note. For the description, see Figure 7 for Monkey S. For monkey G the estimated parameters are $\mu_2 = 0.2$, $\theta = 12$, and T_{ND} =160 ms for Model 1; and $\mu_2 = 0.228$, $\theta = 10$, and $E(T_{ND})$ =139 ms for the uniformly distributed non-decision time. See the online article for the color version of this figure.

than a data-fitting tool: It is eminently testable and is falsifiable already at the level of qualitative features, for example in predicting the bimodality of the *RT* probability distributions for particular gap times and, most importantly, fast errors.

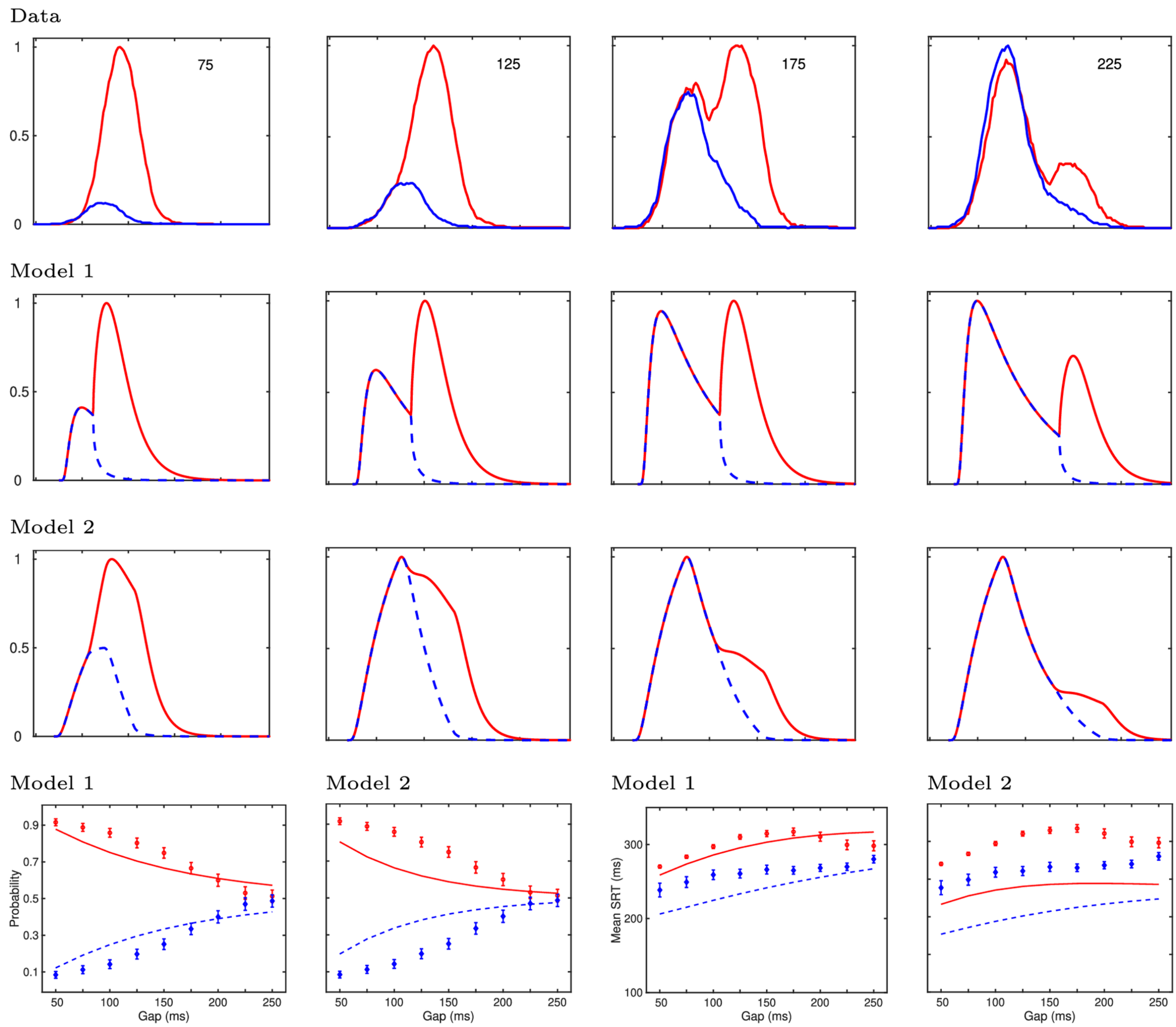
The race-to-threshold model produces a good fit to various data sets from the compelled-response paradigm and quantitatively accounts for the separate components of the task: motor response, perceptual evaluation, and additional cognitive factors. However, it does so by introducing a number of auxiliary assumptions. For example, Shankar et al. (2011) assume that the gap time is normally distributed with a mean of the experimentally administered gap time and a variance identical to the one of the non-decision time. For short gap times, this leads to negative gap times; but eliminating negative gap times would result in a distribution that is no longer Gaussian. In addition Shankar et al. (2011) assume that the transition from stage 1 to stage 2 is interrupted for some time. That is, the process is constant for some time before acceleration or deceleration occurs. How long this interruption lasts is determined by two parameters to be estimated from the data. The probability of "confusion" is another parameter introduced ad-hoc to account for lapses, that is, trajectories going into the wrong direction even for short gap times due to inattention or something else. It is required in the race-to-threshold model because the process is deterministic during one trial. Note that, on the other hand, in a stochastic diffusion process model this is built-in into the process itself. Finally, Stanford et al. (2010) and Shankar et al. (2011) assume a parameter called sensory discrimination time, that is the time to reach the final values of the direction changers for target and distractor. How this parameter is to be interpreted—given its large range (Monkey S 85 ms; Monkey G 1,800 ms)—is not intuitive. Altogether, the race-to-threshold model requires 11 parameters to achieve a good fit to the data. While these additional parameters lead to an improved quantitative fit, we feel that the model is conceived of as a measurement device rather than a model explaining the underlying decision process.

Note that the two-stage diffusion model's fit has been improved upon somewhat by including some of the assumptions stated in the appendix and further specifying the non-decision time. We fitted the model assuming a uniformly distributed non-decision time to explore the effect on the estimation. The results are somewhat mixed. While it improved the fit for Monkey G on some gap conditions, in particular for longer gap times, it worsened it on others. Most obviously, however, the predicted choice probabilities and mean *RT*s show a substantially worsened fit assuming uniformly distributed non-decision times.

To sum up, we propose the two-stage diffusion model as an alternative to the race-to-threshold model for the compelled-response task. This note demonstrates that the data pattern from this task can be described and predicted by a model that accumulates randomly fluctuating sensory information within

Figure 10

Data and Model Account for Monkey G: Predictions



Note. For the description see Figure 7 and Figure 8 for Monkey S. See the online article for the color version of this figure.

each trial, without the need for simulation and with only a few parameters to be estimated. While we do not take up a stance on which model type is better supported by neurophysiological data, there is ample evidence that stochastic-diffusion type models provide a quantitative link between the time-course of behavioral decisions and the growth of stimulus information in neural firing data (see, e.g., Smith & Ratcliff, 2004).

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(*Appendices follow*)

(*Appendix continues*)

Appendix

Further Model Details

Race-to-Threshold Model

According to Shankar et al. (2011), in a given trial both variables x_L and x_R start with a value of 0 and race to a threshold of 1000 units. Build-up rates r_R and r_L are sampled from a truncated bivariate Gaussian distribution with mean r_G (same for both variables), standard deviation s_G (same for both variables) and correlation coefficient r. The first part of the race encompasses the time from the go screen (offset of the fixation point) until the cue screen; the second part from the cue screen until a response it made. Because no information is given in part 1, the winner is random with probability 0.5. In part 2, after the color has been revealed, the racer with the "correct" color direction—the target—accelerates and the one with the "incorrect" color direction—the distractor—slows down.

Stage 1: For a given trial, the rates (change as a function of time)

$$dx_L/dt = r_L \text{ and } dx_R/dt = r_R,$$

are constant and linear but vary across trials.

Stage 2: Let r_L^0 and r_R^0 be the buildup rates drawn initially; after the target color is revealed, the rates in stage 2 change. The competing variables start accelerating according to the locations of the target and distractor. If the target is on the right side, then the buildup rate of x_R approaches a large, positive value r_T (for target) and the buildup rate of x_L approaches a small or negative value r_D (for distractor). The corresponding equations are

$$dr_L/dt = (r_D - r_L^0)/\tau \text{ and } dr_R/dt = (r_T - r_R^0)/\tau,$$

with the added rule that once the buildup rates reach their new target values–that is, once r_L is equal to r_D and r_R is equal to r_T– they stop changing, so the last two derivatives become zero. In this way, the acceleration is constant but lasts a finite amount of time, which is precisely equal to τ, and r_T is the maximum possible buildup rate for the variable that generates a movement toward the target (see Shankar et al., 2011).

When the target is to the left of fixation, the roles of x_L and x_R are reversed: When the sensory information arrives, r_L increases toward the high rate r_T and r_R decreases toward the low rate r_D, so

$$dr_L/dt = (r_T - r_L^0)/\tau \text{ and } dr_R/dt = (r_W - r_R^0)/\tau.$$

Several more parameters are included:

- p_e is the probability of confusion, accounting for lapses. That is, the monkey has enough time to make a correct answer but failed to do so.
- The non-decision time (collapsing afferent and efferent processing times) is drawn from a Gaussian distribution withe mean T_{ND} and standard deviation σ_{ND}.
- The start and end of an interruption time $t = I_1$ and $t = I_2$ from stage 1 to 2 during which no change in the builup rates occurs.

Altogether, the model includes 11 parameters to be estimated from data using simulations and extended grid search. In addition, some further fixed parameters are included in the simulation procedure. Moreover, in the program the gap time is not fixed but follows a normal distribution, $N(\text{gap}, \sigma_{ND})$.

Two-Stage Diffusion Model

The decision process is modeled as a two-stage diffusion model, with stochastic process $X(t)$ representing the numerical value of the accumulated evidence at time t. Note that, as explained above, "evidence" is to be interpreted as an abstract notion depending on the context/stage of accumulation: upon the offset of the fixation point, the decision maker sequentially samples information from the stimulus display over time, retrieves information from memory, or forms preferences, depending on the context.

The small increments of evidence sampled at any moment in time are such that they either favor option left (L) ($dX(t) > 0$) or option right (R) ($dX(t) < 0$). The evidence is incremented according to a diffusion process:

$$dX(t) = \mu(X(t),t)dt + \sigma(X(t),t)dW(t).$$

Here, $\mu(x, t)$ is called the *effective drift rate* describing the instantaneous rate of expected increment change at time t and state $x = X(t)$. The factor $\sigma(x, t)$ in front of the instantaneous increments $dW(t)$ of a standard Wiener process $W(t)$ is called the *diffusion coefficient*, and relates to the variance of the increments.

Here we assume that the process in the first stage is a Wiener process $W(t)$ with drift rate $\mu(x, t) = \mu_1 = 0$ and diffusion coefficient $\sigma(x, t) = \sigma_1$, that is,

$$dX(t) = \sigma_1 dW(t). \tag{2}$$

Because no evidence is given in the first part (from fixation offset and Go screen to the target), the drift rate in the first stage is assumed to be zero, so that it is equally likely to make saccade to the left or to the right side.

With the appearance of the cue screen, target information is revealed, and the second process starts. Depending on the color of the fixation point, a movement to the right side is correct or incorrect (similar for the left side). This part is modeled as a Wiener process with drift $\mu(x, t) = \mu_2$ and diffusion coefficient $\sigma(x, t) = \sigma_2$, that is,

$$dX(t) = \mu_2 + \sigma_2 dW(t), \tag{3}$$

or, with $\mu(x, t) = \delta - \gamma x$ and $\sigma(x, t) = \sigma_2$, that is,

$$dX(t) = (\delta - \gamma X(t))dt + \sigma_2 dW(t) \tag{4}$$

as an Ornstein–Uhlenbeck process (OUP). Setting $\gamma > 0$ yields evidence accumulation toward one of the choice options at a linearly decaying rate, that is, it induces a change of the effective drift rate $\mu(x, t) = \delta - \gamma x$ depending on the current state x. Setting $\gamma < 0$ accelerates the process. Note, however, that the process might become unstable and, strictly speaking, it is not called OUP for

(*Appendix continues*)

$\gamma < 0$. Finally, setting $\gamma = 0$ reduces Equation 4 to a Wiener process with drift (Equation 3). Furthermore note that, the entire processes is non-time-homogeneous.

The process continues until the magnitude of the cumulative evidence exceeds a threshold criterion, θ. Then, the process stops and response L is initiated. Or it stops and an R response is initiated if the accumulated evidence reaches a criterion value for choosing response R [here, $X(t) = \theta < 0$].

The Wiener process with drift and two absorbing boundaries has four basic parameters (drift rate, diffusion coefficient, rating point, boundary) of which one can be set arbitrarily. Here we fix the diffusion coefficient σ which is basically a scaling factor by setting $\sigma_1 = \sigma_2 = 1$ (e.g., Diederich & Mallahi-Karai, 2018). Furthermore, we assume that the participant has no a-priori bias was toward one direction (left or right) and, therefore, we set the starting point of the process to $X(0) = 0$. In this context, if $X(0) > 0$, the participant would a priori favor looking to the left (L); for $X(0) < 0$, the participant has an a-priori tendency (bias) for looking more often to the right than to the left. This initial state could also be governed by a probability distribution.

Similar to the race-to threshold model, we assume a residual time T_{ND} and also distinguish between a correct R and a correct L response. For our purposes, we simply assume T_{ND} to be a constant because we do not know anything about the underlying distribution. With that, the two-stage diffusion model has only three (Wiener process in stage 2) or four (OUP in stage 2) free parameters in total.

Non-Decision Times

Figure 6 shows the predictions for different T_{ND} time assumptions. The decision model assumptions for T_D are the same here. C shows the predictions when a constant non-decision time is assumed. As can be seen, the specific distributions and their parameters can strongly influence the shape of the distribution. This is an important topic but beyond the focus of this theoretical note.

Estimated Parameter Values

Table 1 shows the estimated parameters for Monkey S (left) and Monkey G (right). Both models have three parameters: the drift for the second stage, that is, when the target color has been revealed. θ refers to the decision boundaries, which are very similar for both monkey and and also for the two models. $E(T_{ND})$ is the expected non-decision time. For Model 1, it is actually assumed to be a constant; for Model 2, the non-decision time is uniformly distributed with 50 time units around the mean.

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