Separability in Persistent Petri Nets

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Abstract. We prove that plain, bounded, reversible and persistent Petri nets are weakly and strongly separable.

1 Introduction

Given a place/transition Petri net $N = (N, M_0)$ with initial marking M_0 and a number $k \in \mathbb{N}$, one may consider the k-multiple net $k{\cdot}N = (N, k{\cdot}M_0)$, where every place holds k times the number of tokens it holds in M_0. This paper investigates the relationship between N and $k{\cdot}N$. The net $k{\cdot}N$ will be called *strongly separable* if every firing sequence starting at $k{\cdot}M_0$ belongs to the shuffle product of k firing sequences starting at M_0, and *weakly separable* if the Parikh vector of every firing sequence starting at $k{\cdot}M_0$ is the sum of the Parikh vectors of k firing sequences starting at M_0. To our knowledge, these notions have first been introduced in the context of work-flow nets in [7], where strong and weak separability have been called serializability and separability, respectively. Weak separability was proved in [3] for marked graphs, a strict subclass of persistent nets. In this paper, we prove both weak and strong separability for plain, bounded, reversible and persistent nets (*pbrp-nets*, for short), thus settling a conjecture made in [2]. Boundedness means that the set of reachable markings is finite. Reversibility means that the initial marking is reachable from every other reachable marking. Persistency means that at any reachable marking, an enabled transition is never disabled by the firing of another transition.

The remaining sections of the paper are organized as follows. Section 2 presents the technical background. Section 3 establishes two easy lemmas showing the stability of pbrp nets $k{\cdot}N$ under division by k. Section 4 establishes a crucial lemma stating that, if a pbrp net $k{\cdot}N$ has a single minimal realizable T-invariant X, then $X \leq 1$. Section 5 introduces the properties of weak and strong separability, which are shown to hold in sections 6 and 7, respectively, for pbrp nets $k{\cdot}N$ with a single minimal realizable T-invariant. Both properties are extended to general pbrp nets $k{\cdot}N$ in section 8. It is finally shown in section 9 that if $k{\cdot}N$ is a pbrp net, then $(k-1){\cdot}N$ is also pbrp.

2 Basic definitions, and earlier results

2.1 Petri nets, boundedness, reversibility, and persistency

A Petri net (P, T, F, M_0) consists of two finite and disjoint sets P (*places*) and T (*transitions*), a function $F\colon ((P \times T) \cup (T \times P)) \to \mathbb{N}$ (*flow*) and a marking M_0 (the *initial marking*), where a *marking* is a mapping $M\colon P \to \mathbb{N}$.

A transition $t \in T$ is *enabled by* (or *activated at*, or *firable at*) a marking M, denoted by $M[t\rangle$, if for all places $p \in P$, $M(p) \geq F(p,t)$. If t is enabled at M, then t can *occur* (or *be executed*) in M, leading to the marking M' defined by $M'(p) = M(p) + F(t,p) - F(p,t)$ (notation: $M[t\rangle M'$). These definitions can be extended inductively to transition sequences $\sigma \in T^*$: for the empty sequence ε, $M[\varepsilon\rangle$ and $M[\varepsilon\rangle M$ are always true; and $M[\sigma t\rangle$ (or $M[\sigma t\rangle M'$) iff there is some M'' with $M[\sigma\rangle M''$ and $M''[t\rangle$ (or $M''[t\rangle M'$, respectively).

A marking M' is *reachable* from a marking M if there exists a transition sequence σ such that $M[\sigma\rangle M'$. The *set of reachable markings* from M is denoted by $[M\rangle$. A transition t is called *weakly live* (at M_0) if $\exists M \in [M_0\rangle\colon M[t\rangle$.

The *reachability graph of* N, with initial marking M_0, is the graph with the set of vertices $[M_0\rangle$, i.e., all markings reachable from M_0, and where there is an edge from M to M' labelled with t iff $M[t\rangle M'$.

A Petri net $N = (P, T, F, M_0)$ is *plain* (or *ordinary*) if arc weights do not exceed 1 (i.e., $cod(F) \subseteq \{0, 1\}$). N is *k-bounded* if $M(p) \leq k$ for every place p in every reachable marking $M \in [M_0\rangle$, and *bounded* if it is k-bounded for some k. N is *persistent*, if whenever $M[t_1\rangle$ and $M[t_2\rangle$ for a marking $M \in [M_0\rangle$ and two transitions $t_1 \neq t_2$, then $M[t_1 t_2\rangle$. N is *reversible* if $M_0 \in [M\rangle$ for every $M \in [M_0\rangle$. In the sequel, plain, bounded, reversible and persistent Petri nets are called pbrp-nets for short. Figure 1 shows a pbrp-net and its reachability graph.

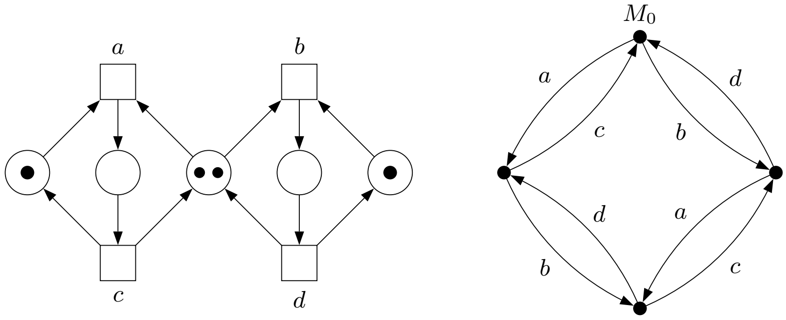


Fig. 1. A pbrp net (l.h.s.) and its reachability graph (r.h.s.)

2.2 Permutation equivalence, and Keller's theorem

Two transition sequences $\sigma \in T^*$ and $\sigma' \in T^*$ are said to arise from each other by a *transposition from* marking M if both are firable at M and they are the same except for the order of an adjacent pair of labels, thus:
$M[\sigma\rangle$ and $M[\sigma'\rangle$ and $\sigma = t_1 \dots t_k t t' \dots t_n$ and $\sigma' = t_1 \dots t_k t' t \dots t_n$.
Two transition sequences σ and σ' are said to be *permutations of each other from* marking M (written $\sigma \equiv_M \sigma'$) if they are both firable at M and they arise out of each other through a sequence of transpositions from M.

By $\tau \overset{\bullet}{-} \sigma$, we denote the *residue* of τ left after cancelling successively in this sequence the leftmost occurrences of all symbols from σ, read from left to right. Formally, $\tau \overset{\bullet}{-} \sigma$ is defined by induction on the length of σ:

$$\tau \overset{\bullet}{-} \varepsilon \quad = \tau$$

$$\tau \overset{\bullet}{-} t \quad = \begin{cases} \tau, \text{ if there is no label } t \text{ in } \tau \\ \text{the sequence obtained by erasing the leftmost } t \text{ in } \tau, \text{ otherwise} \end{cases}$$

$$\tau \overset{\bullet}{-} (t\sigma) = (\tau \overset{\bullet}{-} t) \overset{\bullet}{-} \sigma.$$

Keller's theorem [8] states that in a persistent net, if τ and σ are two transition sequences firable at some reachable marking $M \in [M_0\rangle$, then $\tau(\sigma \overset{\bullet}{-} \tau)$ and $\sigma(\tau \overset{\bullet}{-} \sigma)$ are also firable from M, and $\tau(\sigma \overset{\bullet}{-} \tau) \equiv_M \sigma(\tau \overset{\bullet}{-} \sigma)$. Furthermore, the marking reached after $\tau(\sigma \overset{\bullet}{-} \tau)$ equals the marking reached after $\sigma(\tau \overset{\bullet}{-} \sigma)$.

2.3 T-invariants, Parikh vectors, and cycles

The *incidence matrix* C of a net (P, T, F) is a $P \times T$-matrix of integers where the entry corresponding to a place p and a transition t is, by definition, equal to the number $F(t,p) - F(p,t)$. A *T-invariant* J is a vector of integers with index set T satisfying $C \cdot J = 0$. When comparing vectors with scalars, such as here, we always mean this componentwise. J is called *semipositive* if $J \geq 0$ and J is not the null vector. Throughout the paper, we will only consider semipositive T-invariants, and for succinctness, we will just call them "T-invariants". Two (semipositive) T-invariants J and J' are called *transition-disjoint* if $\forall t \in T\colon\ J(t) = 0 \vee J'(t) = 0$.

For a finite sequence of transitions $\sigma \in T^*$, the *Parikh vector* $\Psi(\sigma)$ of this sequence is a vector of natural numbers with index set T, where $\Psi(\sigma)(t)$ is the number of occurrences of t in σ. The *marking equation* states that if $M[\sigma\rangle M'$, then $M' = M + C \cdot \Psi(\sigma)$.

Two sequences $\tau, \sigma \in T^*$ are called *Parikh-equivalent* if $\Psi(\tau) = \Psi(\sigma)$. In any Petri net, $\sigma \equiv_M \tau$ entails $\Psi(\sigma) = \Psi(\tau)$. In a persistent Petri net, $\Psi(\sigma) = \Psi(\tau)$ entails also $\sigma \equiv_M \tau$ whenever M is a reachable marking and both sequences σ and τ are firable at M.

Let $M \in [M_0\rangle$. A sequence of transitions $M[\tau\rangle M$ is called a *cycle*. By the marking equation, for any cycle $M[\sigma\rangle M$, the Parikh vector $\Psi(\sigma)$ of this cycle is a T-invariant. A T-invariant is called *realizable* if it coincides with the Parikh vector of some cycle.

A cycle $M[\tau\rangle M$ is called *simple* if there is no permutation $\tau' \equiv_M \tau$ such that $\tau' = \tau_1\tau_2$, $M[\tau_1\rangle M$, $M[\tau_2\rangle M$, and $\tau_1 \neq \varepsilon \neq \tau_2$. For example, in Figure 1(r.h.s.), $M_0[ac\rangle M_0$ is simple, but $M_0[abcd\rangle M_0$ is not simple, in view of the permutation $M_0[ac\rangle M_0[bd\rangle M_0$.

The following results from [4] will be used in the sequel.

Theorem 1. Decomposing cycles of reversible persistent nets

Let $N = (P, T, F, M_0)$ be a bounded, reversible, and persistent Petri net. There exists a finite set $\mathcal{B}$ of semipositive T-invariants such that they are transition-disjoint and every cycle $M[\rho\rangle M$ in the reachability graph of N can be decomposed, up to permutations, to some sequence $M[\rho_1\rangle M[\rho_2\rangle M \ldots [\rho_n\rangle M$ of cycles with all Parikh vectors $\Psi(\rho_i)$ in $\mathcal{B}$. Moreover, $\mathcal{B}$ can be chosen as the set of Parikh vectors of simple cycles through any fixed state of N.

Theorem 2. Decomposing reversible persistent nets

Let $N = (P, T, F, M_0)$ be a bounded, reversible, and persistent net. Suppose that $\mathcal{B} = \{X_1, \ldots, X_n\}$, thus at any reachable marking, N generates n simple cycles with transition disjoint Parikh vectors $X_1, \ldots, X_n$. Then there are n bounded, persistent and reversible nets $N_1, \ldots, N_n$, such that each net N_i has exactly one minimal realizable T-invariant X_i and the reachability graph of N is isomorphic to the reachability graph of the disjoint sum of the nets $N_1, \ldots, N_n$.

The respective nets N_i constructed for $i = 1, \ldots, n$ in the proof of Theorem 2 are defined as $N_i = (P, T_i, F_i, M_0)$ where $T_i = \{t \in T \mid X_i(t) \neq 0\}$ and F_i is the induced restriction of F on $(P \times T_i) \cup (T_i \times P)$. In particular, all nets N_i *have the same initial marking M_0 as N*. This remark is crucial to the use of Theorem 2 made in section 8.

3 Multiples of a net, persistency, and the pbrp properties

In this paper, we study k*-multiples* of nets as follows. Let N be a net and let $k \geq 1$ be some positive integer number. For a marking M, the k-multiple marking $k{\cdot}M$ is defined by $(k{\cdot}M)(s) = k{\cdot}(M(s))$ for every place s. The net $k{\cdot}N$ is the same as the net N except that the initial marking $k{\cdot}M_0$ replaces the initial marking M_0 of N (thus, $1{\cdot}N$ is the same as N). The net $k{\cdot}N$ is called a k*-net*, for short. An example is shown in Figure 2. A marking L which is of the form $k{\cdot}M$, that is, which assigns to every place a multiple of k as tokens, is called a k*-marking*.

In this section, we show that the pbrp properties are preserved under scalar division of nets. Similar properties do not hold in general for multiplication. It is easy to construct a net N which is bounded, or persistent, or reversible while $k{\cdot}N$ is not. For persistency, Figure 2 can be taken as a counterexample. Plainness is obviously preserved by division and will henceforth be assumed of all nets.

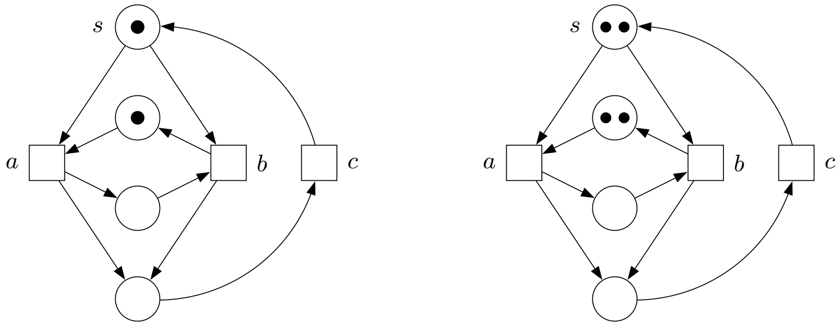


Fig. 2. A persistent Petri net (l.h.s.) and its 2-multiple (r.h.s.)

Lemma 1. DIVISION PRESERVES BOUNDEDNESS AND PERSISTENCY

Let N be plain. Let $k \geq 1$ and let $k{\cdot}N$ be bounded (persistent). Then N is also bounded (respectively, persistent).

Proof: The boundedness of N follows easily from the corresponding property of $k{\cdot}N$ and the fact that every firing sequence of N is also a firing sequence of $k{\cdot}N$ (since $k \geq 1$).

We prove the persistency of N by contraposition. Let $M[\sigma\rangle K$ be a firing sequence of N which violates persistency, i.e. K enables two transitions $a \neq b$ but not the sequence ab.

Then for all places $s \in {}^\bullet a$, $K(s) \geq 1$. Moreover, by plainness, there is some place $s_1 \in {}^\bullet a \cap {}^\bullet b$ such that $K(s_1) = 1$ and $L(s_1) = 0$, where $K[a\rangle L$.

By the firing rule (and by plainness again), σ^k is a firing sequence in $k{\cdot}N$ leading from $k{\cdot}M$ to $k{\cdot}K$. The marking $k{\cdot}K$ satisfies $(k{\cdot}K)(s) \geq k$ for all $s \in {}^\bullet a$ and $(k{\cdot}K)(s_1) = k$. Hence $k{\cdot}K$ activates the sequence a^k. After firing a^{k-1}, i.e. after

$$(k{\cdot}M)\ [\sigma^k\rangle\ (k{\cdot}K)\ [a^{k-1}\rangle\ L_1,$$

a marking $L_1 \geq K$ is reached which thus enables a and b. Now $L_1(s_1) = 1$, hence one further firing of a disables b. Therefore, $k{\cdot}N$ is not persistent. □

Lemma 2. DIVISION PRESERVES REVERSIBILITY

Let $k \geq 1$ and let $k{\cdot}N$ be pbrp. Then N is reversible.

Proof: Let M_0 be the initial marking of N and suppose $M_0[\alpha\rangle M$. As $k \geq 1$, also $k{\cdot}M_0[\alpha\rangle L$ in $k{\cdot}N$ for the marking $L = M + (k{-}1){\cdot}M_0$. Because $k{\cdot}N$ is reversible, $L[\beta\rangle k{\cdot}M_0$ for some sequence β. Combining this with $k{\cdot}M_0[\alpha\rangle L$, we get $k{\cdot}M_0[\alpha\beta\rangle k{\cdot}M_0$.

Executing k times the cycle just found yields $k{\cdot}M_0[(\alpha\beta)^k\rangle k{\cdot}M_0$. Let t_1 be the first transition of $(\alpha\beta)^k$. Because $k{\cdot}M_0[t_1\rangle$ and the net is plain, also $M_0[t_1\rangle$, say that $M_0[t_1\rangle M_1$. Then also $k{\cdot}M_0[t_1^k\rangle$, and of course, $k{\cdot}M_0[t_1^k\rangle k{\cdot}M_1$. Keller's theorem applied in $k{\cdot}N$ yields $k{\cdot}M_0[t_1^k\rangle k{\cdot}M_1[(\alpha\beta)^k \overset{\bullet}{-} t_1^k\rangle k{\cdot}M_0$. As $(\alpha\beta)^k$ contains t_1 a positive multiple of k times, the Parikh vector of the sequence $(\alpha\beta)^k \overset{\bullet}{-} t_1^k$ is again divisible by k. Continuing in this way, therefore, we find some sequence of (not necessarily mutually distinct) transitions $\gamma = t_1 \dots t_n \in T^*$ such that $\Psi(t_1^k \dots t_n^k) = \Psi((\alpha\beta)^k)$ and

$$k{\cdot}M_0[t_1^k\rangle k{\cdot}M_1[t_2^k\rangle k{\cdot}M_2 \;\dots\; k{\cdot}M_{n-1}[t_n^k\rangle k{\cdot}M_n \;\text{ with }\; M_n = M_0.$$

Moreover, $\Psi(\alpha) \leq \Psi(\gamma)$ because $\Psi(\alpha^k) \leq \Psi((\alpha\beta)^k) = \Psi(t_1^k \dots t_n^k) = \Psi(\gamma^k)$. By construction, also, $M_0[t_1\rangle M_1[t_2\rangle M_2 \;\dots\; M_{n-1}[t_n\rangle M_0$. As N is persistent by Lemma 1, Keller's theorem can be applied at M_0 in N. Since $M_0[\alpha\rangle M$ and $M_0[\gamma\rangle M_0$, one obtains both $M_0[\alpha\rangle M[\gamma \overset{\bullet}{-} \alpha\rangle M'$ and $M_0[\gamma\rangle M_0[\alpha \overset{\bullet}{-} \gamma\rangle M'$, for some marking M'. Since $\Psi(\alpha) \leq \Psi(\gamma)$, we have $\alpha \overset{\bullet}{-} \gamma = \varepsilon$, and hence $M' = M_0$. Thus we have found a sequence β', namely $\beta' = \gamma \overset{\bullet}{-} \alpha$, leading back from M to M_0: $M_0[\alpha\rangle M[\beta'\rangle M_0$. Since α was arbitrary, N is reversible. □

4 The minimal cycles of a reversible and persistent *k*-net

Theorem 2 and Lemmas 1 and 2 imply that a pbrp k-net with $n \geq 2$ minimal realizable T-invariants can always be decomposed into n disjoint pbrp k-nets, each of which has exactly one minimal realizable T-invariant X. The latter case is scrutinized in this section, where we will establish the following theorem.

Theorem 3. Simple cycles in $k{\cdot}N$ have Parikh vector 1

Let $k \geq 2$ and let $(N, k{\cdot}M_0)$ be a pbrp k-net with exactly one minimal realizable T-invariant X. Then $X \leq 1$ and for any transition t, $X(t) = 0$ if and only if t is not *weakly live at $k{\cdot}M_0$.*

In the rest of the section, we assume w.l.o.g. that all transitions are weakly live, and we show that $X \leq 1$ under this stronger assumption.

Plainness is important for Theorem 3 to hold. In Figure 3, all simple cycles of the net on the right-hand side have Parikh vector $X = \Psi(abb)$, but $X \neq 1$, contrary to the conclusion of Theorem 3.

Recall that in a (plain) connected marked graph N, all transitions occur an equal number of times in any cycle [5, 6]. Any such marked graph has thus exactly one minimal realizable T-invariant, viz. the vector 1. Theorem 3 extends this behavioural property of connected marked graphs to pbrp-nets $k{\cdot}N$ with exactly one minimal realizable T-invariant. It is worth noting that the statement made in Theorem 3 would not hold under the weaker assumption that N instead of $k{\cdot}N$ is persistent. For instance, let $k = 2$ and consider Figure 2. On the left-hand side, $X = (a \mapsto 1, b \mapsto 1, c \mapsto 2)$ is the unique minimal realizable T-invariant, and it can be realized by the firing sequence $M_0[acbc\rangle M_0$. Note that $X \neq 1$. On the right-hand side, X is also the unique minimal realizable T-invariant. However, the net shown on the right-hand side of Figure 2 is not persistent. Executing

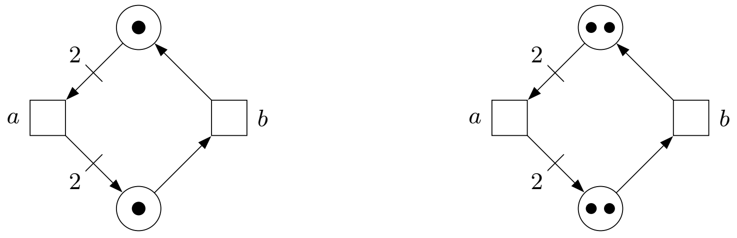


Fig. 3. A weighted Petri net (l.h.s.) and its 2-multiple (r.h.s.)

a in the initial marking leads to a marking in which both a and b are enabled although their shared input place s carries only one token, hence producing a true conflict and destroying persistency. Thus, both requirements that $k{\cdot}N$ be persistent and that $k \geq 2$ are crucial for Theorem 3 to hold.

We shall now give the proof of Theorem 3, which is critical to all results established in the remaining sections. By way of approaching this proof, let $k{\cdot}N$ be a pbrp-net with exactly one minimal realizable T-invariant X. By assumption, all transitions are weakly live, hence we want to show $X = 1$. As every weakly live transition must occur at least once in any firing sequence realizing X, $X \geq 1$ in view of Theorem 1 and the unicity of X. If X is a k-multiple, then, a contradiction of the assumption $k \geq 2$ can be derived easily as we will see later. The complicated case is when X is *not* a k-multiple. In this case, we will *i)* construct a T-invariant which extends X and is a k-multiple; *ii)* show that this new T-invariant is realizable in $k{\cdot}N$; *iii)* and show that a contradiction to the minimality of X ensues unless $X = 1$. In order to construct this T-invariant, we use the fact that X is realized by some firing sequence α in T^* and we introduce an auxiliary function $zip_k \colon T^* \to T^*$ which, given any sequence $\alpha \in T^*$, constructs from α another sequence $zip_k(\alpha)$ with $\Psi(\alpha) \leq \Psi(zip_k(\alpha))$ such that the latter is a k-multiple. Intuitively, zip_k yields a "ceiling" operation on Parikh vectors with respect to divisibility by k.

Definition 1. FUNCTION zip_k

Let $zip_k \colon T^* \to T^*$ be the function inductively defined with the following equations, where $t \in T$:

$$\begin{aligned} zip_k(\varepsilon) &= \varepsilon \\ zip_k(t\alpha') &= t^k\, zip_k(\alpha' \mathbin{\dot{-}} t^{k-1}) \end{aligned} \tag{1}$$

□

It follows directly from this definition that $\Psi(zip_k(\alpha))$ is a k-multiple and more precisely, the least k-multiple larger than or equal to $\Psi(\alpha)$. Moreover, if $\Psi(\alpha) \leq k{-}1$ then $zip_k(\alpha) = a_1^k \ldots a_l^k$, where $a_1 \ldots a_l$ are all distinct letters of α, in the order of their first occurrences. For example, $zip_5(ab^4a^3) = a^5 zip_5(b^4) = a^5b^5$.

Lemma 3. ENABLING $\mathit{zip}_k(\alpha)$

Let $k{\cdot}N$ be plain and persistent, and let $k{\cdot}M$ be a reachable k-marking of $k{\cdot}N$. If a sequence α is enabled at $k{\cdot}M$, then $\mathit{zip}_k(\alpha)$ is also enabled at $k{\cdot}M$.

Proof: We use Keller's theorem and induction on the length of sequences. If $\alpha = \varepsilon$, the claim is obviously true.

Suppose now $\alpha = t\alpha'$ and $k{\cdot}M[t\alpha'\rangle$, with $t \in T$. By plainness, $M[t\rangle M'$ and hence $k{\cdot}M[t^k\rangle k{\cdot}M'$. By Keller's theorem, also $k{\cdot}M[t^k\rangle k{\cdot}M'[(t\alpha') \mathbin{\overset{\bullet}{-}} (t^k)\rangle$, and therefore $k{\cdot}M[t^k\rangle k{\cdot}M'[\alpha' \mathbin{\overset{\bullet}{-}} t^{k-1}\rangle$. By the induction hypothesis, $k{\cdot}M'[\mathit{zip}_k(\alpha' \mathbin{\overset{\bullet}{-}} t^{k-1})\rangle$; hence $k{\cdot}M[t^k\rangle k{\cdot}M'[\mathit{zip}_k(\alpha' \mathbin{\overset{\bullet}{-}} t^{k-1})\rangle$. By the definition of $\mathit{zip}_k(\alpha)$, $k{\cdot}M[\mathit{zip}_k(\alpha)\rangle$. □

In the sequel, we apply the zip_k construction to cycles $k{\cdot}M_0[\gamma\rangle k{\cdot}M_0$ of $k{\cdot}N$, and we use the property that if $\widehat{\gamma} = \mathit{zip}_k(\gamma)$, then the Parikh vector of $\widehat{\gamma}$ may be computed from the Parikh vector of γ. We describe now this computation.

First, we note that the Parikh vector of γ splits (uniquely) as a sum:

$$\Psi(\gamma) \quad = \quad Y_k \quad + \quad Y_{k-1} \quad + \; \ldots \; + \quad Y_1$$

where $k|Y_k$ (read k divides Y_k) and for all $k{-}1 \geq h \geq 1$ and for all transitions t, $Y_h(t) \in \{h, 0\}$. Indeed, let $d_t = \Psi(\gamma)(t) \,\mathbf{div}\, k$ (where **div** denotes integer division) and $h_t = \Psi(\gamma)(t) \,\mathbf{mod}\, k$ for every transition t. Define $Y_k(t) = k{\cdot}d_t$ (thus $k|Y_k(t)$), and for $k{-}1 \geq h \geq 1$, define $Y_{h_t}(t) = h_t$ and $Y_h(t) = 0$ if $h \neq h_t$. We claim that

$$\begin{aligned} \Psi(\widehat{\gamma}) &= Y_k + \textstyle\sum_{h=1}^{k-1} \left(\frac{k}{h}{\cdot}Y_h\right) \\ &= Y_k + \tfrac{k}{k-1}{\cdot}Y_{k-1} + \tfrac{k}{k-2}{\cdot}Y_{k-2} + \ldots + \tfrac{k}{2}{\cdot}Y_2 + k{\cdot}Y_1 \end{aligned} \tag{2}$$

This can be seen by examining the zip_k construction. In fact, $\mathit{zip}_k(\gamma)$ is computed by first moving to the left d_t subwords t^k of γ for each transition t (this does not affect the length of the sequence), and then moving to the left, for each transition t still appearing on the right, all h_t occurrences still untouched, augmented with $k{-}h_t$ new occurrences of t if $h_t \neq 0$ (this may increase the length of the sequence).

Example (with $k = 5$):

$$\begin{aligned} \mathit{zip}_5(a^4ba^3) &= a^5\,\mathit{zip}_5(ba^2) && \text{(the first five } a\text{s are moved left)} \\ &= a^5b^5\,\mathit{zip}_5(a^2) && \text{(one } b \text{ is moved left; four } b\text{s are added)} \\ &= a^5b^5a^5. && \text{(two more } a\text{s are moved left; three } a\text{s are added).} \end{aligned}$$

Writing Parikh vectors as $\begin{pmatrix} x \\ y \end{pmatrix}$ to denote entries x for a and y for b, we have:

$$\begin{aligned} \Psi(a^4ba^3) \quad &= \quad \begin{pmatrix} 7 \\ 1 \end{pmatrix} \quad = \quad \underbrace{\begin{pmatrix} 5 \\ 0 \end{pmatrix}}_{Y_5} + \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{Y_4} + \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{Y_3} + \underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}_{Y_2} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{Y_1} \\ \Psi(a^5b^5a^5) &= \; Y_5 + \tfrac{5}{4}{\cdot}Y_4 + \tfrac{5}{3}{\cdot}Y_3 + \tfrac{5}{2}{\cdot}Y_2 + 5{\cdot}Y_1. \end{aligned}$$

We are now in a position to produce a proof of Theorem 3.

Proof: Let $k{\cdot}M_0[\gamma\rangle k{\cdot}M_0$ be a simple cycle in $k{\cdot}N$, thus γ realizes X. We distinguish two exhaustive and mutually exclusive cases.

Case 1: $k|\Psi(\gamma)$, that is, all entries of the Parikh vector of γ are divisible by k.

If $\gamma = \varepsilon$, then the net has no transitions and there is nothing to prove.

Otherwise, let t be the first transition in γ. Because $\Psi(\gamma)$ is a k-multiple, t occurs at least k times in γ, that is, $\Psi(t^k) \leq \Psi(\gamma)$. As $k{\cdot}M_0[t\rangle$ and $k{\cdot}N$ is a plain net, necessarily $k{\cdot}M_0[t^k\rangle k{\cdot}M_1$ for some k-multiple marking $k{\cdot}M_1$. By Keller's theorem, $k{\cdot}M_1[\gamma \overset{\bullet}{-} t^k\rangle k{\cdot}M_0$. Moreover, $\Psi(\gamma \overset{\bullet}{-} t^k)$ is another k-multiple since $\Psi(t^k)$ is smaller than or equal to $\Psi(\gamma)$ and thus, $\Psi(\gamma \overset{\bullet}{-} t^k) = \Psi(\gamma) - \Psi(t^k)$.

Let $t_1 = t$. Continuing in this way, we find $t_2, \ldots, t_n$ such that

$$k{\cdot}M_0[t_1^k\rangle k{\cdot}M_1[t_2^k\rangle \;\ldots\; [t_n^k\rangle k{\cdot}M_0.$$

As $k{\cdot}M_0[t_1^k \ldots t_n^k\rangle k{\cdot}M_0$, by plainness, $M_0[t_1 \ldots t_n\rangle M_0$, and therefore, *a fortiori*, $k{\cdot}M_0[t_1 \ldots t_n\rangle k{\cdot}M_0$.

Seeing that $\Psi(t_1 \ldots t_n)$ is a realizable T-invariant in $k{\cdot}N$, this Parikh vector must be greater than or equal to X. Therefore,

$$\Psi(\gamma) \;=\; X \;\leq\; \Psi(t_1 \ldots t_n) \;=\; \frac{1}{k}\Psi(t_1^k \ldots t_n^k) \;=\; \frac{1}{k}\Psi(\gamma)\;,$$

yielding a contradiction since $k \geq 2$ (and γ is not empty).

Case 2: $k \nmid \Psi(\gamma)$.

Define $\widehat{\gamma} = \mathit{zip}_k(\gamma)$. Let $\Psi(\gamma) = Y_k + \sum_{h=1}^{k-1} Y_h$ and $\Psi(\widehat{\gamma}) = Y_k + \sum_{h=1}^{k-1} (\frac{k}{h}{\cdot}Y_h)$ be the respective decompositions of these two vectors defined above, thus Y_k is a k-multiple and for every $k-1 \geq h \geq 1$ and $t \in T$, $Y_{h_t}(t) = h_t = \Psi(\gamma)(t) \,\mathbf{mod}\, k$ and $Y_h(t) = 0$ for $h \neq h_t$. Note that $Y_{k-1} + \ldots + Y_1$ is not the null vector, since $k \nmid \Psi(\gamma)$.

From $k{\cdot}M_0[\gamma\rangle$ and by Lemma 3, $k{\cdot}M_0[\widehat{\gamma}\rangle L$ for some marking L. As $k{\cdot}M_0$ is a k-marking and $\Psi(\widehat{\gamma})$ is a k-multiple, L is also a k-marking, say $L = k{\cdot}M_1$. Thus $k{\cdot}M_0[\widehat{\gamma}\rangle k{\cdot}M_1$. Let γ and $\widehat{\gamma}$ be renamed γ_1 and $\widehat{\gamma_1}$, respectively. So far,

$$k{\cdot}M_0[\widehat{\gamma_1}\rangle k{\cdot}M_1.$$

By Theorem 1 and the assumption that X is the only minimal realizable T-invariant of $k{\cdot}N$, $k{\cdot}M_1[\gamma_2\rangle k{\cdot}M_1$ for some simple cycle with Parikh vector $\Psi(\gamma_2) = X$. One may now iterate the construction of $\widehat{\gamma_{i+1}}$ and $k{\cdot}M_{i+1}$ from γ_{i+1} and $k{\cdot}M_i$ (presented above for $i = 0$). By doing so, one obtains an infinite sequence

$$k{\cdot}M_0[\widehat{\gamma_1}\rangle k{\cdot}M_1[\widehat{\gamma_2}\rangle k{\cdot}M_2[\widehat{\gamma_3}\rangle k{\cdot}M_3 \;\ldots$$

where all $\widehat{\gamma_i}$ have the same Parikh vector as $\widehat{\gamma}$, namely the one given by (2), since $\Psi(\gamma_i) = \Psi(\gamma)$ for all i. As the net $k{\cdot}N$ is bounded, the markings $k{\cdot}M_0$, $k{\cdot}M_1, \ldots$

cannot be all different, hence there exists some finite nonempty subsequence of the form

$$k{\cdot}M_{i-1}[\widehat{\gamma_i}\widehat{\gamma_{i+1}}\dots\widehat{\gamma_j}\rangle k{\cdot}M_j, \text{ with } 1 \leq i \leq j \text{ and } k{\cdot}M_{i-1} = k{\cdot}M_j.$$

Between $k{\cdot}M_{i-1}$ and $k{\cdot}M_j$, there are $(j-i+1) \geq 1$ sequences with Parikh vectors equal to $\Psi(\widehat{\gamma})$. Thus, $(j-i+1)\cdot\Psi(\widehat{\gamma})$ is a realizable T-invariant and necessarily, $\Psi(\widehat{\gamma})$ also is, showing that $k{\cdot}M_0[\widehat{\gamma}\rangle k{\cdot}M_0$.

So far, we have constructed two T-invariants, $\Psi(\gamma)$ and $\Psi(\widehat{\gamma})$, such that the latter is a k-multiple and extends the former, which is not a k-multiple. The remaining part of the proof contains an elaborate argument showing that this is possible only when $\Psi(\gamma) = 1$.

Recall that $k{\cdot}M_0[\gamma\rangle k{\cdot}M_0$, and $\Psi(\gamma) \leq \Psi(\widehat{\gamma})$. By Keller's theorem, $k{\cdot}M_0[\widehat{\gamma}\stackrel{\bullet}{-}\gamma\rangle$, and by $\Psi(\gamma) \leq \Psi(\widehat{\gamma})$, $\Psi(\widehat{\gamma}\stackrel{\bullet}{-}\gamma) = \Psi(\widehat{\gamma}) - \Psi(\gamma)$. The latter difference is not null, since $k|\Psi(\widehat{\gamma})$ but $k \nmid \Psi(\gamma)$ (assumption of Case 2). As $\Psi(\widehat{\gamma})$ and $\Psi(\gamma)$ are T-invariants, so is $\Psi(\widehat{\gamma}) - \Psi(\gamma)$. Moreover, $\Psi(\widehat{\gamma}\stackrel{\bullet}{-}\gamma) = \Psi(\widehat{\gamma}) - \Psi(\gamma)$ is realizable since $k{\cdot}M_0[\widehat{\gamma}\stackrel{\bullet}{-}\gamma\rangle$.

Using equation (2) and $X = \Psi(\gamma) = Y_k + \ldots + Y_1$, one obtains

$$\Psi(\widehat{\gamma}\stackrel{\bullet}{-}\gamma) \;=\; \Psi(\widehat{\gamma}) - \Psi(\gamma) \;=\; \sum_{h=1}^{k-1}(\frac{k-h}{h}\cdot Y_h). \tag{3}$$

As $\Psi(\widehat{\gamma}\stackrel{\bullet}{-}\gamma)$ is a realizable T-invariant and X $(= \Psi(\gamma))$ is the unique minimal realizable T-invariant of $k{\cdot}N$, $\Psi(\widehat{\gamma}\stackrel{\bullet}{-}\gamma) = l{\cdot}X$ for some positive integer l. Thus $\Psi(\widehat{\gamma}) = \Psi(\widehat{\gamma}\stackrel{\bullet}{-}\gamma) + X = (l{+}1){\cdot}X$. Combining the above, one obtains:

$$\sum_{h=1}^{k-1}(\frac{k-h}{h}{\cdot}Y_h) + X \;=\; \Psi(\widehat{\gamma}) \;=\; l{\cdot}X + X \;=\; l{\cdot}Y_k + (l{\cdot}\sum_{h=1}^{k-1}Y_h) + X.$$

The first equation follows from (3) and from $\Psi(\gamma) = X$; the second equation follows from $\Psi(\widehat{\gamma}) = (l{+}1){\cdot}X$; the third equation follows from $X = Y_k + \ldots + Y_1$. Comparing the rightmost and leftmost sums in this equation, one gets:

$$l{\cdot}Y_k \;=\; \sum_{h=1}^{k-1}\frac{k-(l+1){\cdot}h}{h}\cdot Y_h \tag{4}$$

We show now that Y_k must be the null vector. For contradiction, assume the contrary. Then $Y_k(t) \geq 1$ for some transition t. As Y_k is a k-multiple, even $Y_k(t) \geq k$ and $l{\cdot}Y_k(t) \geq l{\cdot}k$. As $l > 0$ and in view of equation (4), $Y_{h_t}(t) \neq 0$ since by definition of Y, $Y_h(t) = 0$ for any $1 \leq h \leq k-1$ with $h \neq h_t$. Thus, $Y_{h_t}(t) = h_t$. Combining these two properties and remembering that $k \geq 2$,

$$0 \;<\; l{\cdot}k \;\leq\; l{\cdot}Y_k(t) \;=\; k-(l+1){\cdot}h_t\;, \tag{5}$$

However, $1 \leq h_t$ and $1 \leq l$ entail $k-(l+1){\cdot}h_t \leq k-2$, and with (5), one gets $l{\cdot}k \leq k-2$. As l is a positive integer, we have reached a contradiction. Thus, Y_k is indeed the null vector.

Y_k being the null vector means that $\Psi(\gamma) \leq k-1$. Recall that $\widehat{\gamma} = zip_k(\gamma)$. By the definition of zip_k (and the remark just after Definition 1), $\widehat{\gamma} = t_1^k \dots t_n^k$ where $t_1 \dots t_n$ are all distinct transitions occurring in γ, with $t_i \neq t_j$ for $i \neq j$. As $k{\cdot}M_0[\widehat{\gamma}\rangle k{\cdot}M_0$, by plainness $M_0[t_1 \dots t_n\rangle M_0$, and *a fortiori* $k{\cdot}M_0[t_1 \dots t_n\rangle k{\cdot}M_0$. Seeing that $\Psi(t_1 \dots t_n)$ is a realizable T-invariant, this Parikh vector must be greater than or equal to X. As the transitions $t_1, \dots, t_n$ are mutually distinct, necessarily $\Psi(t_1 \dots t_n) \leq 1$. Therefore, $1 \leq X \leq \Psi(t_1 \dots t_n) \leq 1$. Altogether, $X = 1$ (and also $\Psi(t_1 \dots t_n) = 1$), as was to be shown. ☐

As already mentioned, the property stated for pbrp nets $k{\cdot}N$ in Theorem 3 is a classical property of plain connected marked graphs. A natural question is whether any pbrp net $k{\cdot}N$ with exactly one minimal realizable invariant X can be transformed to a marked graph by just eliminating redundant places. The answer to this question is negative. Indeed, Figure 4 shows a pbrp 2·-net with exactly one minimal realizable T-invariant (the all-ones vector) and with no redundant place (as checked by the tool SYNET [1]). However, SYNET produces also a differently-shaped but language-equivalent marked graph. It is still an open question whether this is always the case.

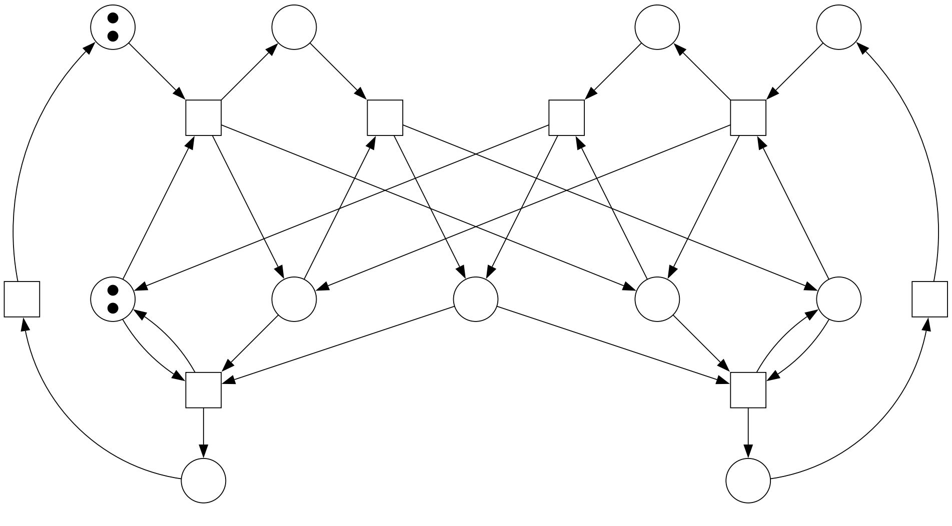


Fig. 4. A persistent 2-net which is not a marked graph

5 Definition of separability

We distinguish two notions of separability.

Definition 2. WEAK AND STRONG SEPARABILITY

Let $k \geq 1$ and let $(N, k{\cdot}M)$ be any net with k-marking $k{\cdot}M$.

A firing sequence $k{\cdot}M[\sigma\rangle$ is *weakly k-separable* from $k{\cdot}M$ (or just weakly separable if k and M are understood from the context) if there exist k sequences $\sigma_1, \ldots, \sigma_k$ such that

$$(\forall j, 1{\leq}j{\leq}k\colon M[\sigma_j\rangle \text{ in } (N, M)) \quad \text{and} \quad (\sum_{j=1}^{k} \Psi(\sigma_j)) = \Psi(\sigma). \tag{6}$$

A firing sequence $k{\cdot}M[\sigma\rangle$ is *strongly k-separable* from $k{\cdot}M$ if there exist k sequences $\sigma_1, \ldots, \sigma_k$ such that

$$(\forall j, 1{\leq}j{\leq}k\colon M[\sigma_j\rangle \text{ in } (N, M)) \quad \text{and} \quad \sigma \in \bigsqcup_{j=1}^{k} \sigma_j, \tag{7}$$

where $\sqcup$ denotes the shuffle product ("arbitrary interleaving") operator. A k-net is weakly (strongly) separable if every sequence firable in its initial marking is weakly (strongly) separable from this k-marking. $\square$

6 Weak separability

In this section and in section 7, we will establish the weak (strong, respectively) separability of pbrp-nets under the special assumption that there exists exactly one minimal realizable T-invariant X. In the rest of this section and in section 7, this assumption applies *implicitly* to all k-nets under consideration. The results will be extended to the general case in section 8.

In the sequel, we usually denote by $N = (N, M_0)$ the net with initial marking M_0 under consideration, by $k{\cdot}N$ the net $(N, k{\cdot}M_0)$ with initial k-marking $k{\cdot}M_0$, and by X the unique minimal realizable T-invariant of $k{\cdot}N$. Note that if $k{\cdot}N$ is a pbrp-net, then $X \leq 1$ by Theorem 3.

Lemma 4. SHIFTING k-MULTIPLE SUBWORDS

Let N be plain. Let $k \geq 2$ and let $k{\cdot}N$, with initial marking $k{\cdot}M_0$, be bounded, reversible, and persistent. Suppose $k{\cdot}M_0[\sigma\rangle$. Then there is some sequence of transitions $t_1 \ldots t_n$ such that

$$k{\cdot}M_0[t_1^k \ldots t_n^k\rangle k{\cdot}M_1[\sigma'\rangle \text{ with } \Psi(\sigma') \leq k-1 \text{ and } \sigma \equiv_{k\cdot M_0} t_1^k \ldots t_n^k \sigma'.$$

Proof: Choose a transition t_1 which is enabled at M_0 and satisfies $\Psi(\sigma)(t_1) \geq k$, i.e., such that there are at least k occurrences of t_1 in σ, if such a transition exists. By plainness and by Keller's theorem,

$$k{\cdot}M_0[t_1^k\rangle k{\cdot}M_0'[\sigma \mathbin{\dot{-}} t_1^k\rangle.$$

Choose a transition t_2 which is enabled at M_0' and satisfies $\Psi(\sigma \mathbin{\dot{-}} t_1^k)(t_2) \geq k$, if such a transition exists. Again by plainness and by Keller's theorem,

$$k{\cdot}M_0[t_1^k\rangle k{\cdot}M_0'[t_2^k\rangle k{\cdot}M_0''[\sigma \mathbin{\dot{-}} (t_1^k t_2^k)\rangle.$$

Repeating this reordering procedure as long as possible, one constructs a sequence

$$k{\cdot}M_0[t_1^k \dots t_n^k\rangle k{\cdot}M_1[\sigma'\rangle$$

where $\sigma' = \sigma \mathbin{\dot{-}} (t_1^k \dots t_n^k)$ (possibly $n = 0$, in which case $\sigma' = \sigma$ and $M_1 = M_0$) and $\Psi(\sigma')(t) \leq k{-}1$ for every transition t enabled at M_1.

We show that no transition (not just the ones enabled at M_1) can occur more than $k-1$ times in σ'. To this end, let $k{\cdot}M_1[\gamma\rangle k{\cdot}M_1$ be any cycle such that $\Psi(\gamma) \leq 1$. Such a cycle must exist because, on the one hand, X is a realizable T-invariant of $k{\cdot}N$ and $X \leq 1$ by Theorem 3, and on the other hand, this T-invariant can be realized at every reachable marking of $k{\cdot}N$ (by Theorem 1). Repeating this cycle $k-1$ times gives a cycle $k{\cdot}M_1[\gamma^{k-1}\rangle k{\cdot}M_1$.

Applying now Keller's theorem to $k{\cdot}M_1[\gamma^{k-1}\rangle$ and $k{\cdot}M_1[\sigma'\rangle$ yields

$$k{\cdot}M_1[\sigma' \mathbin{\dot{-}} \gamma^{k-1}\rangle \tag{8}$$

If $\sigma' \mathbin{\dot{-}} \gamma^{k-1} \neq \varepsilon$ then the first transition of $\sigma' \mathbin{\dot{-}} \gamma^{k-1}$ is firable at $k{\cdot}M_1$ (due to (8)) and it occurs at least k times in σ' (due to $\sigma' \mathbin{\dot{-}} \gamma^{k-1} \neq \varepsilon$ and the fact, stated in Theorem 3, that $\Psi(\gamma)(t) = 1$ for any transition t firable!em at $k{\cdot}M_1$). This contradicts the fact that the reordering procedure (extracting such t^k from σ) has been repeated as long as possible.

Hence $\sigma' \mathbin{\dot{-}} \gamma^{k-1} = \varepsilon$, which, by $\Psi(\gamma) \leq 1$, implies that σ' contains every transition at most $k-1$ times. By construction, $\sigma \equiv_{k\cdot M_0} t_1^k \dots t_n^k \sigma'$. This establishes the claims of the lemma. ☐

By applying Lemma 4, a sequence σ fired at $k{\cdot}M_0$ can be transformed into a permutation-equivalent sequence, viz. $t_1^k \dots t_n^k \sigma'$, consisting of an initial segment (leading to $k{\cdot}M_1$) in which every transition occurs a multiple of k times (where the $t_1, \dots, t_n$ are not necessarily all distinct), followed by a tail, denoted by σ', in which every transition occurs at most $k-1$ times. The next lemma, applied with $j = k-1$ and $L = M = M_1$ (thus $L + j \cdot M = k \cdot M_1[\sigma'\rangle$), and with $\tau = \sigma'$ and $\chi = \varepsilon$ (thus $k \cdot M_1[\tau\rangle$ and $k \cdot M_1[\chi\rangle k \cdot M_1$), shows that σ' can be further transformed into an initial segment in which every transition occurs *exactly* $k-1$ times and a new tail in which every transition occurs *at most* $k-2$ times.

Lemma 5. Shifting j-multiple subwords for $1 \leq j < k$

Let N be plain. Let $k \geq 2$ and let $k{\cdot}N$, with initial marking $k{\cdot}M_0$, be bounded, reversible and persistent. Let j be a fixed number such that $1 \leq j < k$. Then the following implication is valid:

if *a transition sequence τ satisfying $\Psi(\tau) \leq j$ is firable in $k{\cdot}N$ at a reachable marking of the form $L{+}j{\cdot}M$, and if moreover $(L{+}j{\cdot}M)[\chi\rangle k{\cdot}M$ for some sequence χ such that τ and χ are transition-disjoint,*

then *$M[t_1 \dots t_p\rangle$ where $t_1 \dots t_p$ is an enumeration of the set $\{t_1, \dots, t_p\} = \{t \mid \Psi(\tau)(t) = j\}$, and $\tau \equiv_{L+j\cdot M} t_1^j \dots t_p^j \, \tau'$ for a sequence τ' satisfying $\Psi(\tau') \leq j-1$ and not containing $t_1, \dots, t_p$. Moreover, $L{+}j{\cdot}M[t_1^j \dots t_p^j\rangle L{+}j{\cdot}M'[\chi'\rangle k{\cdot}M'$ for some sequence χ' such that τ' and χ' are transition-disjoint.*

For explaining the meaning of this lemma, examine the arrows τ and χ emanating from the North-Western corner, labelled $L + j{\cdot}M$, of Figure 5. According to the lemma, all instances of the transitions $t_1, \dots, t_p$, which occur exactly j times in τ, may be shifted towards the beginning, thus forming an initial segment $t_1^j \dots t_p^j$ after which the residual sequence $\tau' = \tau \overset{\bullet}{-} (t_1^j \dots t_p^j)$ is executed. In τ', every transition occurs now at most $j-1$ times, and since τ' and χ' are transition disjoint, the lemma be be applied again to τ', $j-1$ and χ'.

$$\begin{tikzcd}
\boxed{L+j{\cdot}M} \arrow[r, "t_1^j \dots t_p^j"] \arrow[rrrr, bend left, "\tau \text{ with } \Psi(\tau) \leq j"] \arrow[d, "\chi"] & \boxed{L+j{\cdot}M'} & = & \boxed{\overbrace{L+M'}^{L'}+(j-1){\cdot}M'} \arrow[r, "\tau' = \tau \overset{\bullet}{-} (t_1^j \dots t_p^j)"] \arrow[d, "\chi' = \chi\, t_1^{k-j} \dots t_p^{k-j}"] & {} \\
\boxed{k{\cdot}M} \arrow[rrr, "t_1^k \dots t_p^k"] & & & \boxed{k{\cdot}M'} &
\end{tikzcd}$$

Fig. 5. Explanation of Lemma 5

Proof: We use an induction on p. If $p = 0$, then $\Psi(\tau) \leq j-1$, and apart from setting $\tau' = \tau$, there is nothing to prove. Otherwise, if $p > 0$, we claim that some transition t' occurring j times in τ is enabled at M in N. We establish this claim by producing such t'.

As $(L{+}j{\cdot}M)[\tau\rangle$ and $(L{+}j{\cdot}M)[\chi\rangle k{\cdot}M$ in $k{\cdot}N$, $(L{+}j{\cdot}M)[\chi\rangle k{\cdot}M[\tau \overset{\bullet}{-} \chi\rangle$ by Keller's theorem. Therefore, seeing that τ and χ are transition-disjoint, $k{\cdot}M[\tau\rangle$.

As $k{\cdot}M$ is a reachable marking of $k{\cdot}N$ and $X \leq 1$ is the least realizable T-invariant of $k{\cdot}N$, by Theorem 1, $k{\cdot}M[\gamma\rangle k{\cdot}M$ for some sequence γ satisfying $\Psi(\gamma) = X \leq 1$. Repeating this cycle $j-1$ times yields the cycle $k{\cdot}M[\gamma^{j-1}\rangle k{\cdot}M$.

By Keller's theorem (applied in $k{\cdot}M$ with $k{\cdot}M[\tau\rangle$ and $k{\cdot}M[\gamma^{j-1}\rangle$), $k{\cdot}M[\sigma\rangle$ with $\sigma = \tau \overset{\bullet}{-} \gamma^{j-1}$, and since $\Psi(\tau)(t) \neq 0 \Rightarrow X(t) = \Psi(\gamma)(t) = 1$ (by Theorem 3), $\Psi(\sigma)(t) = \mathbf{max}\{0, \Psi(\tau)(t) - (j-1)\}$ for all t. Now $\Psi(\tau)(t) = j$ for some t (since $p > 0$), hence σ differs from the empty sequence. Let $\sigma = t'\sigma'$. Then $k{\cdot}M[t'\rangle$, hence $M[t'\rangle$ by plainness. Moreover, $\Psi(\tau)(t') \geq 1 + (j-1)$, hence $\Psi(\tau)(t') = j$, which establishes our claim.

Let t_1 $(= t')$ be some transition enabled at M and occurring j times in τ. Let $M[t_1\rangle M'$ in N, then $(L + j{\cdot}M)[t_1^j\rangle(L + j{\cdot}M')$ in $k{\cdot}N$. As also $(L + j{\cdot}M)[\tau\rangle$, by Keller's theorem, $(L + j{\cdot}M')[\tau'\rangle$ with $\tau' = \tau \dot{-} t_1^j$. Thus, $\Psi(\tau')(t) = \Psi(\tau)(t)$ for $t \neq t_1$ and $\Psi(\tau')(t_1) = 0$, and if we let $\{t \,|\, \Psi(\tau)(t) = j\} = \{t_1, \ldots, t_p\}$, then $\{t \,|\, \Psi(\tau')(t) = j\} = \{t_2, \ldots, t_p\}$.

In order to get a full proof of the lemma by the induction on p, it suffices to construct χ' such that $(L + j{\cdot}M')[\chi'\rangle\, k{\cdot}M'$ and χ' and τ' are transition disjoint. We show that both conditions are fulfilled if we set $\chi' = \chi\, t_1^{k-j}$. Transition disjointness is clear since t_1 does not occur in $\tau' = \tau \dot{-}\, t_1^j$ and τ and χ are transition disjoint. Now $(L + j{\cdot}M)[\chi\rangle\, k{\cdot}M$, $(L + j{\cdot}M)[t_1^j\rangle\, (L + j{\cdot}M')$, and t_1 does not occur in χ since it occurs in τ. By Keller's theorem and the fundamental equation, $(L + j{\cdot}M')[\chi\rangle\, (L + j{\cdot}M)' + (k{\cdot}M - (L + j{\cdot}M)) = (k - j){\cdot}M + j{\cdot}M'$. As $M[t_1\rangle\, M'$, $(k - j){\cdot}M + j{\cdot}M'[T_1^{k-j}\rangle\, k{\cdot}M'$. Thus, the proof is complete. □

Iterating the application of Lemma 5 after one application of Lemma 4, is the principle of the proof of our first separability result.

Theorem 4. WEAK SEPARABILITY

Let N be plain. Let $k \geq 2$ and let $k{\cdot}N$, with initial marking $k{\cdot}M_0$, be bounded, reversible, and persistent. If $k{\cdot}N$ has only one minimal realizable T-invariant, then $(N, k{\cdot}M_0)$ is weakly separable.

Note that both reversibility and plainness are important for Theorem 4 to hold. Figure 6 shows on the left-hand side a plain, bounded, non-reversible, persistent Petri net with a 2-marking $2{\cdot}M_0$ such that the firing sequence $2{\cdot}M_0[bcac\rangle$ is not weakly 2-separable. The right-hand side of Figure 6 displays a non-plain, bounded, reversible, persistent 2-net with a 2-marking $2{\cdot}M_0$ in which the firing sequence $2{\cdot}M_0[a\rangle$ cannot be separated for obvious reasons.

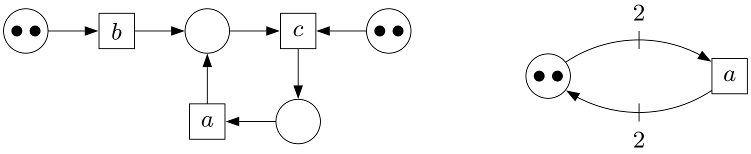


Fig. 6. Two non-separable nets: not reversible (l.h.s.) and not plain (r.h.s.)

Proof: Let $k{\cdot}M_0[\sigma\rangle$ be given. We show that applying once Lemma 4 and $k{-}1$ times Lemma 5 produces a decomposition of $k{\cdot}M_0[\sigma\rangle$ into k sequences $M_0[\sigma_j\rangle$ $(j = 1, \ldots, k)$ such that $\Psi(\sigma) = \sum_{j=1}^{k} \Psi(\sigma_j)$. This decomposition is depicted in Table 1, where the j-th horizontal line shows the "process" $M_0[\sigma_j\rangle$. To give a rough idea, the application of Lemma 4 produces the part of the tableau between the first two columns $M_0{+}\ldots{+}M_0$ and $M_1{+}\ldots{+}M_1$. The l-th application of Lemma 5 $(1 \leq l \leq k - 1)$ produces the part of the tableau between the columns $M_l{+}\ldots{+}M_l$ and $M_{l+1}{+}\ldots{+}M_{l+1}$.

$$\begin{tikzcd}
\sigma_1\colon & M_0 \arrow[r, "t_{1,1}\cdots t_{1,n_1}"] & M_1 & & & & \\
 & + & + & & & & \\
\sigma_2\colon & M_0 \arrow[r, "t_{1,1}\cdots t_{1,n_1}"] & M_1 \arrow[r, "t_{2,1}\cdots t_{2,n_2}"] & M_2 & & & \\
 & + & + & + & & & \\
\sigma_3\colon & M_0 \arrow[r, "t_{1,1}\cdots t_{1,n_1}"] & M_1 \arrow[r, "t_{2,1}\cdots t_{2,n_2}"] & M_2 \arrow[r, "t_{3,1}\cdots t_{3,n_3}"] & M_3 & & \\
 & + & + & + & + & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \\
 & + & + & + & + & & \\
\sigma_k\colon & M_0 \arrow[r, "t_{1,1}\cdots t_{1,n_1}", "\underbrace{\hphantom{xxxx}}_{\tau_1}"'] & M_1 \arrow[r, "t_{2,1}\cdots t_{2,n_2}", "\underbrace{\hphantom{xxxx}}_{\tau_2}"'] & M_2 \arrow[r, "t_{3,1}\cdots t_{3,n_3}", "\underbrace{\hphantom{xxxx}}_{\tau_3}"'] & M_3 & \cdots \arrow[r, "t_{k,1}\cdots t_{k,n_k}", "\underbrace{\hphantom{xxxx}}_{\tau_k}"'] & M_k \\
h_t\colon & & h_t = k{-}1 & h_t = k{-}2 & & h_t = 1 &
\end{tikzcd}$$

Table 1. A tableau explaining the weak separation of σ

We describe now more precisely the successive phases of the decomposition.

Step 1: This step consists of applying Lemma 4 to $k{\cdot}M_0[\sigma\rangle$.

The lemma yields $k{\cdot}M_0[t_1^k \dots t_n^k\rangle k{\cdot}M_1[\sigma'\rangle$, with $\Psi(\sigma') \leq k{-}1$ and $\sigma \equiv_{k\cdot M_0} t_1^k \dots t_n^k \sigma'$. Putting $n_1{=}n$ and $t_{1,1}{=}t_1$, $t_{1,2}{=}t_2$, ..., $t_{1,n_1}{=}t_n$, one obtains the part of the tableau to the left of $M_1{+}\dots{+}M_1$. (**End of Step 1.**)

Step 2: This step consists of $k{-}1$ successive applications of Lemma 5 (substeps 2.l for $l = 1, \dots, k{-}1$).

For every transition t, let $h_t = \Psi(\sigma)(t) \,\mathbf{mod}\, k$, thus h_t is the remainder left after dividing $\Psi(\sigma)(t)$ by k. For each transition t occurring in σ' (produced in Step 1), if $h_t = k - l$, then the $k - l$ remaining occurrences of t in σ' are grouped and shifted to the left in the l-th application (substep 2.l) of Lemma 5, yielding $k - l$ subprocesses starting at M_l and stopping at M_{l+1}.

More precisely, in substep 2.1, Lemma 5 is applied to

$$\begin{array}{rl}
 & (L + j{\cdot}M)[\tau\rangle \text{ and } (L + j{\cdot}M)[\chi\rangle k{\cdot}M \\
\text{with} & j = k - 1, \;\; L = M_1, \;\; M = M_1, \\
 & \tau = \sigma' \; = \; \sigma \stackrel{\bullet}{-} (t_{1,1}^k \dots t_{1,n_1}^k), \\
\text{and} & \chi = \varepsilon.
\end{array}$$

The lemma yields $M_1[t_1 \dots t_p\rangle$ where $t_1, \dots, t_p$ is an enumeration of the set $\{t \mid \Psi(\sigma')(t) = k - 1\}$, i.e. of the set $\{t \mid h_t = k - 1\}$. Putting $n_2{=}p$ and $t_{2,1}{=}t_1$, ..., $t_{2,n_2}{=}t_p$, one obtains a decomposition

$$(k-1){\cdot}(M_1[t_{2,1} \dots t_{2,n_2}\rangle M_2) \;\text{ of }\; ((k-1){\cdot}M_1)\,[t_{2,1}^{k-1} \dots t_{2,n_2}^{k-1}\rangle\,((k-1){\cdot}M_2).$$

In substep 2.l for $l = 2 \ldots, k{-}1$, Lemma 5 is similarly applied to

$$\begin{array}{rl} & (L + j{\cdot}M)[\tau\rangle \;\text{ and }\; (L + j{\cdot}M)[\chi\rangle k{\cdot}M \\ \text{with} & j = k - l,\;\; L = M_1 + \ldots + M_l,\;\; M = M_l, \\ & \tau = \sigma \overset{\bullet}{-} (t^{k}_{1,1} \ldots t^{k}_{1,n_1} t^{k-1}_{2,1} \ldots t^{k-1}_{2,n_2} \ldots t^{k-l+1}_{l,1} \ldots t^{k-l+1}_{l,n_l}), \\ \text{and} & \chi = t_{2,1} \ldots t_{2,n_2} \ldots t^{2}_{3,1} \ldots t^{2}_{3,n_3} \ldots t^{l-1}_{l,1} \ldots t^{l-1}_{l,n_l}. \end{array}$$

(**End of Step 2.**)

Finally, the sequences $\sigma_1, \ldots, \sigma_k$ are defined in accordance with the lines 1 to k of Table 1. More precisely, for $1 \leq l \leq k$ let

$$\sigma_l \;=\; (t_{1,1} \ldots t_{1,n_1})\,(t_{2,1} \ldots t_{2,n_2}) \;\ldots\; (t_{l,1} \ldots t_{l,n_l}) \;.$$

Then clearly, $M_0[\sigma_l\rangle M_l$ for $l = 1, \ldots, k$ and $\Psi(\sigma) = \Psi(\sigma_1) + \ldots + \Psi(\sigma_k)$ by construction. Thus, the $\sigma_1, \ldots, \sigma_k$ provide the weak separation of σ that was claimed to exist. □

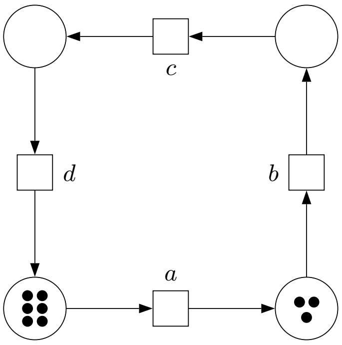
It may be observed that for $i \neq i'$ and for any transition t, $\Psi(\sigma_i)(t)$ and $\Psi(\sigma_{i'})(t)$ differ at most by 1. Thus, the decomposition of firing sequences given by Theorem 4 is, in fact, a balanced decomposition. More precisely, depending on the value of h_t, any transition t may occur in at most one column of Table 1 after the column defined by markings $M_1 + \ldots + M_1$, and at most once in every line in this column. So, if h_t is zero, then t does not occur at all on the right of the column $M_1 + \ldots + M_1$, and if h_t differs from zero, then it occurs once in each line in the column indicated by h_t and in no other column (except possibly between $M_0 + \ldots + M_0$ and $M_1 + \ldots + M_1$).

A simple example with $k = 3$ is shown in Figure 7. Consider $t = a$. Since a occurs seven times in σ and $k = 3$, we have $h_a = 1$. Hence a occurs (once) in the column determined by $h_t = 1$. The remaining six occurrences of a are spread evenly in the lines between $M_0 + M_0 + M_0$ and $M_1 + M_1 + M_1$. Similarly, b occurs five times in σ. Thus $h_b = 2$, and b occurs (twice, but only once per line) in the column specified by $h_t = 2$.

This example also shows that the weak separation which exists by Theorem 4 is not necessarily a strong separation, since $\sigma \notin (\sigma_1 \sqcup\!\sqcup\, \sigma_2 \sqcup\!\sqcup\, \sigma_3)$.

7 Strong separability

Weak separability will now be used in an essential way in order to prove the stronger version, viz. strong separability. In the remainder of this section, we refer to the decomposition constructed in the proof of Theorem 4 and shown in Table 1, relative to a firing sequence σ. In particular, M_0, M_1, M_2, ..., M_k refer to the markings shown in this table. To avoid excessive indexing, let $\tau_i = t_{i,1} \ldots t_{i,n_i}$ for $i = 1, \ldots, k$. Thus $M_{i-1}[\tau_i\rangle M_i$, and $M_0[\sigma_i\rangle M_i$ rewrites as $M_0[\tau_1\rangle M_1[\tau_2\rangle M_2 \ldots M_{i-1}[\tau_i\rangle M_i$.



σ to be separated: $3{\cdot}M_0[abbbbcaaacaadbac\rangle$							
σ_1 :	M_0	$[aabc\rangle$	M_1				
σ_2 :	M_0	$[aabc\rangle$	M_1	$[b\rangle$	M_2		
σ_3 :	M_0	$[aabc\rangle$	M_1	$[b\rangle$	M_2	$[da\rangle$	M_3
h_t :				2		1	

Fig. 7. A 3-net (l.h.s.) and a firing sequence together with a weak separation (r.h.s.)

Note that any two transitions $t_{i,j}$ and $t_{i',j'}$ with $i, i' \geq 2$ and $i \neq i'$ or $j \neq j'$ are different. In particular, also, $\Psi(\tau_i) \leq 1$ for every τ_i.

If some k-marking enables a transition t, then in view of the plainness assumption, one k'th of this marking also enables t. We have used this argument several times. The next two lemmata extend this property first from transitions to cycles and next from k-markings to arbitrary reachable markings.

Lemma 6. Individual enabling part 1

Let N be plain. Let $k \geq 2$ and let $k{\cdot}N$ be the multiple of N with initial marking $k{\cdot}M_0$. Suppose that $k{\cdot}N$ is bounded, reversible and persistent, and that $X \leq 1$ is the unique minimal T-invariant realized in this net.
If $k{\cdot}M_0[\alpha\rangle k{\cdot}M_0$ is a cycle in $k{\cdot}N$ and $\Psi(\alpha) \leq 1$, then also $M_0[\alpha\rangle M_0$ in N.

Proof: Executing k times the cycle α in $k{\cdot}N$ yields $k{\cdot}M_0[\alpha^k\rangle k{\cdot}M_0$. Let t_1 be the first transition of α^k and hence also of α. Since $k{\cdot}M_0[t_1\rangle$, also $M_0[t_1\rangle$, and then also $k{\cdot}M_0[t_1^k\rangle$. By Keller's theorem, $k{\cdot}M_0[t_1^k(\alpha^k \,\dot{-}\, t_1^k)\rangle$. As t_1 occurs exactly k times in α^k (because $\Psi(\alpha) \leq 1$), this firing sequence is of the form:

$$k{\cdot}M_0[t_1^k\rangle k{\cdot}M_1[\alpha^k \,\dot{-}\, t_1^k\rangle k{\cdot}M_0.$$

As $\Psi(\alpha) \leq 1$, the first transition of $\alpha^k \,\dot{-}\, t_1^k$ is also the second transition of α. Continuing as above, we get a sequence $t_1 \dots t_n$ of transitions with $k{\cdot}M_0[t_1^k \dots t_n^k\rangle k{\cdot}M_0$, and then also $M_0[t_1 \dots t_n\rangle M_0$ in N, and by construction, $t_1 \dots t_n = \alpha$. ☐

Lemma 7. Individual enabling part 2

Under the same assumptions as in Lemma 6, let $k{\cdot}M_0[\sigma\rangle L$ be any firing sequence and let

$$M_0[\sigma_1\rangle M_1\ ,\ \ \dots\ \ ,\ M_0[\sigma_i\rangle M_i\ ,\ \ \dots\ \ ,\ M_0[\sigma_k\rangle M_k$$

be the weak separation of this firing sequence given by Table 1 (i.e., $L = M_1 + \ldots + M_k$ and $\sigma_i = \tau_1 \dots \tau_i$ with $\tau_i = t_{i,1} \dots t_{i,n_i}$). If $L[t\rangle$ for some transition t, then $M_h[t\rangle$ for some index $1 \leq h \leq k$. Moreover, if $t \neq t_{i,l}$ for all $i \geq 2$ and $1 \leq l \leq n_i$ then $h = k$, else $t \in \{t_{h+1,1}, \dots, t_{h+1,n_{h+1}}\}$.

Proof: Suppose that $L[t\rangle$ with $t \neq t_{i,j}$ for all $i \geq 2$ and for all j. Let $\tau = \tau_2(\tau_3)^2 \dots (\tau_k)^{k-1}$, then by construction, $L[\tau\rangle k{\cdot}M_k$ (intuitively, τ is what is missing in the North-Eastern corner of Table 1). As t does not occur in τ, it follows by persistency that $k{\cdot}M_k[t\rangle$, hence $M_k[t\rangle$ by plainness.

Suppose that $L[t\rangle$ with $t = t_{i,j}$ and $i \geq 2$. Then t occurs in the sequence τ_i and in no other $\tau_{i'}$ with $i' \neq i$. As all transitions $t_{i',j'}$ are different provided that $i' \geq 2$, $\Psi(\tau_2\tau_3 \dots \tau_k) \leq 1$. As $(N, k{\cdot}M_0)$ is pbrp, $(N, k{\cdot}M_1)$ is pbrp. By Theorem 1, both nets have the same (unique) minimal realizable T-invariant X, and X is realized at $k{\cdot}M_1$. By Lemma 6, the T-invariant $X \leq 1$ (of $k{\cdot}N$) is realized in M_1 (in N). By Theorem 3, $\Psi(\tau_2\tau_3 \dots \tau_k) \leq X$. By Keller's theorem, there must exist a sequence α such that $M_k[\alpha\rangle M_1$ and $\Psi(\tau_2\tau_3 \dots \tau_k\alpha) = X \leq 1$. Since t occurs in τ_i and hence also in $\tau_2\tau_3 \dots \tau_k$, it does not occur in α.

We claim now that

$$\begin{array}{lll} L = M_1 + \ldots + M_k & [\tau'\rangle & ((i{-}1){\cdot}M_{i-1} + M_i + \ldots + M_k) \\ & [\tau''\rangle & ((i{-}1){\cdot}M_{i-1} + (k{-}i{+}1){\cdot}M_k) \\ & [\tau'''\rangle & ((i{-}1){\cdot}M_{i-1} + (k{-}i{+}1){\cdot}M_1) \\ & [\tau''''\rangle & k{\cdot}M_{i-1} \end{array}$$

with $\tau' = \tau_2(\tau_3)^2 \dots (\tau_{i-1})^{i-2}$, $\tau'' = \tau_{i+1}(\tau_{i+2})^2 \dots (\tau_k)^{k-i}$, $\tau''' = \alpha^{k-i+1}$, and $\tau'''' = (\tau_2 \dots \tau_{i-1})^{k-i+1}$.

This may be seen by inspecting Table 1. The sequence τ' produces $i-1$ copies of M_{i-1} out of $M_1 + M_2 + \ldots + M_{i-1}$ in the first $i-1$ lines of the table. Then $\tau'' = \tau_{i+1}(\tau_{i+2})^2 \dots (\tau_k)^{k-i}$ produces $k-i+1$ copies of M_k on lines i to k of the table. After this, $k-i+1$ copies of M_1 are produced by $\tau''' = \alpha^{k-i+1}$ on lines i to k. Finally, $k-i+1$ copies of M_{i-1} are produced by τ'''' on the same lines.

Now $L[t\rangle$ and if we let $\tau = \tau'\tau''\tau'''\tau''''$, then $L[\tau\rangle k{\cdot}M_{i-1}$ and t does not occur in τ since it appears neither in α nor in any τ_j for $j \neq i$. By persistency, $k{\cdot}M_{i-1}[t\rangle$. By plainness, $M_{i-1}[t\rangle$. ☐

Theorem 5. Strong separability

Under the same assumptions as in Lemma 6, every firing sequence $k{\cdot}M_0[\sigma\rangle L$ has a strong separation.

Proof: We will prove by induction on σ that, if $k{\cdot}M_0[\sigma\rangle L$ has the weak separation $M_0[\sigma_1\rangle M_1$, …, $M_0[\sigma_k\rangle M_k$, where $\sigma_i = \tau_1 \dots \tau_i$ and $\tau_i = t_{i,1} \dots t_{i,n_i}$ as indicated in Table 1 (cf. proof of Theorem 4), then $k{\cdot}M_0[\sigma\rangle L$ belongs to the shuffle of k firing sequences $M_0[\zeta_1\rangle M_1, \dots M_0[\zeta_k\rangle M_k$, such that $\Psi(\sigma_i) = \Psi(\zeta_i)$ for all i.[3]

For σ with length 0, there is nothing to prove. Now let $\sigma' = \sigma t$ and suppose that the firing sequence $k{\cdot}M_0[\sigma\rangle L$ matches both the weak separation $M_0[\sigma_1\rangle M_1$, …, $M_0[\sigma_k\rangle M_k$ (given by Theorem 4) and the strong separation $M_0[\zeta_1\rangle M_1$, …, $M_0[\zeta_k\rangle M_k$ (given by induction), such that $\Psi(\sigma_i) = \Psi(\zeta_i)$ for all i.

---

[3] Thus, as was noted after the proof of Theorem 4, the resulting separation is balanced.

Note that $\Psi(\tau_1)$ is the integer part of $\frac{1}{k} \cdot \Psi(\sigma)$ and for $l > 1$, $\Psi(\tau_l)(t) = 1$ if and only if l is the rest of the integer division of $\Psi(\sigma)(t)$ by k.

The properties under consideration hold clearly for σ with length 0. Assuming they hold for σ, we show now that they hold for $\sigma' = \sigma t$. By Theorem 4 and its proof, the firing sequence $k{\cdot}M_0[\sigma'\rangle$ has a similar weak decomposition $M_0[\tau'_1\rangle M'_1$, …, $M_0[\tau'_1\rangle M'_1[\tau'_2\rangle M'_2 \dots [\tau'_k\rangle M'_k$, where $\Psi(\tau'_1 \dots \tau'_l) = \Psi(\tau_1 \dots \tau_l)$ for *all* $l \geq 1$ *except one*, for which $\Psi(\tau'_1 \dots \tau'_l) = \Psi(\tau_1 \dots \tau_l) + \Psi(t)$. Fix this index l. By persistency of N (Lemma 1), and by Keller's theorem, applied to $M_0[\tau_1 \dots \tau_l\rangle$ and $M_0[\tau'_1 \dots \tau'_l\rangle$, necessarily $M_0[\tau_1 \dots \tau_l t\rangle$. Therefore, $M_l[t\rangle$, showing that one may obtain a strong decomposition of $k{\cdot}M_0[\sigma t\rangle$, i.e. of $k{\cdot}M_0[\sigma'\rangle$, by setting $\zeta'_i = \zeta_i$ for $i \neq l$ and $\zeta'_l = \zeta_l t$. As ζ'_j is a permutation of $\sigma'_j = \tau'_1 \dots \tau'_j$ for all j, the proof of the theorem follows by the induction on σ. □

Remark 1.

When σ is extended to σt, one and exactly one of the sequences ζ_j found in the strong decomposition of $k{\cdot}M_0[\sigma\rangle$ is changed to a longer sequence $\zeta_j\, t$ in the strong decomposition of $k{\cdot}M_0[\sigma\, t\rangle$. Thus, if $k{\cdot}M_0[\sigma\, t\rangle$ and $k{\cdot}M_0[\sigma\, t'\rangle$, then either a common sequence ζ_j is extended both to $\zeta_j\, t$ and to $\zeta_j\, t'$ in the strong decompositions of σt and $\sigma t'$, respectively, or two distinct sequences ζ_j and $\zeta_{j'}$ are extended separately to $\zeta_j\, t$ and to $\zeta_j\, t'$ in the strong decompositions of σt and $\sigma t'$, respectively. As a consequence, if $k{\cdot}N$ is strongly separable and N is persistent, then $k{\cdot}N$ is persistent. This property will be used in section 9. □

The reader may recall from Figure 7 that Theorem 4 does not necessarily yield the sequences ζ_i whose shuffle realizes σ. On the other hand, the sequences ζ_i yield a weak decomposition of σ, but this weak decomposition does not necessarily enjoy the uniformity and orthogonality properties shown by Table 1.

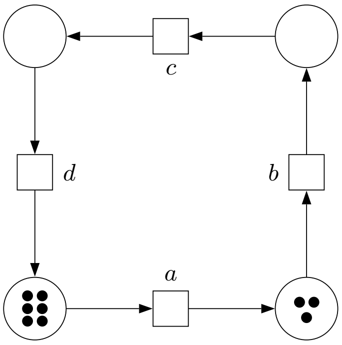
As an example, consider Figure 8. It shows one step in the proof of Theorem 5, constructing a new strong separation ζ'_j and then also a new weak separation σ'_j (of σ') from the given separations σ_j and ζ_j (of σ). Note that the initial weak separation is also a strong one, while the new weak separation is no longer strong.

8 The general case

With the help of Theorem 2, we can now extend the strong separability result to pbrp-nets with several incomparable realizable T-invariants.

Theorem 6. STRONG SEPARABILITY (FOR GENERAL PBRP-NETS)

Let N be plain. Let $k \geq 2$ and let $k{\cdot}N$, with initial marking $k{\cdot}M_0$, be bounded, reversible, and persistent. Then $(N, k{\cdot}M_0)$ is strongly separable.



Separate $3\cdot M_0\ [\ \overbrace{\underbrace{ab}_{\sigma}\ b}^{\sigma'}\ \rangle$ with $\sigma = ab$ and $\sigma' = abb$

$$\boxed{\begin{array}{l} M_0[\varepsilon\rangle M_1 \\ M_0[\varepsilon\rangle M_1[\varepsilon\rangle M_2 \\ M_0[\varepsilon\rangle M_1[\varepsilon\rangle M_2[ab\rangle M_3 \end{array}} \rightsquigarrow \boxed{\begin{array}{l} M'_0[\varepsilon\rangle M'_1 \\ M'_0[\varepsilon\rangle M'_1[b\rangle M'_2 \\ M'_0[\varepsilon\rangle M'_1[b\rangle M'_2[a\rangle M'_3 \end{array}}$$

$\tau_1 = \varepsilon,\ \tau_2 = \varepsilon,\ \tau_3 = ab \qquad \tau'_1 = \varepsilon,\ \tau'_2 = b,\ \tau'_3 = a$

$\sigma_1 = \varepsilon,\ \sigma_2 = \varepsilon,\ \sigma_3 = ab \qquad \sigma'_1 = \varepsilon,\ \sigma'_2 = b,\ \sigma'_3 = ba$

$\zeta_1 = \varepsilon,\ \zeta_2 = \varepsilon,\ \zeta_3 = ab \qquad \zeta'_1 = \varepsilon,\ \zeta'_2 = b,\ \zeta'_3 = ab$

Fig. 8. Illustration of the proof of Theorem 5

Proof: Let $\{X_1, \ldots, X_n\}$ be the set of mutually transition-disjoint T-invariants of $k\cdot N$ given by Theorem 1. According to Theorem 2, there are n bounded, reversible and persistent nets $k\cdot N_1, \ldots, k\cdot N_n$ such that the reachability graph of $k\cdot N$ is isomorphic to the reachability graph of the disjoint sum of the nets $k\cdot N_1, \ldots, k\cdot N_n$. Moreover, these nets $k\cdot N_i$ are given by $k\cdot N_i = (P, T_i, F_i, k\cdot M_0)$ where $T_i = \{t \in T \mid X_i(t) \neq 0\}$ and F_i is the induced restriction of F on $(P \times T_i) \cup (T_i \times P)$. Thus all nets $k\cdot N_i$ have similar initial markings $k\cdot M_0$ (but for separate copies of the set of places P), and $\{T_1, \ldots, T_n\}$ is a partition of the set of transitions T.

Let $k\cdot M_0[\sigma\rangle$ be a given firing sequence of $k\cdot N$. For $i = 1, \ldots, n$, let σ_i be the projection of σ on T_i^*. Thus, $\sigma \in \mathop{\sqcup\!\sqcup}_{i=1}^{n} \sigma_i$, and in particular, $\Psi(\sigma) = \sum_{i=1}^{n} \Psi(\sigma_i)$. In view of the isomorphism of reachability graphs described above, there must exist corresponding firing sequences $k\cdot M_0[\sigma_i\rangle$ of nets $k\cdot N_i$.

Consider some fixed net $k\cdot N_i$. As $k\cdot N_i$ is the induced (subnet) restriction of $k\cdot N$ on P and T_i, and both nets have the same initial marking, the reachability graph of $k\cdot N_i$ embeds into the reachability graph of $k\cdot N$, and it is isomorphic to the reachable restriction of this labelled graph induced on the subset of labels T_i. Therefore, the T-invariant X_i which is realized at $k\cdot M_0$ in $k\cdot N$ is also realized at $k\cdot M_0$ in $k\cdot N_i$. Moreover, it is the only minimal realizable T-invariant of $k\cdot N_i$. Indeed, any T-invariant which is realized in $k\cdot N_i$ is also realized in $k\cdot N$ due to the embedding of reachability graphs, and we know from Theorem 1 that X_i is the only minimal realizable T_i-invariant of $k\cdot N$. Now, $k\cdot N_i$ is bounded, reversible and persistent, and it is moreover a k-net since it has the initial marking $k\cdot M_0$. By Theorem 5, $k\cdot N_i$ is strongly separable, hence there exist k firing sequences $M_0[\sigma_{i,1}\rangle, \ldots, M_0[\sigma_{i,k}\rangle$ of the net $N_i = (P, T_i, F_i, M_0)$ such that $\sigma_i \in \mathop{\sqcup\!\sqcup}_{j=1}^{k} \sigma_{i,j}$ for each i from 1 to n. Thus, $\sigma \in \mathop{\sqcup\!\sqcup}_{i=1}^{n} \mathop{\sqcup\!\sqcup}_{j=1}^{k} \sigma_{i,j}$. By associativity and commutativity of the shuffle product, $\sigma \in \mathop{\sqcup\!\sqcup}_{j=1}^{k} \mathop{\sqcup\!\sqcup}_{i=1}^{n} \sigma_{i,j}$, hence one may choose specific words $\tau_j \in \mathop{\sqcup\!\sqcup}_{i=1}^{n} \sigma_{i,j}$ $(j = 1, \ldots, k)$ such that $\sigma \in \mathop{\sqcup\!\sqcup}_{j=1}^{k} \tau_j$. In order to complete the proof of the theorem, it suffices to show that $M_0[\tau_j\rangle$ in $N =$

(P, T, F, M_0) for each j from 1 to k. Fix j with $1 \leq j \leq k$. As $k{\cdot}N$ is bounded, reversible and persistent, by Lemmata 1 and 2, N enjoys similar properties. For $i = 1, \ldots, n$, as $k{\cdot}N_i$ is bounded, reversible and persistent, by Lemmata 1 and 2, N_i enjoys similar properties. Therefore, by Theorem 2, the reachability graph of N (with initial marking M_0) is isomorphic to the reachability graph of the disjoint sum of nets $N_1 + \ldots + N_n$ (each of them also with the initial marking M_0). In view of this isomorphism, as τ_j projects (on T_i^*) to $\sigma_{i,j}$ and $M_0[\sigma_{i,j}\rangle$ in (N_i, M_0) for all i with $1 \leq i \leq n$, necessarily, $M_0[\tau_j\rangle$ in N. □

As was noted after Theorem 4, the strong decomposition of firing sequences given by Theorem 6 is balanced.

9 On $(k-1){\cdot}N$

In this section, we prove that if $k{\cdot}N$ is a pbrp-net, then so is $(k-1){\cdot}N$, and then, of course, also $(k-2){\cdot}N$ and so on, down to $1{\cdot}N$ (for $1{\cdot}N$, the pbrp-property was already known from section 3). We consider first the case where $k{\cdot}N$ has a unique minimal T-invariant. We show that in this case $(k-1){\cdot}N$ is weakly separable. On this basis, we establish that $(k-1){\cdot}N$ enjoys also reversibility, strong separability, and the other pbrp-properties. We finally extend all results to general pbrp-nets with several minimal realizable T-invariants. For this purpose, we show the strong separability of $(k-1){\cdot}N$ in the general case. All other properties follow indeed from strong separability.

Theorem 7. $(k-1){\cdot}N$ IS WEAKLY SEPARABLE

Let N be plain. Let $k \geq 2$ and let $k{\cdot}N$ be the multiple of N with initial marking $k{\cdot}M_0$. Suppose that $k{\cdot}N$ is bounded, reversible and persistent, and that it has a unique minimal realizable T-invariant. Then $(k-1){\cdot}N$ is weakly separable.

Corollary 1. $(k-1){\cdot}N$ IS REVERSIBLE

Under the same assumptions, $(k-1){\cdot}N$ is reversible. □

The corollary can be proved as follows. Given a firable sequence $(k-1){\cdot}M_0[\sigma\rangle$ in $(k-1){\cdot}N$, consider a weak separation of σ into $k-1$ sequences, each of which is enabled at M_0. By lemma 2, N is reversible, hence one can extend each of these firing sequences so that it reaches on its own M_0, thus the $(k-1)$ sequences reach jointly $(k-1){\cdot}M_0$.

Proof: Let $(k-1){\cdot}M_0[\sigma\rangle$. Clearly, $k{\cdot}M_0[\sigma\rangle$. By Theorem 4, $k{\cdot}M_0[\sigma\rangle$ splits to k firing sequences $M_0[\sigma_i\rangle M_i$ $(i = 1, \ldots, k)$, each of which is defined as a concatenation $M_0[\tau_1\rangle M_1[\tau_2\rangle M_2 \ldots\ M_{i-1}[\tau_i\rangle M_i$ where by construction (see the proof of Theorem 4 and Table 1), $\Psi(\tau_1)$ is the largest integer vector x such that $k \cdot x \leq \Psi(\sigma)$. Note that this vector is unique and can be expressed explicitly as

$$x(t) \;=\; \Psi(\sigma)(t) \;\div\; k\,, \quad \text{for all } t.$$

As $M_0[\tau_1\rangle M_1$ and $(k-1)\cdot M_0[\sigma\rangle$, clearly, $k\cdot M_0[\tau_1\sigma\rangle$. By Theorem 4 applied to $k\cdot N$, $k\cdot M_0[\tau_1\sigma\rangle$ splits to k firing sequences $M_0[\sigma'_i\rangle M'_i$ $(i=1,\ldots,k)$, each of which is defined as a concatenation $M_0[\tau'_1\rangle M'_1[\tau'_2\rangle M'_2\ldots M'_{i-1}[\tau'_i\rangle M'_i$ where $\Psi(\tau'_1)$ is the largest integer vector x such that $k\cdot x\leq\Psi(\tau_1\sigma)$, hence necessarily

$$\Psi(\tau_1) \;\leq\; \Psi(\tau'_1) \;\leq\; \frac{1}{k}\cdot\Psi(\tau_1\sigma) \;\leq\; \frac{1}{k}\cdot(1+\frac{1}{k})\cdot\Psi(\sigma).$$

The first inequality is due to the maximality of τ'_1, since by $k\cdot\Psi(\tau_1)\leq\Psi(\sigma)\leq\Psi(\tau_1\sigma)$, $\Psi(\tau_1)$ is amongst the vectors x satisfying $k\cdot x\leq\Psi(\tau_1\sigma)$. The second inequality stems from $k\cdot\Psi(\tau'_1)\leq\Psi(\tau_1\sigma)$, and the third inequality comes from $\Psi(\tau_1\sigma)=\Psi(\tau_1)+\Psi(\sigma)\leq\frac{1}{k}\cdot\Psi(\sigma)+\Psi(\sigma)$.

As $M_0[\tau'_1\rangle M'_1$ and $(k-1)\cdot M_0[\sigma\rangle$, clearly, $k\cdot M_0[\tau'_1\sigma\rangle$. By Theorem 4, $k\cdot M_0[\tau'_1\sigma\rangle$ splits to k firing sequences $M_0[\sigma''_i\rangle M''_i$ $(i=1,\ldots,k)$, each of which is defined as a concatenation $M_0[\tau''_1\rangle M''_1[\tau''_2\rangle M''_2\ldots M''_{i-1}[\tau''_i\rangle M''_i$ where $\Psi(\tau''_1)$ is the largest integer vector x such that $k\cdot x\leq\Psi(\tau'_1\sigma)$, hence necessarily

$$\Psi(\tau'_1) \;\leq\; \Psi(\tau''_1) \;\leq\; \frac{1}{k}\cdot\Psi(\tau'_1\sigma) \;\leq\; \frac{1}{k}\cdot(1+\frac{1}{k}\cdot(1+\frac{1}{k}))\cdot\Psi(\sigma).$$

Continuing in this way, one builds a sequence $(\tau_1^{[n]})_{n\geq 0}$ of firing sequences (along with sequences $(\tau_2^{[n]})_{n\geq 0}$, $(\tau_3^{[n]})_{n\geq 0}$, $\ldots$), yielding an increasing sequence of Parikh vectors $(\Psi(\tau_1^{[n]}))_{n\geq 0}$, bounded from above by

$$\frac{1}{k}\cdot(1+\frac{1}{k}\cdot(1+\frac{1}{k}\cdot(1+\frac{1}{k}\cdot(\cdots))))\cdot\Psi(\sigma),$$

which is equal to $\frac{1}{k-1}\cdot\Psi(\sigma)$. Sooner or later, $\Psi(\tau_1^{[n]})=\Psi(\tau_1^{[n+1]})$. At this stage,

$$\begin{array}{lll}
\sigma_2 & = & \tau_1^{[n+1]}\,\tau_2^{[n+1]}\\
\sigma_3 & = & \tau_1^{[n+1]}\,\tau_2^{[n+1]}\,\tau_3^{[n+1]}\\
\vdots & & \vdots\\
\sigma_k & = & \tau_1^{[n+1]}\,\tau_2^{[n+1]}\,\tau_3^{[n+1]}\;\;\cdots\;\;\tau_k^{[n+1]}
\end{array}$$

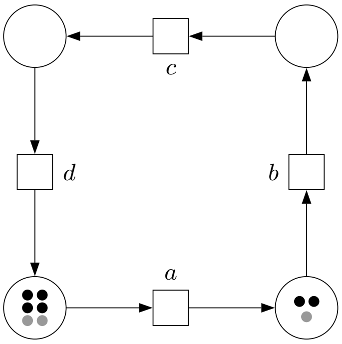
is a weak separation of $(k-1)\cdot M_0[\sigma\rangle$. To see this, note first that every σ_i (for $2\leq i\leq k$) is firable from M_0, since it belongs by construction to a separation of $k\cdot M_0[\tau_1^{[n]}\sigma\rangle$. Note, secondly, that

$$\Psi(\tau_1^{[n+1]})+\Psi(\sigma_2)+\ldots+\Psi(\sigma_k) \;=\; \Psi(\tau_1^{[n]}\sigma) \;=\; \Psi(\tau_1^{[n]})+\Psi(\sigma),$$

for the same reason and because $\tau_1^{[n+1]}$ is the first sequence in the considered separation of $k\cdot M_0[\tau_1^{[n]}\sigma\rangle$. As $\Psi(\tau_1^{[n]})=\Psi(\tau_1^{[n+1]})$, it follows that $\Psi(\sigma_2)+\ldots+\Psi(\sigma_k)=\Psi(\sigma)$. Thus, $\{\sigma_2,\ldots,\sigma_k\}$ defines indeed a weak separation of $(k-1)\cdot M_0[\sigma\rangle$. □

Remark 2.

It may be observed that in the weak decomposition of $(k-1){\cdot}M_0[\sigma\rangle$ described above, $\Psi(\tau_1^{[n]}\tau_2^{[n]})$ is the integer part of $\frac{1}{k-1}\cdot\Psi(\sigma)$, and similarly for $l > 2$, $\Psi(\tau_l^{[n]})(t) = 1$ if and only if l is the remainder of the integer division $\frac{1}{k-1}{\cdot}\Psi(\sigma)(t)$. This is illustrated by the example shown in Figure 9 where one copy of M_0 has been grayed. ☐



Separate $2{\cdot}M_0[\sigma\rangle$ with $\sigma = abbbcaaacdbac$

$\tau_1 = abc,\ \tau_1' = aabc,\ \tau_1'' = aabc,\ M_0{=}M_0''.$

$$\begin{array}{r}\text{sep.}\\ \text{of}\\ 3{\cdot}M_0\\ [aabc\,\sigma\rangle\end{array}\left\{\begin{array}{l}\qquad M_0''[aabc\rangle M_1''\\ \left.\boxed{\begin{array}{l}\sigma_2 : M_0''[aabc\rangle M_1''[b\rangle M_2''\\ \sigma_3 : M_0''[aabc\rangle M_1''[b\rangle M_2''[dac\rangle M_3''\end{array}}\right\}\begin{array}{l}\text{sep.}\\ \text{of}\\ 2{\cdot}M_0[\sigma\rangle\end{array}\end{array}\right.$$

Fig. 9. Illustration of the proof of Theorem 7

Theorem 8. $(k-1){\cdot}N$ IS STRONGLY SEPARABLE

Let N be plain. Let $k \geq 2$ and let $k{\cdot}N$ be the multiple of N with initial marking $k{\cdot}M_0$. Suppose that $k{\cdot}N$ is bounded, reversible and persistent, and that it has a unique minimal realizable T-invariant. Then $(k-1){\cdot}N$ is strongly separable.

Proof: Similar to the proof of Theorem 5 and therefore omitted. ☐

Corollary 2. $(k-1){\cdot}N$ IS PERSISTENT

Under the same assumptions, $(k-1){\cdot}N$ is persistent. ☐

The corollary follows from the persistency of N (Lemma 1) and Remark 1.

Corollary 3. $(k-1){\cdot}N$ IS PBRP

Under the same assumptions, $(k-1){\cdot}N$ is a pbrp-net. ☐

This follows from Corollaries 1 and 2, together with the fact that $(k-1){\cdot}N$ inherits plainness and boundedness directly from $k{\cdot}N$.

Finally, the results of this section can be extended to arbitrary pbrp-k-nets.

Corollary 4. $(k-1)\cdot N$ IS SEPARABLE AND PBRP (GENERAL CASE)

Let N be plain. Let $k \geq 2$ and let $k\cdot N$ be the multiple of N with initial marking $k\cdot M_0$. Suppose that $k\cdot N$ is bounded, reversible and persistent. Then $(k-1)\cdot N$ is (weakly and strongly) separable and it has the pbrp property. ☐

The proof for strong separation is similar to the proof of Theorem 6. Persistence, boundedness and reversibility follow from strong separation and the corresponding properties of N stated in Lemmas 1 and 2.

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