$\mathcal{M},\mathcal{N}$-Adhesive Transformation Systems

Annegret Habel[1] and Detlef Plump[2]

[1] Carl von Ossietzky Universität Oldenburg
annegret.habel@informatik.uni-oldenburg.de
[2] The University of York
detlef.plump@york.ac.uk

Abstract. The categorical framework of $\mathcal{M}$-adhesive transformation systems does not cover graph transformation with relabelling. Rules that relabel nodes are natural for computing with graphs, however, and are commonly used in graph transformation languages. In this paper, we generalise $\mathcal{M}$-adhesive transformation systems to $\mathcal{M},\mathcal{N}$-adhesive transformation systems, where $\mathcal{N}$ is a class of morphisms containing the vertical morphisms in double-pushouts. We show that the category of partially labelled graphs is $\mathcal{M},\mathcal{N}$-adhesive, where $\mathcal{M}$ and $\mathcal{N}$ are the classes of injective and injective, undefinedness-preserving graph morphisms, respectively. We obtain the Local Church-Rosser Theorem and the Parallelism Theorem for graph transformation with relabelling and application conditions as instances of results which we prove at the abstract level of $\mathcal{M},\mathcal{N}$-adhesive systems.

1 Introduction

The double-pushout approach to graph transformation, which was invented in the early 1970's, is the best studied framework for graph transformation [20, 5, 10, 4]. As applications of graph transformation come with a large variety of graphs and graph-like structures, the double-pushout approach has been generalised to the abstract settings of high-level replacement systems [9], adhesive categories [17] and $\mathcal{M}$-adhesive categories [8, 6, 7].

The categories of labelled graphs, typed graphs, and typed attributed graphs, for example, are known to be $\mathcal{M}$-adhesive categories if one chooses $\mathcal{M}$ to be the class of injective graph morphisms [8]. Each such category induces a class of $\mathcal{M}$-adhesive transformation systems for which several classical results of the double-pushout approach hold. Specifically, the Local Church-Rosser Theorem, the Parallelism Theorem, the Concurrency Theorem, the Amalgamation Theorem, the Embedding Theorem and the Local Confluence Theorem have been established for rules with nested application conditions [6, 7].

However, $\mathcal{M}$-adhesive transformation systems do not cover graph transformation systems with rules that relabel nodes. Such rules are natural for computing with graphs and are used as a foundation for the graph transformation language GP [18, 19]. The double-pushout approach can be extended with relabelling by introducing rules with partially labelled interface graphs [14], providing a theoretical foundation for graph transformation languages that is much simpler than attributed graph transformation in the sense of [4]. In the latter approach, attributed graphs contain the algebra underlying the operations in the attributes as well as special edges which connect nodes and edges with their attributes. Hence they are (usually) complex infinite objects which are difficult to comprehend and which do not directly correspond to the graph data structures used to implement graph transformation languages.

In this paper, we study transformation systems over the category PLG of partially labelled graphs and the class $\mathcal{M}$ of injective graph morphims (which are used in rules). It turns out that PLG violates two of the properties required for $\mathcal{M}$-adhesive categories: pushouts along $\mathcal{M}$-morphisms do not always exist and, when they exist, need not be pullbacks. We therefore generalise $\mathcal{M}$-adhesive categories to $\mathcal{M}, \mathcal{N}$-adhesive categories, where $\mathcal{N}$ is a class of morphisms containing the vertical morphisms in double-pushouts. $\mathcal{M}$-adhesive categories are then the special case where $\mathcal{N}$ is the class of all morphisms.

For $\mathcal{M}, \mathcal{N}$-adhesive transformation systems with (nested) application conditions, we prove two classical results of the double-pushout approach: the Local Church-Rosser Theorem and the Parallelism Theorem. We then show that PLG is $\mathcal{M}, \mathcal{N}$-adhesive, where $\mathcal{N}$ is the class of injective morphisms that preserve unlabelled nodes and edges. As a result, we obtain both theorems for the setting of graph transformation with relabelling and application conditions.

The paper is structured as follows. In Section 2, we generalise $\mathcal{M}$-adhesive categories to $\mathcal{M}, \mathcal{N}$-adhesive categories, prove that they satisfy the so-called HLR properties, and identify two additional factorization properties. In Section 3, we present the Local Church-Rosser Theorem and the Parallelism Theorem for $\mathcal{M}, \mathcal{N}$-adhesive transformation systems with application conditions. In Section 4, we show that the category PLG is $\mathcal{M}, \mathcal{N}$-adhesive for suitable classes $\mathcal{M}$ and $\mathcal{N}$ of morphisms. As a consequence, we obtain the Local Church-Rosser Theorem and the Parallelism Theorem for graph transformation with relabelling. In Section 5, we conclude and give some topics for future work.

The proofs omitted in this paper are given in [15], as well as the Concurrency Theorem for $\mathcal{M}, \mathcal{N}$-adhesive transformation systems with application conditions.

2 $\mathcal{M}, \mathcal{N}$-Adhesive Categories

In [8] an overview is given on some categorical frameworks for double-pushout transformations. It is shown that adhesive categories [17], weak adhesive HLR categories [4], and partial map adhesive categories [16] are special cases of so-called $\mathcal{M}$-adhesive categories. A large number of results have been proved for $\mathcal{M}$-adhesive transformation systems, such as the Local Church-Rosser Theorem, the Parallelism Theorem, the Concurrency Theorem, the Amalgamation Theorem, the Embedding Theorem, and the Local Confluence Theorem [6, 7].

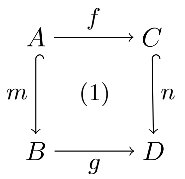
In this section, we generalize $\mathcal{M}$-adhesive categories as defined in [8, 6] to $\mathcal{M}, \mathcal{N}$-adhesive categories.

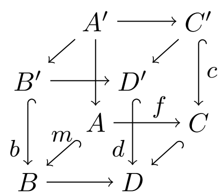
Definition 1 ($\mathcal{M},\mathcal{N}$-adhesive category). A category $\mathcal{C}$ is $\mathcal{M},\mathcal{N}$*-adhesive*, where $\mathcal{M}$ is a class of monomorphisms and $\mathcal{N}$ a class of morphisms, if the following properties are satisfied:

1. $\mathcal{M}$ and $\mathcal{N}$ contain all isomorphisms and are closed under composition and decomposition (see [6]). Moreover, $\mathcal{N}$ is closed under $\mathcal{M}$-decomposition, that is, $g \circ f \in \mathcal{N}$, $g \in \mathcal{M}$ implies $f \in \mathcal{N}$.
2. $\mathcal{C}$ has pushouts along $\mathcal{M},\mathcal{N}$-morphisms and pullbacks along $\mathcal{M}$-morphisms. Also, $\mathcal{M}$ and $\mathcal{N}$ are stable under $\mathcal{M},\mathcal{N}$-pushouts and $\mathcal{M}$-pullbacks (see below).
3. Pushouts along $\mathcal{M},\mathcal{N}$-morphisms are $\mathcal{M},\mathcal{N}$-van Kampen squares (see below).

Remark 1. A pushout *along* $\mathcal{M},\mathcal{N}$-morphisms, or $\mathcal{M},\mathcal{N}$*-pushout*, is a pushout where one of the given morphisms is in $\mathcal{M}$ and the other morphism is in $\mathcal{N}$. A pullback *along* an $\mathcal{M}$-morphism, or $\mathcal{M}$*-pullback*, is a pullback where at least one of the given morphisms is in $\mathcal{M}$. A class $\mathcal{X}$ of morphisms is *stable under* $\mathcal{M},\mathcal{N}$*-pushouts* if, given the $\mathcal{M},\mathcal{N}$-pushout (1) in the diagram below, $m \in \mathcal{X}$ implies $n \in \mathcal{X}$. Class $\mathcal{X}$ is *stable under* $\mathcal{M}$*-pullbacks* if, given the $\mathcal{M}$-pullback (1) in the diagram below, $n \in \mathcal{X}$ implies $m \in \mathcal{X}$.

A pushout along $\mathcal{M},\mathcal{N}$-morphisms is an $\mathcal{M},\mathcal{N}$*-van Kampen square* if for the commutative cube in the diagram below with the pushout as bottom square, $b, c, d, m \in \mathcal{M}$, $f \in \mathcal{N}$, and the back faces being pullbacks, we have that the top square is a pushout if and only if the front faces are pullbacks.





Fact 1. Let $\mathcal{C}$ be any category and let $\mathcal{N}$ be the class of all morphisms in $\mathcal{C}$. Then $\mathcal{C}$ is $\mathcal{M},\mathcal{N}$-adhesive if and only if $\mathcal{C}$ is $\mathcal{M}$-adhesive in the sense of [6].

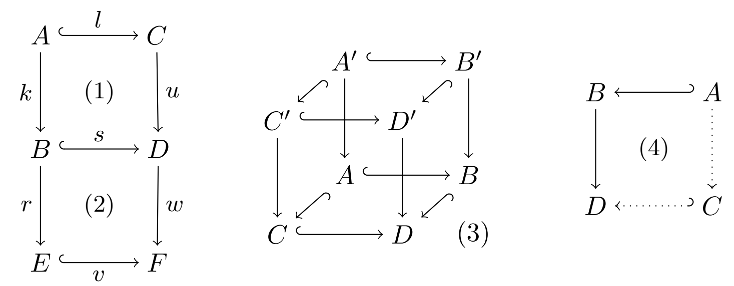
Proof. This follows from the definition of an $\mathcal{M}$-adhesive category because if $\mathcal{N}$ contains all morphisms, then $\mathcal{M},\mathcal{N}$-pushouts and $\mathcal{M},\mathcal{N}$-van Kampen squares are precisely the $\mathcal{M}$-pushouts and $\mathcal{M}$-van Kampen squares of [6], respectively. □

In Section 4, we show that the category PLG of partially labelled graphs is $\mathcal{M},\mathcal{N}$-adhesive but not $\mathcal{M}$-adhesive. In this case, $\mathcal{M}$ is the class of injective graph morphisms and $\mathcal{N}$ is the class of injective, undefinedness preserving graph morphisms.

$\mathcal{M},\mathcal{N}$-adhesive categories satisfy generalised versions of the so-called HLR-properties [9] of $\mathcal{M}$-adhesive categories.

Theorem 1 (HLR-properties). Every $\mathcal{M}, \mathcal{N}$-adhesive category satisfies the following *HLR-properties*:

1. Pushouts along $\mathcal{M}, \mathcal{N}$-morphisms are pullbacks.
2. $\mathcal{M}, \mathcal{N}$-pushout-pullback decomposition: If (1)+(2) in the diagram below is a pushout, (2) is a pullback, $l \in \mathcal{M}$, and $k, w \in \mathcal{N}$, then (1) and (2) are pushouts as well as pullbacks.
3. Cube $\mathcal{M}, \mathcal{N}$-pushout-pullback decomposition: If in the commutative cube (3) of the diagram below, all morphisms in the top square and in the bottom square are in $\mathcal{M}$, all vertical morphisms are in $\mathcal{N}$, the top square is a pullback, and the front faces are pushouts, then the bottom square is a pullback if and only if the back faces are pushouts.
4. Uniqueness of pushout complements: Given morphisms $A \hookrightarrow B$ in $\mathcal{M}$ and $B \to D$ in $\mathcal{N}$, there is, up to isomorphism, at most one object C with morphisms $A \to C$ and $C \hookrightarrow D$ such that (4) in the diagram below is a pushout.

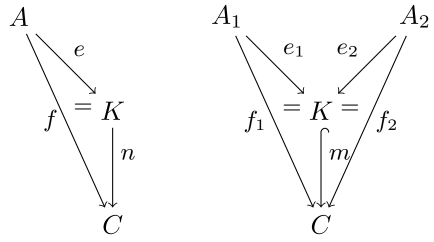


In order to prove the desired results for $\mathcal{M}, \mathcal{N}$-adhesive transformation systems, three more properties will be needed.

Definition 2 (HLR$^+$**-properties).** Let $\mathcal{C}$ be an $\mathcal{M}, \mathcal{N}$-adhesive category, $\mathcal{E}$ a class of morphisms, and $\mathcal{E}'$ a class of pairs of morphism with the same codomain. Then the following properties are the *HLR$^+$-properties* with respect to $\mathcal{M}$, $\mathcal{N}$, $\mathcal{E}$ and $\mathcal{E}'$.

1. $\mathcal{C}$ has binary coproducts.
2. $\mathcal{C}$ has an $\mathcal{E}$-$\mathcal{N}$ *factorization* if for each coproduct morphism $f\colon A_1 + A_2 \to C$ induced by morphisms $f_i\colon A_i \to C$ in $\mathcal{N}$ $(i = 1, 2)$, there is a decomposition, unique up to isomorphism, $f = n \circ e$ with $e \in \mathcal{E}$ and $n \in \mathcal{N}$.
3. $\mathcal{C}$ has an $\mathcal{E}'$-$\mathcal{M}$ *pair factorization* if, for each pair of morphisms $f_1\colon A_1 \to C$ and $f_2\colon A_2 \to C$, there exist a unique (up to isomorphism) object K and unique (up to isomorphism) morphisms $e_1\colon A_1 \to K$, $e_2\colon A_2 \to K$, and $m\colon K \hookrightarrow C$ with $(e_1, e_2) \in \mathcal{E}'$ and $m \in \mathcal{M}$ such that $m \circ e_1 = f_1$ and

$m \circ e_2 = f_2$.



General Assumption. We assume that $\mathcal{C}$ is an $\mathcal{M},\mathcal{N}$-adhesive category and that $\mathcal{E}$ and $\mathcal{E}'$ are classes of morphisms and morphisms pairs, respectively, such that $\mathcal{C}$ satisfies the HLR$^+$-properties.

The $\mathcal{E}$-$\mathcal{N}$ factorization is used in the proof of the Parallelism Theorem. The $\mathcal{E}'$-$\mathcal{M}$ pair factorization is used in the proof of a shift lemma for application conditions and in the construction of E-related transformations in [15].

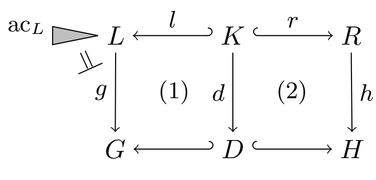
Example 1. The category PLG considered in Section 4 satisfies the HLR$^+$-properties, where $\mathcal{M}$ is the class of injective morphisms, $\mathcal{N}$ is the class of injective, undefinedness preserving morphisms, $\mathcal{E}$ is the class of surjective, undefinedness preserving morphisms, and $\mathcal{E}'$ is the class of pairs of jointly surjective, undefinedness preserving morphisms.

3 $\mathcal{M},\mathcal{N}$-Adhesive Transformation Systems

In this section, we introduce $\mathcal{M},\mathcal{N}$-adhesive transformation systems and present the Local Church-Rosser Theorem and the Parallelism Theorem in this setting.

We start by defining rules, direct transformations, and transformation systems.

Definition 3 (Rules, transformations, and systems). Given an $\mathcal{M},\mathcal{N}$-adhesive category, a *rule* $\varrho = \langle p, \mathrm{ac}_L \rangle$ consists of a *plain rule* $p = \langle L \hookleftarrow K \hookrightarrow R \rangle$ with morphisms $l\colon K \hookrightarrow L$ and $r\colon K \hookrightarrow R$ in $\mathcal{M}$, and an application condition ac_L over L (see below). A *direct transformation* from an object G to an object H via the rule ϱ consists of two pushouts (1) and (2) as below where the vertical morphisms[3] are in $\mathcal{N}$ and $g \models \mathrm{ac}_L$. We write $G \Rightarrow_{\varrho,g} H$ if there exists such a direct transformation. For a set of rules $\mathcal{R}$, we write $G \Rightarrow_{\mathcal{R}} H$, if $G \Rightarrow_{\varrho} H$ with $\varrho \in \mathcal{R}$.



An $\mathcal{M},\mathcal{N}$*-adhesive transformation system* consists of an $\mathcal{M},\mathcal{N}$-adhesive category and a set $\mathcal{R}$ of rules.

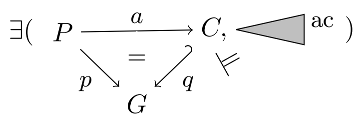
[3] By stability of $\mathcal{N}$ under $\mathcal{M},\mathcal{N}$-pushouts, it is equivalent to require $d \in \mathcal{N}$.

Remark 2. Every $\mathcal{M}$-adhesive transformation system in the sense of [6] is an $\mathcal{M}, \mathcal{N}$-adhesive transformation system if we choose $\mathcal{N}$ as the class of all morphisms in $\mathcal{C}$. Our notion of transformation system is more flexible because it allows to restrict the class of morphisms that are used to match rules. For example, one can show that every $\mathcal{M}$-adhesive category is $\mathcal{M}, \mathcal{M}$-adhesive and hence gives rise to an $\mathcal{M}, \mathcal{M}$-adhesive transformation system. A concrete example for this is the category of totally labelled graphs together with the class of injective graph morphisms (see also [12] for this setting).

Application conditions are nested constructs which can be represented as trees of morphisms equipped with quantifiers and Boolean connectives.

Definition 4 (Application condition). *Application conditions* are inductively defined as follows. For every object P, true is an application condition over P. For every morphism $a\colon P \to C$ and every application condition ac over C, $\exists(a, \mathrm{ac})$ is an application condition over P. For application conditions $\mathrm{ac}, \mathrm{ac}_i$ over P with $i \in I$ (for a given index set I), $\neg\mathrm{ac}$ and $\wedge_{i\in I}\mathrm{ac}_i$ are application conditions over P.

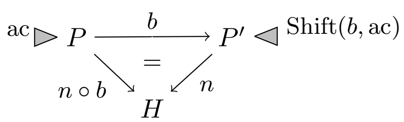
Satisfiability of application conditions is also defined inductively. Every morphism satisfies true. A morphism $p\colon P \to G$ satisfies $\exists(a, \mathrm{ac})$ over P if there exists a morphism $q\colon C \hookrightarrow G$ in $\mathcal{M}$ such that $q \circ a = p$ and q satisfies ac.



A morphism $p\colon P \to G$ satisfies $\neg\mathrm{ac}$ over P if p does not satisfy ac, and p satisfies $\wedge_{i\in I}\mathrm{ac}_i$ over P if p satisfies each ac_i $(i \in I)$. We write $p \models \mathrm{ac}$ to express that p satisfies ac.

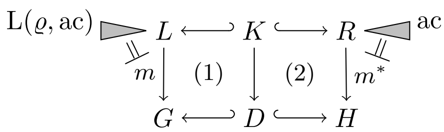
Next we state two important technical results. The first lemma allows to shift application conditions over arbitrary morphisms.

Lemma 1 (Shift of application conditions over morphisms [6]). There is a construction Shift such that, for each application condition ac over P and for each morphism $b\colon P \to P'$, Shift transforms ac via b into an application condition $\mathrm{Shift}(b, \mathrm{ac})$ over P' such that, for each morphism $n\colon P' \to H$, $n \circ b \models \mathrm{ac} \iff n \models \mathrm{Shift}(b, \mathrm{ac})$.



The other technical result that we need is that application conditions can be shifted over rules.

Lemma 2 (Shift of application conditions over rules [13]). There is a construction L such that, for each rule ϱ and each application condition ac over R, L transforms ac via ϱ into an application condition $\mathrm{L}(\varrho, \mathrm{ac})$ over L such that, for each direct transformation $G \Rightarrow_{\varrho,m,m^*} H$, we have $m \models \mathrm{L}(\varrho, \mathrm{ac}) \iff m^* \models \mathrm{ac}$.

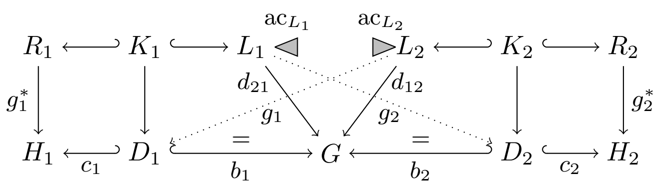


Remark 3. There is a construction R with $\mathrm{R}(\varrho, \mathrm{ac}) = \mathrm{L}(\varrho^{-1}, \mathrm{ac})$ that transforms left application conditions ac via the rule ϱ into right application conditions.

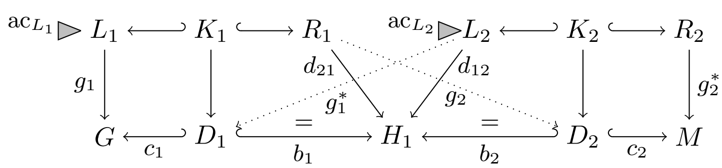
Assumption. For $i = 1, 2$, let $\varrho_i = \langle p_i, \mathrm{ac}_{L_i} \rangle$ be a rule with plain rule $p_i = \langle L_i \hookleftarrow K_i \hookrightarrow R_i \rangle$. Also, let $\varrho = \langle p, \mathrm{ac}_L \rangle$ and $\varrho' = \langle p', \mathrm{ac}_{L'} \rangle$ be rules with plain rules $p = \langle L \hookleftarrow K \hookrightarrow R \rangle$ and $p' = \langle L' \hookleftarrow K' \hookrightarrow R' \rangle$, respectively.

First, we formulate the notions of parallel and sequential independence and present the Local Church-Rosser Theorem.

Definition 5 (Parallel and sequential independence). Two direct transformations $H_1 \Leftarrow_{\varrho_1,g_1} G \Rightarrow_{\varrho_2,g_2} H_2$ are *parallelly independent* if in the diagram below there are morphisms $d_{ij} \colon L_i \to D_j$ such that $g_i = b_j \circ d_{ij}$, $g'_i = (c_j \circ d_{ij}) \in \mathcal{N}$, and $g'_i \models \mathrm{ac}_{L_i}$ $(i, j \in \{1, 2\}$ and $i \neq j)$.



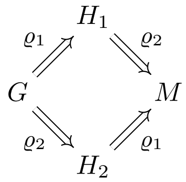
Two direct transformations $G \Rightarrow_{\varrho_1,g_1} H_1 \Rightarrow_{\varrho_2,g_2} M$ are *sequentially independent* if in the diagram below there are morphisms $d_{12} \colon R_1 \to D_2$ and $d_{21} \colon L_2 \to D_1$ such that $g_1^* = b_2 \circ d_{12}$, $g_2 = b_1 \circ d_{21}$, $g'_2 = (c_1 \circ d_{21}) \in \mathcal{N}$, $g'_1 = (c_2 \circ d_{12}) \in \mathcal{N}$, $g'_2 \models \mathrm{ac}_{L_2}$, and $g'_1 \models \mathrm{R}(\varrho_1, \mathrm{ac}_{L_1})$.



The following Local Church-Rosser Theorem generalises the corresponding result in [6] from $\mathcal{M}$-adhesive transformation systems to $\mathcal{M},\mathcal{N}$-adhesive transformation systems.

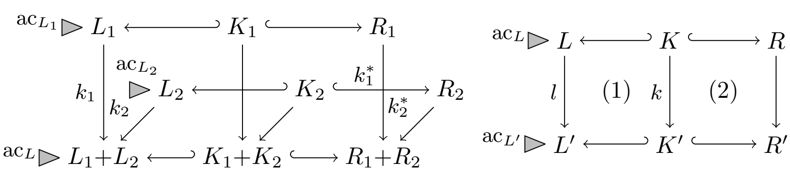
Theorem 2 (Local Church-Rosser Theorem). Given parallelly independent direct transformations $H_1 \Leftarrow_{\varrho_1,g_1} G \Rightarrow_{\varrho_2,g_2} H_2$, there are an object M and direct transformations $H_1 \Rightarrow_{\varrho_2,g_2'} M \Leftarrow_{\varrho_1,g_1'} H_2$ such that $G \Rightarrow_{\varrho_1,g_1} H_1 \Rightarrow_{\varrho_2,g_2'} M$ and $G \Rightarrow_{\varrho_2,g_2} H_2 \Rightarrow_{\varrho_1,g_1'} M$ are sequentially independent.

Given sequentially independent direct transformations $G \Rightarrow_{\varrho_1,g_1} H_1 \Rightarrow_{\varrho_2,g_2} M$, there are an object H_2 and direct transformations $G \Rightarrow_{\varrho_2,g_2'} H_2 \Rightarrow_{\varrho_1,g_1'} M$ such that $H_1 \Leftarrow_{\varrho_1,g_1} G \Rightarrow_{\varrho_2,g_2'} H_2$ are parallelly independent:



Next we consider parallel rules, quotients rules, and parallel transformations. The parallel rule $\varrho_1 + \varrho_2$ of the rules ϱ_1 and ϱ_2 is defined by using the binary coproducts of the components of the rules (which exist by the General Assumption).

Definition 6 (Parallel rule, quotient rule, parallel transformation). The *parallel rule* of ϱ_1 and ϱ_2 is the rule $\varrho_1+\varrho_2 = \langle p, \mathrm{ac}_L \rangle$ where $p = \langle L_1+L_2 \hookleftarrow K_1+K_2 \hookrightarrow R_1+R_2 \rangle$ is the parallel rule of p_1 and p_2 and $\mathrm{ac}_L = \wedge_{i=1}^{2} \mathrm{Shift}(k_i, \mathrm{ac}_{L_i}) \wedge \mathrm{L}(p, \mathrm{Shift}(k_i^*, \mathrm{R}(\varrho_i, \mathrm{ac}_{L_i})))$.



The rule ϱ' is a *quotient rule* of a parallel rule ϱ if there are two pushouts (1) and (2) as in the figure above where $k\colon K \to K'$ is an epimorphism in the class of coproduct morphisms induced by $\mathcal{N}$ and $\mathrm{ac}_{L'} = \mathrm{Shift}(l, \mathrm{ac}_L)$. The set of quotient rules of ϱ is denoted by $\mathrm{Q}(\varrho)$.

A direct transformation via a quotient of a parallel rule is called *parallel* direct transformation or parallel transformation, for short.

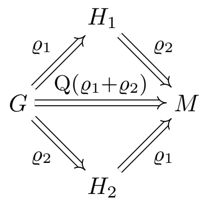
Fact 2 ([6]). $K_1+K_2 \hookrightarrow L_1+L_2$ and $K_1+K_2 \hookrightarrow R_1+R_2$ are in $\mathcal{M}$.

The connection between sequentially independent direct transformations and parallel direct transformations is given in the Parallelism Theorem.

Theorem (Parallelism Theorem).
1. Synthesis. Given two sequentially independent direct transformations $G \Rightarrow_{\varrho_1,g_1} H_1 \Rightarrow_{\varrho_2,g_2'} M$, there is a parallel transformation $G \Rightarrow_{\mathrm{Q}(\varrho_1+\varrho_2),g} M$.

2. Analysis. Given a parallel transformation $G \Rightarrow_{\mathrm{Q}(\varrho_1+\varrho_2),m} M$, there are sequentially independent direct transformations $G \Rightarrow_{\varrho_1,g_1} H_1 \Rightarrow_{\varrho_2,g'_2} M$ and $G \Rightarrow_{\varrho_2,g_2} H_i \Rightarrow_{\varrho_1,g'_1} M$.

3. Bijective correspondence. The synthesis and analysis constructions are inverse to each other up to isomorphism:



We conclude this section by mentioning that the Concurrency Theorem for $\mathcal{M},\mathcal{N}$-adhesive transformation systems is established in [15].

4 Category PLG is $\mathcal{M},\mathcal{N}$-Adhesive

In this section, we consider the category PLG of partially labelled graphs [14]. We first show that PLG is not $\mathcal{M}$-adhesive for the class $\mathcal{M}$ of injective graph morphisms. We then prove that PLG is $\mathcal{M},\mathcal{N}$-adhesive, though, and satisfies the HLR$^+$-properties if we choose $\mathcal{N}$ as a suitable class of morphisms. As a consequence, we obtain the Local Church-Rosser Theorem and the Parallelism Theorem as new results for the setting of graph transformation with relabelling and application conditions.

We start by recalling the basic notions of partially labelled graphs and their morphisms.

Definition 7 (Graphs and morphisms). A *(partially labelled) graph* is a system $G = (\mathrm{V}_G, \mathrm{E}_G, \mathrm{s}_G, \mathrm{t}_G, \mathrm{l}_{G,\mathrm{V}}, \mathrm{l}_{G,\mathrm{E}})$ consisting of finite sets V_G and E_G of *nodes* and *edges*, source and target functions $\mathrm{s}_G, \mathrm{t}_G \colon \mathrm{E}_G \to \mathrm{V}_G$, and partial labelling functions $\mathrm{l}_{G,\mathrm{V}} \colon \mathrm{V}_G \to \mathrm{C}_\mathrm{V}$ and $\mathrm{l}_{G,\mathrm{E}} \colon \mathrm{E}_G \to \mathrm{C}_\mathrm{E}$,[4] where C_V and C_E are fixed sets of node and edge labels. A graph G is *totally labelled* if $\mathrm{l}_{G,\mathrm{V}}$ and $\mathrm{l}_{G,\mathrm{E}}$ are total functions.

A *morphism* $g \colon G \to H$ between graphs G and H consists of two functions $g_\mathrm{V} \colon \mathrm{V}_G \to \mathrm{V}_H$ and $g_\mathrm{E} \colon \mathrm{E}_G \to \mathrm{E}_H$ that preserve sources, targets and labels, that is, $\mathrm{s}_H \circ g_\mathrm{E} = g_\mathrm{V} \circ \mathrm{s}_G$, $\mathrm{t}_H \circ g_\mathrm{E} = g_\mathrm{V} \circ \mathrm{t}_G$, and $\mathrm{l}_H(g(x)) = \mathrm{l}_G(x)$ for all x in $\mathrm{Dom}(\mathrm{l}_G)$[5]. Such a morphism *preserves undefinedness* if it maps unlabelled items in G to unlabelled items in H. Morphism g is *injective* (*surjective*) if g_V and g_E are injective (surjective), and an *isomorphism* if it is injective, surjective and

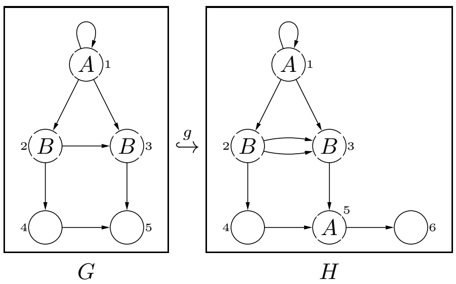
[4] Given sets A and B, a partial function $f \colon A \to B$ is a function from some subset A' of A to B. The set A' is the *domain* of f and is denoted by $\mathrm{Dom}(f)$. We say that $f(x)$ is *undefined*, and write $f(x) = \bot$, if x is in $A - \mathrm{Dom}(f)$.

[5] We often do not distinguish between nodes and edges in statements that hold analogously for both sets.

preserves undefinedness. In the latter case G and H are *isomorphic*, which is denoted by $G \cong H$. Furthermore, g is an *inclusion* if $g(x) = x$ for all x in G (note that inclusions need not preserve undefinedness). The *composition* $h \circ g$ of g with a morphism $h\colon H \to M$ consists of the composed functions $h_V \circ g_V$ and $h_E \circ g_E$. We write PLG for the category having partially labelled graphs as objects and graph morphisms as arrows.

In pictures of graphs, nodes are drawn as circles with their labels (if existent) inside, and edges are drawn as arrows with their labels (if existent) placed next to them. Graph morphisms are graphically represented by attaching the same number to nodes and their images.

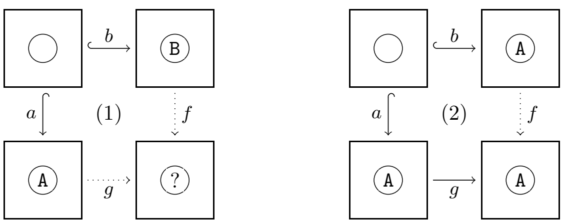
Example 2. Consider the partially labelled graphs G and H below. Nodes 4 and 5 in G, nodes 4 and 6 in H, and all edges are unlabelled. The graph morphism $g\colon G \hookrightarrow H$ is injective but not undefinedness preserving, because it maps the unlabelled node 5 in G to a labelled node in H.



While the category of labelled graphs with aribitrary morphisms has pushouts [3], the category of partially labelled graphs with injecitive morphisms has no pushouts [14]. As a consequence, the category PLG with the class $\mathcal{M}$ of injective morphisms is not $\mathcal{M}$-adhesive.

Fact 3 (PLG is not $\mathcal{M}$-adhesive). Let $\mathcal{M}$ be the class of injective graph morphisms. Then PLG does not have pushouts along $\mathcal{M}$-morphisms. Moreover, pushouts along $\mathcal{M}$-morphisms need not be pullbacks.

Example 3. The morphisms a and b in square (1) below are injective but their pushout does not exist: it is impossible to make both morphisms f and g label preserving. Square (2) is a pushout along $\mathcal{M}$, but not a pullback.



Assumption. For the rest of this section, we consider the category PLG and let $\mathcal{M}$ be the class of injective graph morphisms and $\mathcal{N}$ the class of injective, undefinedness preserving graph morphisms.

Theorem 3. The category PLG is $\mathcal{M}, \mathcal{N}$-adhesive.

To prove Theorem 3, we establish the properties required by Definition 1 in the following five lemmata.

Lemma 3 (Closure properties). $\mathcal{M}$ and $\mathcal{N}$ contain all isomorphisms and are closed under composition and decomposition. Moreover, $\mathcal{N}$ is closed under $\mathcal{M}$-decomposition.

Proof. Straightforward. □

Lemma 4 (Pushouts along $\mathcal{M}, \mathcal{N}$-morphisms). Given graph morphisms $r\colon K \hookrightarrow R$ in $\mathcal{M}$ and $d\colon K \hookrightarrow D$ in $\mathcal{N}$, there exist a graph H and graph morphisms $c\colon D \hookrightarrow H$ and $h\colon R \to H$ such that square (2) below is a pushout.

$$\begin{array}{ccc} K & \overset{r}{\hookrightarrow} & R \\ d \downarrow & (2) & \downarrow h \\ D & \underset{c}{\hookrightarrow} & H \end{array}$$

Construction. The sets of nodes and edges are defined by $H = (D - d(K)) + R$. The source function s_H is defined by $\mathrm{s}_H(e) =$ if $e \in E_R$ then $\mathrm{s}_R(e)$ else $\mathrm{s}_D(e)$; the target function t_D is defined analogously. The labelling functions l_H are defined by

$$\mathrm{l}_H(x) = \begin{cases} \mathrm{l}_R(x) & \text{if } x \in R \text{ and } \mathrm{l}_R(x) \neq \bot, \\ \mathrm{l}_D(d(x')) & \text{if } x \in R,\ \mathrm{l}_R(x) = \bot,\ r(x') = x \text{ and } \mathrm{l}_D(d(x')) \neq \bot, \\ \bot & \text{if } x \in R,\ \mathrm{l}_R(x) = \bot,\ r(x') = x \text{ and } \mathrm{l}_D(d(x')) = \bot, \\ \mathrm{l}_D(x) & \text{if } x \in (D - d(K)). \end{cases}$$

Morphism $h\colon R \to H$ is the inclusion of R in H and $c\colon D \hookrightarrow H$ is defined by $c(x) =$ if $x \in D - d(K)$ then x else $r(k)$ for the unique $k \in K$ with $d(k) = x$.

Proof. See [14]. □

The category PLG has not only pullbacks along $\mathcal{M}$-morphisms but possesses all pullbacks.

Lemma 5 (Pullbacks). Let $c\colon D \to H$ and $h\colon R \to H$ be graph morphisms. Then there exist a graph K and graph morphisms $d\colon K \to D$ and $r\colon K \to R$ such that square (2) abvove is a pullback.

Construction. The sets of nodes and edges are defined by

$$K = \{\langle x, y\rangle \in D \times R \mid c(x) = h(y)\}.$$

The source function s_K is defined by $\mathrm{s}_K(\langle x, y\rangle) = \langle \mathrm{s}_D(x), \mathrm{s}_R(y)\rangle$, the target function t_K is defined analogously. The labelling functions l_K are defined by

$$\mathrm{l}_K(\langle x, y\rangle) = \text{ if } (\mathrm{l}_D(x) = \mathrm{l}_R(y) \neq \bot) \text{ then } \mathrm{l}_R(x) \text{ else } \bot.$$

The morphisms $d\colon K \to D$ and $r\colon K \to R$ are the projections from $D \times R$ to D and R, that is, they are given by $d(\langle x, y\rangle) = x$ and $r(\langle x, y\rangle) = y$.

Proof. See [14]. □

Lemma 6 ($\mathcal{M}$ and $\mathcal{N}$ are stable). The classes $\mathcal{M}$ and $\mathcal{N}$ are stable under $\mathcal{M},\mathcal{N}$-pushouts and $\mathcal{M}$-pullbacks.

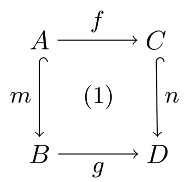
Proof. This follows from the construction of $\mathcal{M},\mathcal{N}$-pushouts and pullbacks in Lemma 4 and Lemma 5, and the fact that pushouts and pullbacks are unique up to isomorphism. □

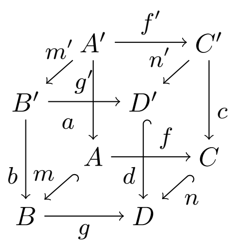
Lemma 7 ($\mathcal{M},\mathcal{N}$-van Kampen squares). Pushouts along $\mathcal{M},\mathcal{N}$-morphisms are $\mathcal{M},\mathcal{N}$-van Kampen squares.

Proof. We exploit the fact that the category ULG of unlabelled graphs is $\mathcal{M}$-adhesive. (This follows from Fact 4.1.6 for labelled graphs in [4], by restricting the label alphabet to a single label.)

Consider the pushout (1) below where $m \in \mathcal{M}$ and $f \in \mathcal{N}$. We have to show that, given a commutative cube (2) with (1) as bottom face, $b, c, d \in \mathcal{M}$, and pullbacks as back faces, the following holds:

the top face is a pushout ⇔ the front faces are pullbacks.





(2)

Part 1 ("⇒"). Assume that the top face of cube (2) is a pushout. Since pullback objects are unique up to isomorphism, it is sufficient to prove that B' and C' are isomorphic to the corresponding pullback objects. Let B'' be the pullback object of g and d with morphisms $b''\colon B'' \to B$ and $g''\colon B'' \to D'$. By the universal property of pullbacks, there is a unique morphism $u\colon B' \to B''$ such that $b'' \circ u = b$ and $g'' \circ u = g$. By forgetting all labels, cube (2) becomes a cube in ULG. Since ULG is $\mathcal{M}$-adhesive, every pushout in ULG is a van Kampen square.

Consequently, the morphism u is injective and surjective. It remains to show that u is $\perp$-preserving. Let $x \in B' - \mathrm{Dom}(\mathrm{l}_{B'})$. Suppose that $u(x) \in \mathrm{Dom}(\mathrm{l}_{B''})$. Then $b(x) \in \mathrm{Dom}(\mathrm{l}_B)$ and $g'(x) \in \mathrm{Dom}(\mathrm{l}_{D'})$.

Since the top is a pushout in PLG and $x \in B' - \mathrm{Dom}(\mathrm{l}_{B'})$, there exists $y \in \mathrm{Dom}(\mathrm{l}_{C'})$ with $g'(x) = n'(y)$. Since the bottom is a pushout in PLG and $m \in \mathcal{M}$, by Theorem 1, it is also a pullback, $b(x) \in \mathrm{Dom}(\mathrm{l}_B)$, $c(y) \in \mathrm{Dom}(\mathrm{l}_C)$, and the left front face commutes, $g(b(x)) = d(g'(x)) = d(n'(y))$ and there exists $z \in \mathrm{Dom}(\mathrm{l}_A)$ such that $m(z) = b(x)$ and $f(z) = c(y)$. Since the back right face is a pullback, $y \in \mathrm{Dom}(\mathrm{l}_{C'})$ and $z \in \mathrm{Dom}(\mathrm{l}_A)$ with $c(y) = m(z)$, there is some $x' \in \mathrm{Dom}(\mathrm{l}_{A'})$ with $m'(x') = x$. Then $x \in \mathrm{Dom}(\mathrm{l}_{B'})$, a contradiction. Thus u is $\perp$-preserving and B' and B'' are isomorphic. Similarly, it is shown that C' and the pullback object C'' of d and n are isomorphic. Thus, the back faces of cube (2) are pullbacks.

Part 2 ("$\Leftarrow$ ") Assume that the front faces of cube (2) are pullbacks in PLG. Since pushout objects are unique up to isomorphism, it is sufficient to prove that D' is isomorphic to the corresponding pushout object. Let D'' be the pushout object of m' and f' in PLG with morphisms $g'' \colon B' \to D''$ and $n'' \colon C' \to D''$. By the universal property of pushouts, there is a unique morphism $u \colon D'' \to D'$ such that $g' = u \circ g''$ and $n' = u \circ n''$. Consider now the underlying pushout in ULG. Since ULG is $\mathcal{M}$-adhesive, every pushout in ULG is a van Kampen square. Consequently, the morphism u is injective and surjective. It remains to show that u is $\perp$-preserving. Let $x \in D'' - \mathrm{Dom}(\mathrm{l}_{D''})$. Suppose that $u(x) \in \mathrm{Dom}(\mathrm{l}_{D'})$. Then $d(u(x)) \in \mathrm{Dom}(\mathrm{l}_D)$. Since the bottom is a pushout, there are two cases. In the first case, there exists an item $y \in \mathrm{Dom}(\mathrm{l}_B)$ such that $g(y) = d(u(x))$. Since the left front face is a pullback, $y \in \mathrm{Dom}(\mathrm{l}_B)$ and $u(x) \in \mathrm{Dom}(\mathrm{l}_{D'})$ with $g(y) = d(u(x))$, there is some $z \in \mathrm{Dom}(\mathrm{l}_{B'})$ with $b(z) = y$ and $g'(z) = u(x)$. By commutativity of the left front face, $d(g'(z)) = g(b(z)) = g(y) = d(u(x))$. By $d \in \mathcal{M}$, $g'(z) = u(x) \in \mathrm{Dom}(\mathrm{l}_{D''})$, a contradiction. In the second case, there exists an item $y \in \mathrm{Dom}(\mathrm{l}_C)$ such that $n(y) = d(u(x))$. Since the right front face is a pullback, we obtain a contradiction. Thus, the morphism u is $\perp$-preserving and the top face is a pushout. Since the back faces are pullbacks and $\mathcal{M}$ and $\mathcal{N}$ are stable under $\mathcal{M}$-pullbacks, $m \in \mathcal{M}$ and $f \in \mathcal{N}$ imply $m' \in \mathcal{M}$ and $f' \in \mathcal{N}$, i.e. the top face is an $\mathcal{M}, \mathcal{N}$-pushout. □

Proof of Theorem 3. See Lemma 3 to Lemma 7. □

Lemma 8 (HLR$^+$-properties). PLG has binary coproducts, an $\mathcal{E}$-$\mathcal{N}$ factorization, and an $\mathcal{E}'$-$\mathcal{M}$ pair factorization, where $\mathcal{E}$ is the class of surjective, undefinedness preserving morphisms and $\mathcal{E}'$ is the class of pairs of jointly surjective, undefinedness preserving morphisms.

Proof. Routine. □

By Theorem 3 and Lemma 8, we obtain the following corollary.

Corollary 1. The Local Church-Rosser Theorem and the Parallelism Theorem hold for $\mathcal{M}, \mathcal{N}$-adhesive tranformation systems over PLG.

Remark 4. $\mathcal{M},\mathcal{N}$-adhesive transformation systems over PLG provide a foundation for the semantics of the graph programming language GP [18, 19]. The graphs on which GP programs operate are totally labelled, and instances of GP's conditional rule schemata are rules with application conditions whose left- and right-hand graphs L and R are also totally labelled. The interface graph K consists of unlabelled nodes and hence enables relabelling of nodes. Moreover, the requirement that the vertical morphisms in double-pushouts must preserve unlabelled nodes guarantees that pushout complements are unique (see [14]).

In comparison with the approach of [14], $\mathcal{M},\mathcal{N}$-adhesive tranformation systems over PLG are more restrictive in that unlabelled nodes in rules must not match labelled nodes in host graphs. However, to allow certain nodes in rules to match nodes with arbitrary labels, one can use rule schemata with label variables instead of unlabelled nodes. As in GP, rule schemata are instantiated to rules with totally labelled left- and right-hand graphs, while unlabelled nodes are solely used for relabelling. Indeed, label variables in left-hand graphs are more versatile than unlabelled nodes because they can be typed in order to match only subsets of labels.

5 Conclusion

Double-pushout graph transformation with relabelling is not covered by $\mathcal{M}$-adhesive transformation systems. Relabelling is natural for computing with graphs, though, and provides a foundation for graph transformation languages such as GP. We have generalised $\mathcal{M}$-adhesive transformation systems to $\mathcal{M},\mathcal{N}$-adhesive transformation systems which do cover graph transformation with relabelling. We have proved the Local Church-Rosser Theorem and the Parallelism Theorem for $\mathcal{M},\mathcal{N}$-adhesive transformation systems with application conditions, and hence these results hold for graph transformation with relabelling. The Concurrency Theorem is proved in the long version of this paper [15].

We hope to establish the Amalgamation Theorem, the Embedding Theorem and the Local Confluence Theorem in our new framework, too. These results have recently been proved for $\mathcal{M}$-adhesive transformation systems with application conditions [6, 7].

In future work, we expect to be able to show that the category of term graphs is $\mathcal{M},\mathcal{N}$-adhesive. This category is known to be not $\mathcal{M}$-adhesive, too, but has been shown to be quasi-adhesive [2]. Indeed the categories of term graphs and partially labelled graphs are similar in that PLG can also be shown to be quasi-adhesive. In PLG, the regular monomorphisms are precisely the undefinedness preserving injective morphisms.

An extension of $\mathcal{M},\mathcal{N}$-adhesive transformation systems with rules that have a non-monomorphic right-hand morphism, allowing to merge items, may be possible. In the context of graph transformation with relabelling, the approach of [14] already includes such rules. Independently, in [1] a class of categories is identified for which the local Church-Rosser property holds for certain classes of rules with non-monomorphic right-hand morphisms.

Finally, the $\mathcal{W}$-adhesive transformation systems introduced in [11] provide a general framework for attributed objects. They allow undefined attributes in the interface of a rule to change attributes, which is similar to relabelling. But the precise relationship to $\mathcal{M}, \mathcal{N}$-adhesive transformation systems remains to be worked out.

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