Orthogonal Transformations for Coloured Petri Nets

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Abstract. In this paper we describe a multirelational approach for performing linear algebraic manipulations of (coloured) Petri nets. In particular, we show how orthogonal transformations can be used in order to describe the net folding and unfolding operations.

1 Introduction

In this paper we apply linear algebra on high-level (in particular, coloured) Petri nets [1, 3, 5, 6, 9, 10, 13] in order to obtain a linear-algebraic characterisation of folding and unfolding by means of orthogonal transformations. For this purpose, we require two properties:

1. The basic semantic model underlying coloured nets has to be made symmetric. This is achieved by switching from the use of arc-labelling functions (as in [6]) to the use of multirelations (as in [8]). While this may seem a small step, it leads to a significantly simplified linear algebra.
2. There needs to be a consistent way of incorporating value sets of places and mode sets of transitions. We achieve this by assuming an infinite supply of colour sets to be available and by requiring simple set-theoretical conditions which imply that the transformation matrices form a group. This yields a close linear-algebraic relationship between a coloured net and the set of its (partial) (un)foldings.

The paper is structured as follows. In Section 2, we describe the linear-algebraic setup. Section 3 contains the definition of a group of matrices which effect transformations between unfolding-equivalent nets. In Section 4, we define (coloured) nets and their behaviour, using linear algebra on multirelations. In Section 5, we show that unfolding-equivalent nets are equivalent in terms of structure and behaviour. In Section 6, we show that the set of unfolding-equivalent nets forms a lattice. Section 7 contains concluding remarks.

2 Basic sets, multisets, vectors, and matrices

To start the calculus, we fix certain sets. We need the *empty set* $\emptyset$, the set $\mathbf{Z}$ of *integer numbers* and a *singleton set*, say $\mathbf{1} = \{1\}$. Furthermore, let VAL and

MOD denote two infinite, disjoint sets. VAL is the set of *values* and MOD is the set of *modes*. The former plays the same role as the set of (all possible) place colours in coloured nets. The latter plays the same role as the set of (all possible) transition colours in coloured nets. We give these sets different names for easier reference, because otherwise we should often need a remark about what kind of colour is meant. The name 'value' to denote a place colour is convenient because place colour sets play the same role as types, i.e. value sets, of program variables. The name 'mode' to denote a transition colour is already well established in high-level net theory. Finally, we need two disjoint sets $\mathbf{S}$ and $\mathbf{T}$. $\mathbf{S}$ is a sufficiently large supply of *place names* and $\mathbf{T}$ is a sufficiently large supply of *transition names*.

We use mnemonic conventions as follows. The capital letters X, Y, Z are used for sets which are made from $\mathbf{1}$, VAL and MOD by the set-theoretic operations of Cartesian product, union, and the subset relation. The capital letters A, B, C, ..., are used for sets which are made in a similar way from $\mathbf{1}$, $\mathbf{S}$ and $\mathbf{T}$ by the set-theoretic operations of Cartesian product, union, and the subset relation. For instance, X could be a subset of $(VAL \times MOD) \cup (MOD \times VAL)$ and A could be a subset of $(\mathbf{S}\times\mathbf{T}) \cup (\mathbf{T}\times\mathbf{S})$.

We will assume that place names s in $\mathbf{S}$ and transition names t in $\mathbf{T}$ satisfy the property that a *value set* $\iota(s) \subseteq VAL$ is associated with s and a *mode set* $\iota(t) \subseteq MOD$ is associated with t. The sets $\iota(s)$ and $\iota(t)$ are called the *types* of s and t, respectively. Types are generalised in a straightforward way; for instance, the type of a pair $(a,b)\in(\mathbf{1}\cup\mathbf{S}\cup\mathbf{T})\times(\mathbf{1}\cup\mathbf{S}\cup\mathbf{T})$ is $\iota(a,b) = \iota(a)\times\iota(b)$, the type of a subset $A \subseteq (\mathbf{1}\cup\mathbf{S}\cup\mathbf{T})$ is $\bigcup\{\iota(a) | a\in A\}$, and the type of a Cartesian product $A\times B$ is $\iota(A)\times\iota(B)$. If two places or transitions have the same types, then we will often identify them by saying that they are the same 'up to renaming'.

Let X and Y be sets as above. An X-*multiset* is a function $\mu\colon X \to \mathbf{Z}$ associating an integer with every element of X. The set X is called the *base* of μ. Y-multisets are defined analogously as functions from Y to $\mathbf{Z}$. An (X,Y)-*multirelation* is an $(X\times Y)$-multiset, i.e. a function ρ associating an integer number with every pair $(x,y)\in(X\times Y)$. The special case $X{=}Y$ is allowed. An X-multiset μ is called *finite* iff the set $\{x\in X | \mu(y)\neq 0\}$ is finite.

X-multisets are analogues of (integer-valued) vectors. Hence such multisets will sometimes be called X-*vectors*. (X,Y)-multirelations are analogues of (integer-valued) matrices. Hence such multirelations will sometimes be called (X,Y)-*matrices*.

Remark 2.1. Singleton-base multisets

Consider the special case $X{=}\mathbf{1}$. Every $\mathbf{1}$-multiset μ associates a number $k{=}\mu(1)$ with the single element 1 of $\mathbf{1}$. In this way every $\mathbf{1}$-multiset (and more generally every multiset over a singleton) can be viewed as an integer number. Conversely, every integer number $k\in\mathbf{Z}$ can be viewed as the $\mathbf{1}$-multiset mapping 1 to k. Thus, the set of integers and the set of $\mathbf{1}$-multisets can be identified, and so can any other set of X-multisets where X is a singleton. ■ 2.1

Every X-multiset μ can be viewed as an $(X,\mathbf{1})$-multirelation ρ_μ by putting

$\rho_\mu(x,1){=}\mu(x)$ for all $x{\in}X$. Conversely, every $(X,\mathbf{1})$-multirelation ρ can be viewed as an X-multiset μ_ρ by putting $\mu_\rho(x){=}\rho(x,1)$ for all $x{\in}X$. This correspondence is bijective, so that we are justified in identifying μ and ρ_μ; we will just call the latter again μ. This corresponds to the analogue of viewing any vector canonically as a matrix with only one column. In a similar way, μ can also be identified with the $(\mathbf{1},X)$-multirelation σ_μ defined by $\sigma_\mu(1,x){=}\mu(x)$ for all $x{\in}X$. We will write σ_μ as μ' in order to distinguish it from ρ_μ. Then μ' corresponds to the multiset μ viewed as a matrix with only one row; however, as multisets, μ and μ' are equal.

Let ρ be an (X,Y)-multirelation. Then the *transpose* of ρ, denoted by ρ', is defined as a (Y,X)-multirelation with $\rho'(y,x){=}\rho(x,y)$ for all $x{\in}X$ and $y{\in}Y$. Of course, we have $(\rho')' = \rho$. This use of dashes is consistent with the above notation, introduced for μ and μ'.

The following operations and predicates are defined on multisets. Let μ and ν be two Y-multisets. Then $\mu{\le}\nu$ (i.e. μ is a *submultiset* of ν), by definition, iff $\forall y \in Y$: $\mu(y) \le \nu(y)$. The *multiset sum* $\mu{+}\nu$ and the *multiset difference* $\mu{-}\nu$ are defined by $(\mu{+}\nu)(y) = \mu(y)+\nu(y)$ and $(\mu{-}\nu)(y) = \mu(y)-\nu(y)$, respectively, for all $y{\in}Y$. The (additive) *inverse* of μ is defined as $(-\mu) = (\emptyset{-}\mu)$. The *empty Y-multiset* is denoted by $\emptyset_Y$ (or simply $\emptyset$ if its base is clear from the context) and is defined by $\emptyset(y){=}0$ for all $y{\in}Y$. A Y-multiset μ is called *semipositive* iff $\emptyset_Y{\le}\mu$.

All these notions are consistent with their counterparts in $\mathbf{Z}$. To see this, it is only necessary to apply the above definitions in the special case $|Y| = 1$. Because of this consistency, we are justified in using the same symbols ($+$ for addition both in $\mathbf{Z}$ and in the set of multisets, ...).

The *(inner) matrix product* is defined, for finite multisets, as follows. Let ρ be an (X,Y)-multirelation and σ be a (Y,Z)-multirelation. Then $\rho{\otimes}\sigma$ is an (X,Z)-multirelation defined by the following formula:

$$(\rho{\otimes}\sigma)(x,z) \;=\; \sum_{y\in Y}\rho(x,y){\cdot}\sigma(y,z) \qquad \text{for all } x{\in}X \text{ and } z{\in}Z. \tag{1}$$

In this formula, $\cdot$ is the normal product in $\mathbf{Z}$ and $\sum$ is the normal sum in $\mathbf{Z}$.

The matrix product defined in formula 1 specialises to vector/matrix product, matrix/vector product and vector/vector product in the following way. Let μ be a Y-multiset and let σ be a (Y,Z)-multirelation. Then the product $\mu'{\otimes}\sigma$ is a well-defined special case of the above (consider μ' as the same as μ, but seen as a row vector, i.e. a $(\mathbf{1},Y)$-multirelation σ_μ); thus, $\mu'{\otimes}\sigma$ is a $(\mathbf{1},Z)$-multirelation, i.e. a Z-multiset seen as a row vector. The defining formula (1) becomes

$$(\mu'{\otimes}\sigma)(z) \;=\; \sum_{y\in Y}\mu(y){\cdot}\sigma(y,z) \qquad \text{for all } z{\in}Z. \tag{2}$$

Similarly, let ρ be an (X,Y)-multirelation and ν a Y-multiset seen as a column vector, i.e. a $(Y,\mathbf{1})$-multirelation. Then the inner product $\rho{\otimes}\nu$ is a well-defined

X-multiset seen as a column vector, for which formula (1) specialises thus:

$$(\rho\otimes\nu)(x) \quad = \quad \sum_{y\in Y} \rho(x,y)\cdot\nu(y) \qquad \text{for all } x\in X. \tag{3}$$

Finally, let μ' be a Y-multiset (seen as a row vector) and ν a Y-multiset (seen as a column vector). Then the product $\mu'\otimes\nu$ is a $(\mathbf{1},\mathbf{1})$-multirelation, which corresponds to a number in $\mathbf{Z}$ since $\mathbf{1}\times\mathbf{1}$ is a singleton set just as well as $\mathbf{1}$. Formula (1) specialises as follows in this case:

$$(\mu'\otimes\nu) \quad = \quad \sum_{y\in Y} \mu(y)\cdot\nu(y). \tag{4}$$

Under these conditions, the product $\mu\otimes\nu'$ is also well defined. It is a (Y,Y)-matrix. We leave it to the reader to derive its defining formula from (1). Note finally that the inner product defined in formula (1) is a generalisation of the product in $\mathbf{Z}$: if all of X, Y and Z are singletons then both ρ and σ are multisets over a singleton (i.e. integer numbers), and (1) specialises to their product.

To represent specific (finite) multisets we often use 'multiset notation'. For instance, $\{x,x,y\}$ denotes the multiset μ with $\mu(x)$=2, $\mu(y)$=1 and $\mu(z)$=0 for any $z\in X\backslash\{x,y\}$ (where it is assumed that the base X is understood from the context). This works nicely for semipositive multisets $\mu\geq 0$. For non-semipositive multisets, we will use linear combinations of multisets liberally. For example, $-\{x\}+\{y\}-\{x\}$ denotes the multiset μ with $\mu(x)=-2$, $\mu(y)$=1 and $\mu(z)$=0 for any $z\in X\backslash\{x,y\}$. In this case, $-\mu$ is the multiset with $\mu(x)$=2, $\mu(y)=-1$ and $\mu(z)$=0 for any $z\in X\backslash\{x,y\}$, i.e., $-(-\{x\}+\{y\}-\{x\}) = 2\cdot\{x\}-\{y\}$.

Thus far, we have considered functions (vectors, matrices) from domains made from the sets $\mathbf{1}$, VAL and MOD to codomains in $\mathbf{Z}$. To help understand the hierarchical construction we embark on, let us call such functions 'of the first kind'. Now, we consider functions whose domains A, ..., are similarly made from the sets $\mathbf{1}$, $\mathbf{S}$ and $\mathbf{T}$ and whose codomains are functions of the first kind. Let us call them 'functions (vectors, matrices etc.) of the second kind'. Previously we have lifted operations and predicates from $\mathbf{Z}$ to functions of the first kind. Now we will lift them from functions of the first kind to functions of the second kind.

Let A be a set, as above, with type $\iota(A)$. An A-*multiset* is any function from A to a $\iota(A)$-multiset which is *type-respecting*, i.e. such that, by definition, for every argument $a\in A$, the function value $f(a)$ is a $\iota(a)$-multiset.

Remark 2.2. The hierarchy explained (part 1)

Functions of the second kind are thus type-respecting functions from certain sets to certain functions. We can make functions of the first kind look exactly the same in the following way. Associate a type $\iota(x)=\mathbf{1}$ with every element $x\in VAL\cup MOD$ and consider $\mathbf{Z}$ as the set of functions from $\mathbf{1}$ to $\mathbf{Z}$ (and thus, as 'functions of the zero'th kind'). Then functions of the first kind are of exactly the same shape as functions of the second kind. In particular, they are type-respecting due to $\mathbf{Z}$ being considered as the set of $\mathbf{1}$-multisets. We may build functions of the third, fourth, etc., kind on top of functions of the second kind, but we do not need those in the calculus. ■ 2.2

By double linear extension, all operations on $\mathbf{Z}$ (addition, subtraction, comparisons, the empty multiset, etc.) can be lifted first to X-multisets – as seen above – and then to A-multisets. For example, let α_1 and α_2 be two A-multisets. Then $\alpha_1 + \alpha_2$, which is defined by the formula $(\alpha_1 + \alpha_2)(a) = \alpha_1(a) + \alpha_2(a)$ for all $a \in A$, is again an A-multiset. The sum on the right-hand side of this definition is the multiset sum defined previously for X-multisets. Note that because both α_1 and α_2 are type-respecting, both $\alpha_1(a)$ and $\alpha_2(a)$ are $\iota(a)$-multisets, and therefore this sum is well defined. Now we show that the matrix product can be lifted from functions of the first kind (for which it is given by formula (1)) to functions of the second kind.

Let A, B and C be sets, let α be an (A, B)-matrix and let β be a (B, C)-matrix. Then $\alpha \otimes \beta$ is defined as an (A, C)-matrix by the following formula:

$$(\alpha \otimes \beta)(a, c) \quad = \quad \sum_{b \in B} \alpha(a, b) \otimes \beta(b, c) \qquad \text{for all } a \in A \text{ and } c \in C. \tag{5}$$

This formula is well defined: on the right-hand side, $\sum$ is multiset sum and $\otimes$ is multiset product. The latter is well-defined by formula (1), since $\alpha(a, b)$ is a $(\iota(a), \iota(b))$-multirelation and $\beta(b, c)$ is a $(\iota(b), \iota(c))$-multirelation.

Remark 2.3. The hierarchy explained (part 2)

The similarity of (1) and (5) suggests that the former should be considered a special case of the latter. It may be checked that under the interpretation of functions of the first kind described in Remark 2.2, formula (5) indeed specialises to formula (1). ■ 2.3

The distinction between functions of the first kind and functions of the second kind was introduced for heuristic purposes. Having lifted (explicitly and implicitly) all necessary operations on all functions we need, we may speak of A-multisets and X-multisets in analogous terms. In particular, if A and B are subsets of $\mathbf{S}$ or $\mathbf{T}$ then the notion of an A-vector, the notion of an (A, B)-matrix, the notion of a transposed matrix, etc., are analogues of the same notions defined for X-multisets and $X \times Y$-multisets.

Lemma 2.4. *Algebraic properties*

> On the set of multisets, $+$ is associative and commutative, the scalar product satisfies $k \cdot (l \cdot \alpha) = (k \cdot l) \cdot \alpha$, $\otimes$ is associative and distributive over $+$ as well as over $-$, and $\emptyset$ is a Unit with respect to $+$ and a Zero with respect to $\otimes$. Moreover, $(F \otimes G)' = G' \otimes F'$.

Proof: By reduction to the analogous properties holding in the ring of integers. The last claim comes from the commutativity of the product in $\mathbf{Z}$. ■ 2.4

3 The group of transformation matrices

In this section we investigate a certain multiplicative subgroup of matrices. This group will be useful for transforming different representatives of the same (elementary) net into each other. Let A be a subset of **S** or **T**. A is called *normal* iff all types of elements are mutually disjoint, i.e. iff

$$\forall a_1, a_2 \in A\colon\ a_1 \neq a_2 \Rightarrow \iota(a_1) \cap \iota(a_2) = \emptyset.$$

Furthermore, let A and B be called *compatible* iff they have the same types, i.e. $\iota(A) = \iota(B)$.

Let A and B be two normal, compatible subsets of **S** or **T**. The *transformation matrix with respect to A and B*, $\mathcal{T}_{A,B}$, is defined as an (A, B)-matrix as follows:

$$\mathcal{T}_{A,B}(a, b) \;=\; \{\, (w, w) \mid w \in \iota(a) \wedge w \in \iota(b) \,\} \qquad \text{for all } a \in A \text{ and } b \in B.$$

Since both $\iota(a)$ and $\iota(b)$ are sets, the right-hand side of this definition is again a set (i.e. a multiset with codomain $\{0, 1\}$). Note that $\mathcal{T}_{A,B}$ is type-respecting, semipositive, and satisfies $(\mathcal{T}_{A,B})' = \mathcal{T}_{B,A}$. The transformation matrix $\mathcal{T}_{A,A}$ will be called the *A-unit-matrix* and denoted by $\mathcal{E}_{A,A}$. In the following, we list the group properties of transformation matrices.

Lemma 3.1. *Transformation matrices are closed under multiplication*

Let A and B be normal and compatible, and let B and C be normal and compatible. Then $\mathcal{T}_{A,B} \otimes \mathcal{T}_{B,C}$ is a transformation matrix and $\mathcal{T}_{A,B} \otimes \mathcal{T}_{B,C} = \mathcal{T}_{A,C}$.

Proof: Note first that the compatibility of A and B, and of B and C, implies that A and C are also compatible. Hence $\mathcal{T}_{A,B} \otimes \mathcal{T}_{B,C}$ has the form of a transformation matrix.

It remains to prove $\mathcal{T}_{A,B} \otimes \mathcal{T}_{B,C} = \mathcal{T}_{A,C}$. To this end, we fix $a \in A$, $c \in C$, $x \in \iota(a)$, $z \in \iota(c)$ and evaluate:

$$
\begin{aligned}
((\mathcal{T}_{A,B}\otimes\mathcal{T}_{B,C})(a,c))(x,z) \;=\;& \text{(by formula (5))}\\
& (\textstyle\sum_{b\in B}\mathcal{T}_{A,B}(a,b)\otimes\mathcal{T}_{B,C}(b,c))(x,z)\\
=\;& \text{(function evaluation)}\\
& \textstyle\sum_{b\in B}((\mathcal{T}_{A,B}(a,b)\otimes\mathcal{T}_{B,C}(b,c))(x,z))\\
=\;& \text{(formula (1) and function evaluation)}\\
& \textstyle\sum_{b\in B}\sum_{y\in\iota(b)}((\mathcal{T}_{A,B}(a,b))(x,y)\cdot(\mathcal{T}_{B,C}(b,c))(y,z))\\
=\;& \text{(definitions of } \mathcal{T}_{A,B} \text{ and } \mathcal{T}_{B,C})\\
& \textstyle\sum_{b\in B}\sum_{y\in\iota(b)}\left\{\begin{matrix}1 \text{ if } x{=}y{\in}\iota(a)\cap\iota(b)\\ 0 \text{ otherwise}\end{matrix}\right\}\cdot\left\{\begin{matrix}1 \text{ if } y{=}z{\in}\iota(b)\cap\iota(c)\\ 0 \text{ otherwise}\end{matrix}\right\}\\
=\;& \text{(the product is 1 only for } x{=}y{=}z)\\
& \textstyle\sum_{b\in B}\left\{\begin{matrix}1 \text{ if } x{=}z{\in}\iota(a)\cap\iota(b)\cap\iota(c)\\ 0 \text{ otherwise}\end{matrix}\right\}\\
=\;& \text{(see justification below)}\\
& \left\{\begin{matrix}1 \text{ if } x{=}z{\in}\iota(a)\cap\iota(c)\\ 0 \text{ otherwise}\end{matrix}\right\}\\
=\;& \text{(definition of } \mathcal{T}_{A,C})\\
& (\mathcal{T}_{A,C}(a,c))(x,z).
\end{aligned}
$$

The last but one equation follows from the fact that, by the normality of B, the sum contains a single 1 iff $x{=}z{\in}\iota(a)\cap\iota(c)$, and only 0's otherwise. ■ 3.1

Lemma 3.2. *Unit-matrices are units with respect to matrix multiplication*

Let F be any (A,B)-matrix. Then $\mathcal{E}_{A,A}\otimes F = F = F\otimes\mathcal{E}_{B,B}$.

Proof: We show only that $\mathcal{E}_{A,A}\otimes F = F$ as the proof of $F = F\otimes\mathcal{E}_{B,B}$ is analogous.
We fix $a{\in}A$, $b{\in}B$, $x{\in}\iota(a)$, $y{\in}\iota(b)$ and evaluate:

$$
\begin{aligned}
((\mathcal{E}_{A,A}\otimes F)(a,b))(x,y) \;=\;& \text{(formulae (5), (1) and function evaluation)}\\
& \textstyle\sum_{a'\in A}\sum_{x'\in\iota(a')}((\mathcal{E}_{A,A}(a,a'))(x,x')\cdot(F(a',b))(x',y))\\
=\;& \text{(definition of } \mathcal{E}_{A,A})\\
& \textstyle\sum_{a'\in A}\sum_{x'\in\iota(a')}\left\{\begin{matrix}1 \text{ if } x{=}x'{\in}\iota(a)\cap\iota(a')\\ 0 \text{ otherwise}\end{matrix}\right\}\cdot(F(a',b))(x',y))\\
=\;& (F(a,b))(x,y).
\end{aligned}
$$

The last equation follows from the fact that, by the normality of A, the first inner factor equals 1 iff $x{=}x'{\in}\iota(a){=}\iota(a')$, and hence (by $x{\in}\iota(a)$) iff $(a',x'){=}(a,x)$, and that it equals 0 for all other pairs (a',x'). ■ 3.2

Let A and B be normal and compatible. A (B,A)-matrix G is called a *right-inverse* of an (A,B)-matrix F if $F\otimes G = \mathcal{E}_{A,A}$, and a *left-inverse* of F if $G\otimes F = \mathcal{E}_{B,B}$. An (A,B)-matrix F is called *orthogonal* iff its transpose F' is a right-inverse of F.

Corollary 3.3. *Transformation matrices are orthogonal*

Suppose that A and B are normal and compatible. Then $\mathcal{T}_{A,B}$ is orthogonal.

Proof: By Lemma 3.1, $\mathcal{T}_{A,B}\otimes\mathcal{T}_{B,A} = \mathcal{T}_{A,A} = \mathcal{E}_{A,A}$ and $\mathcal{T}_{B,A}\otimes\mathcal{T}_{A,B} = \mathcal{T}_{B,B} = \mathcal{E}_{B,B}$. ■ 3.3

Corollary 3.4. *Uniqueness of inverses*

The right-inverse and the left-inverse of $F{=}\mathcal{T}_{A,B}$ exist and are unique.

Proof: Existence follows directly from the orthogonality of F. We prove uniqueness for the right-inverse:

$$
\begin{aligned}
F{\otimes}G_1{=}\mathcal{E}_{A,A}{=}F{\otimes}G_2 \Rightarrow\ & \text{(multiply from the left with } F'\text{)} \\
& F'{\otimes}(F{\otimes}G_1) = F'{\otimes}\mathcal{E}_{A,A} = F'{\otimes}(F{\otimes}G_2) \\
\Rightarrow\ & \text{(Lemma 3.2, Corollary 3.3, algebraic properties)} \\
& G_1 = F' = G_2.
\end{aligned}
$$

■ 3.4

4 Nets

A *net* is a triple (S, T, ω) such that:

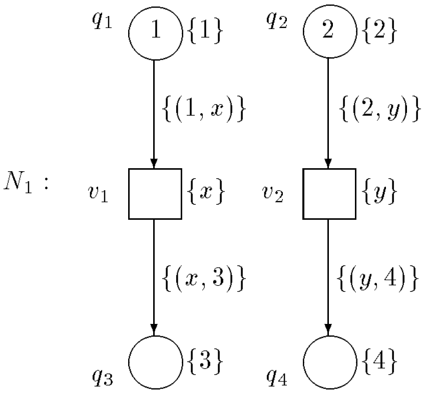
- S is a normal subset of $\mathbf{S}$. Recall from Section 2 that every $s{\in}S$ has an associated value set $\iota(s){\subseteq}\mathit{VAL}$ and from Section 3 that (by normality) those sets are mutually disjoint.
- T is a normal subset of $\mathbf{T}$. A similar remark as for S applies.
- ω is a semipositive $(S{\times}T){\cup}(T{\times}S)$-multirelation (called the 'flow relation'). Recall that by definition, ω is type-respecting.

Thanks to the disjointness of S and T, ω can be decomposed additively as $\omega = (\omega|_{S\times T}) + (\omega|_{T\times S})$, where $|$ denotes restriction of functions. The first summand is an (S,T)-matrix and the second summand is a (T,S)-matrix. The *incidence matrix* of N is defined as an (S,T)-matrix $\Gamma = -\omega|_{S\times T} + (\omega|_{T\times S})'$. In this definition, Γ determines a multirelation-valued (rather than integer-valued) (S,T)-multirelation. It is thus a 'matrix of matrices' similar to the ones described in [4, 11].

Four examples are shown in Figures 1–4. The marked nets N_1–N_4 are shown on the left-hand sides, and their respective incidence matrices Γ_1–Γ_4 are shown on the right-hand sides of these figures. The following mnemonic conventions are used: $\mathit{VAL} = \{1, 2, 3, \ldots\}$ and $\mathit{MOD} = \{x, y, z, \ldots\}$. Place names in S are $q, r, s, \ldots$ and transition names in T are $t, u, v, \ldots$

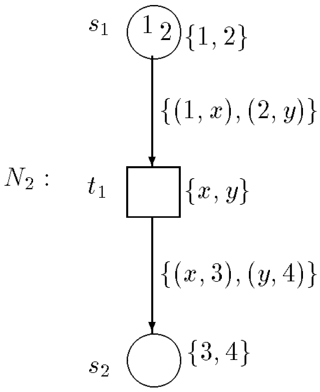
Let us see, for instance, how Figure 4 conforms to the formal definitions given in the previous sections. N_4 is (S_4, T_4, ω_4) with

$$
\begin{aligned}
S_4 &= \{s\} \quad \text{with} \quad \iota(s) = \{1, 2, 3, 4\} \\
T_4 &= \{t\} \quad \text{with} \quad \iota(t) = \{x, y\} \\
\omega_4 \ &: \left\{
\begin{array}{ccl}
\{s\}{\times}\{t\} \cup \{t\}{\times}\{s\} & \to & \{\mu \mid \mu\colon (\iota(s){\times}\iota(t) \cup \iota(t){\times}\iota(s)) \to \mathbf{Z}\} \\
(s,t) & \mapsto & \{(1,x), (2,y)\} \\
(t,s) & \mapsto & \{(x,3), (y,4)\},
\end{array}
\right.
\end{aligned}
$$



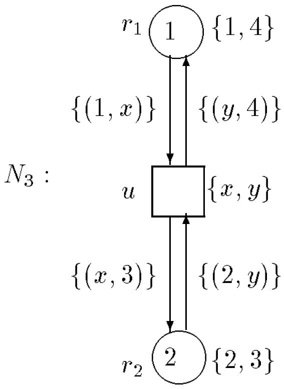
Γ_1 :	v_1 with $\iota(v_1)=\{x\}$	v_2 with $\iota(v_2)=\{y\}$
q_1 with $\iota(q_1)=\{1\}$	$-\{(1,x)\}$	$\emptyset$
q_2 with $\iota(q_2)=\{2\}$	$\emptyset$	$-\{(2,y)\}$
q_3 with $\iota(q_3)=\{3\}$	$+\{(3,x)\}$	$\emptyset$
q_4 with $\iota(q_4)=\{4\}$	$\emptyset$	$+\{(4,y)\}$

Figure 1. N_1 (left) and Γ_1 (right)



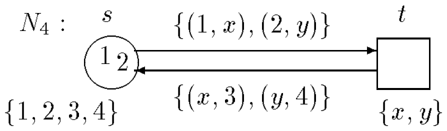
Γ_2 :	t_1 with $\iota(t_1)=\{x,y\}$
s_1 with $\iota(s_1)=\{1,2\}$	$-\{(1,x),(2,y)\}$
s_2 with $\iota(s_2)=\{3,4\}$	$+\{(3,x),(4,y)\}$

Figure 2. N_2 (left) and Γ_2 (right)



Γ_3 :	u with $\iota(u)=\{x,y\}$
r_1 with $\iota(r_1)=\{1,4\}$	$-\{(1,x)\}+\{(4,y)\}$
r_2 with $\iota(r_2)=\{2,3\}$	$-\{(2,y)\}+\{(3,x)\}$

Figure 3. N_3 (left) and Γ_3 (right)



Γ_4 :	t with $\iota(t)=\{x,y\}$
s with $\iota(s)$	$+\{(3,x),(4,y)\}$
$=\{1,2,3,4\}$	$-\{(1,x),(2,y)\}$

Figure 4. N_4 (left) and Γ_4 (right)

and Γ_4 is the multirelation

$$\Gamma_4 : \begin{cases} \{s\} \times \{t\} \to \{\mu \mid \mu\colon (\iota(s)\times\iota(t)) \to \mathbf{Z}\} \\ \qquad (s,t) \quad \mapsto \begin{cases} (1,x) \mapsto -1 \\ (3,x) \mapsto +1 \\ (2,y) \mapsto -1 \\ (4,y) \mapsto +1 \\ (w,z) \mapsto \ \ 0 \ \text{ for all other } (w,z) \in \iota(s)\times\iota(t). \end{cases} \end{cases}$$

All other figures conform similarly to the formal definitions.

Remark 4.1. Relation to coloured nets

In coloured nets [6, 10], $\omega(s,t)$ is a function from the colour set of t to multisets over the colour set of s. This is equivalent with our definition because every function from A to multisets over B can be viewed as an (A,B)-multirelation and conversely, and hence, by symmetry, also to viewing $\omega(s,t)$ as a function from the colour set of s to multisets over the colour set of t. The same holds for $\omega(t,s)$. In the original definition, value sets and mode sets of coloured nets need not be normal, but it is easy to see that this property can be assumed without loss of generality; simply consider the type of a place (transition) to be 'tagged' by its name. For practical purposes, it is often desirable to have non-normality, but for theoretical purposes, it is a property that can, without loss of generality, conveniently be assumed to hold without hindering practitioners to violate it.

■ 4.1

We define the behaviour of nets in the usual way, but using the linear algebraic calculus defined above. Let $N = (S,T,\omega)$ be a net. A *marking* M of N is a semipositive S-multiset. Let a net $N = (S,T,\omega)$, a marking M of N, a transition t of N and a mode $m\in\iota(t)$ be given. We say that M *activates* t *in mode* m iff

$$M \ \geq \ (\omega|_{S\times T}) \otimes \underline{(t,m)} \tag{6}$$

where $\underline{(t,m)}$ is a T-vector defined as follows:

$$\underline{(t,m)} : \begin{cases} T \to \{\mu \mid \mu\colon \iota(T) \to \mathbf{Z}\} \\ t \ \mapsto \{m\} \ \text{ and } \ u \mapsto \emptyset_{\iota(u)} \ \text{ for } u{\neq}t. \end{cases}$$

Other behavioural concepts relating to nets (occurrence sequences, reachability graph, etc.) are based on the above.

5 Unfolding-equivalence of nets

In this section we assume a net $N = (S, T, \omega)$ to be given; recall that $S \subseteq \mathbf{S}$ and $T \subseteq \mathbf{T}$ are both normal. We will show how (un)folding equivalence is described in the calculus. Let R be a normal subset of $\mathbf{S}$ compatible with S and let U be a normal subset of $\mathbf{T}$ compatible with T. From $N = (S, T, \omega)$ we construct a transformed net $N_0 = (S_0, T_0, \omega_0)$ by the following rule:

$$\begin{aligned} S_0 &= R \\ T_0 &= U \\ \omega_0 &= (\mathcal{T}_{R,S} \otimes \omega|_{S\times T} \otimes \mathcal{T}_{T,U}) + (\mathcal{T}_{U,T} \otimes \omega|_{T\times S} \otimes \mathcal{T}_{S,R}). \end{aligned} \tag{7}$$

We call N and N_0 *unfolding-equivalent* iff they are related to each other by two transformation matrices $\mathcal{T}_{R,S}$ and $\mathcal{T}_{U,T}$ as in (7). As an immediate corollary of the fact that transformation matrices form a group, unfolding equivalence is reflexive, symmetric and transitive; i.e., it is indeed an equivalence relation. We claim (and substantiate this claim in the remainder of this section) that unfolding-equivalent nets such as N and N_0 are equivalent in all respects: structure (e.g. concerning invariants) and behaviour, and that they can, in fact, be viewed as different representations of the same net.

Lemma 5.1. *Transformation of incidence matrices*

Let Γ, Γ_0 be the incidence matrices of N and N_0, respectively.
Then $\Gamma_0 = \mathcal{T}_{R,S} \otimes \Gamma \otimes \mathcal{T}_{T,U}$.

Proof:

$$\begin{aligned} \Gamma_0 &= \text{(definition of } \Gamma_0) \\ &\quad -\omega_0|_{R\times U} + (\omega_0|_{U\times R})' \\ &= \text{(definition of } \omega_0) \\ &\quad -(\mathcal{T}_{R,S} \otimes \omega|_{S\times T} \otimes \mathcal{T}_{T,U}) + (\mathcal{T}_{U,T} \otimes \omega|_{T\times S} \otimes \mathcal{T}_{S,R})' \\ &= \text{(property of transposition w.r.t. matrix multiplication)} \\ &\quad -(\mathcal{T}_{R,S} \otimes \omega|_{S\times T} \otimes \mathcal{T}_{T,U}) + (\mathcal{T}_{R,S} \otimes (\omega|_{T\times S})' \otimes \mathcal{T}_{T,U}) \\ &= \text{(left and right distributivity of } \otimes \text{ over } +) \\ &\quad \mathcal{T}_{R,S} \otimes (-\omega|_{S\times T} + (\omega|_{T\times S})') \otimes \mathcal{T}_{T,U} \\ &= \text{(definition of } \Gamma) \\ &\quad \mathcal{T}_{R,S} \otimes \Gamma \otimes \mathcal{T}_{T,U}. \end{aligned}$$

■ 5.1

Corollary 5.2. *Inverse transformation*

With the same premises, $\Gamma = \mathcal{T}_{S,R} \otimes \Gamma_0 \otimes \mathcal{T}_{U,T}$.

Proof: Multiplying Γ_0 in Lemma 5.1 from the left by $\mathcal{T}_{S,R}$ and from the right by $\mathcal{T}_{U,T}$. ■ 5.2

Transformation matrices cast the usual folding/unfolding operations, described net theoretically, for instance, in [11], in linear-algebraic form. However, a single pair of transformation matrices can describe a mixture of folding and unfolding. We show this on the above example, for which we claim that N_1–N_4 are unfolding-equivalent.

Consider, e.g., N_2 and N_3. N_2 is (S, T, ω_2) with $S = \{s_1, s_2\}$ and $T = \{t_1\}$ while N_3 is (R, U, ω_3) with $R = \{r_1, r_2\}$ and $U = \{u\}$. Thus we consider the transformation matrices $\mathcal{T}_{R,S}$ and $\mathcal{T}_{T,U}$. First, we give them in the form of multirelations:

$$\mathcal{T}_{R,S} : \begin{cases} R \times S \to \{\rho \mid \rho\colon (\iota(R)\times\iota(S))\to\mathbf{Z}\} \\ (r_1, s_1) \mapsto \{(1,1)\} \\ (r_1, s_2) \mapsto \{(4,4)\} \\ (r_2, s_1) \mapsto \{(2,2)\} \\ (r_2, s_2) \mapsto \{(3,3)\} \end{cases} \qquad \mathcal{T}_{T,U} : \begin{cases} T \times U \to \{\rho \mid \rho\colon (\iota(T)\times\iota(U))\to\mathbf{Z}\} \\ (t_1, u) \mapsto \{(x,x),(y,y)\} \end{cases}$$

and thus, in matrix form:

$\mathcal{T}_{R,S}$:	s_1 with $\iota(s_1)=\{1,2\}$	s_2 with $\iota(s_2)=\{3,4\}$
r_1 with $\iota(r_1)=\{1,4\}$	$\{(1,1)\}$	$\{(4,4)\}$
r_2 with $\iota(r_2)=\{2,3\}$	$\{(2,2)\}$	$\{(3,3)\}$

and

$\mathcal{T}_{T,U}$:	u with $\iota(u)=\{x,y\}$
t_1 with $\iota(t_1)=\{x,y\}$	$\{(x,x),(y,y)\}$

Note that $\mathcal{T}_{T,U}$ is the T-unit-matrix. It is also the U-unit-matrix, as T and U are the same up to renaming. First, we check the truth of Corollary 3.3 which states that $\mathcal{T}_{R,U}$ is orthogonal:

$$\begin{array}{|c|c|}\hline \{(1,1)\} & \{(4,4)\} \\ \hline \{(2,2)\} & \{(3,3)\} \\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline \{(1,1)\} & \{(2,2)\} \\ \hline \{(4,4)\} & \{(3,3)\} \\ \hline\end{array} = \begin{array}{|c|c|}\hline \{(1,1),(4,4)\} & \emptyset \\ \hline \emptyset & \{(2,2),(3,3)\} \\ \hline\end{array}$$

To verify this equation, note that every single entry on the right-hand side involves the sum of two multisets (of the first kind) which are themselves inner products formed according to the rule (1). Thus, as expected, $\mathcal{T}_{R,S}\otimes\mathcal{T}_{S,R} = \mathcal{E}_{R,R}$ and similarly (but not shown above), $\mathcal{T}_{S,R}\otimes\mathcal{T}_{R,S} = \mathcal{E}_{S,S}$.

Next, we consider the product $\mathcal{T}_{R,S}\otimes\Gamma_2\otimes\mathcal{T}_{T,U}$, which is well-defined by the associativity of $\otimes$ and because the three matrices fit together:

$$\begin{array}{|c|c|}\hline \{(1,1)\} & \{(4,4)\} \\ \hline \{(2,2)\} & \{(3,3)\} \\ \hline\end{array} \otimes \begin{array}{|c|}\hline -\{(1,x),(2,y)\} \\ \hline +\{(3,x),(4,y)\} \\ \hline\end{array} \otimes \begin{array}{|c|}\hline \{(x,x),(y,y)\} \\ \hline\end{array} = \begin{array}{|c|}\hline -\{(1,x)\}+\{(4,y)\} \\ \hline -\{(2,y)\}+\{(3,x)\} \\ \hline\end{array}$$

This calculation shows that, as expected, Γ_3 results from the transformation. Conversely, it may be verified that $\Gamma_2 = \mathcal{T}_{S,R}\otimes\Gamma_3\otimes\mathcal{T}_{U,T}$ (but this is left to the reader). Note that the relationship between Γ_2 and Γ_3 is neither pure unfolding nor pure folding. Transformation matrices that describe pure (un)folding satisfy special properties; see Section 6.

As another example, we consider the unfolding of N_4 into N_1. N_4 is based on $\{s\}$ with $\iota(s) = \{1,2,3,4\}$ and $\{t\}$ with $\iota(t) = \{x,y\}$, while N_1 is based on $Q = \{q_1,\ldots,q_4\}$ with $\iota(q_i) = \{i\}$ $(1\leq i\leq 4)$ and $V = \{v_1, v_2\}$ with $\iota(v_1) = \{x\}$

and $\iota(v_2) = \{y\}$. The transformation matrices are, therefore:

$\mathcal{T}_{Q,\{s\}}$:	s with $\iota(s)=\{1,2,3,4\}$
q_1 with $\iota(q_1)=\{1\}$	$\{(1,1)\}$
q_2 with $\iota(q_2)=\{2\}$	$\{(2,2)\}$
q_3 with $\iota(q_3)=\{3\}$	$\{(3,3)\}$
q_4 with $\iota(q_4)=\{4\}$	$\{(4,4)\}$

and

$\mathcal{T}_{\{t\},V}$:	v_1 with $\iota(v_1)=\{x\}$	v_2 with $\iota(v_2)=\{y\}$
t with $\iota(t)=\{x,y\}$	$\{(x,x)\}$	$\{(y,y)\}$

For instance, $\Gamma_1 = \mathcal{T}_{Q,\{s\}} \otimes \Gamma_4 \otimes \mathcal{T}_{\{t\},V}$ and $\Gamma_4 = \mathcal{T}_{\{s\},Q} \otimes \Gamma_1 \otimes \mathcal{T}_{V,\{t\}}$. Similar transformations can be devised between any pair of the nets shown in Figures 1–4. Note that these four nets do not cover the whole spectrum of equivalent nets. For instance, there are also several equivalent nets with two places and two transitions, as well as several nets with three places.

We now show that linear net invariants are transformed into each other. Referring to a net $N = (S, T, \omega)$, an S-vector I is called an S-*invariant* iff $I' \otimes \Gamma = \emptyset_{\mathbf{1}\times T}$, and a T-vector J is called a T-*invariant* iff $\Gamma \otimes J = \emptyset_{S\times\mathbf{1}}$. We show coherence in the sense that the set of S-invariants of a net is transformed exactly into the set of S-invariants of the transformed net, and the same for T-invariants. Let $N = (S, T, \omega)$ and $N_0 = (R, U, \omega_0)$ be related as above, i.e. as in (7). In particular, $\Gamma_0 = \mathcal{T}_{R,S} \otimes \Gamma \otimes \mathcal{T}_{T,U}$, and hence, by Corollary 5.2, $\Gamma = \mathcal{T}_{S,R} \otimes \Gamma_0 \otimes \mathcal{T}_{U,T}$.

Lemma 5.3. *Coherence of unfolding-equivalence and invariants (part 1)*

(I): I is an S-invariant of N if and only if $\mathcal{T}_{R,S} \otimes I$ is an S-invariant of N_0.
(J): J is a T-invariant of N if and only if $\mathcal{T}_{U,T} \otimes J$ is a T-invariant of N_0.

Proof: (I($\Rightarrow$)):
$$\begin{aligned}
(\mathcal{T}_{R,S} \otimes I)' \otimes \Gamma_0 &= \text{(with Lemma 5.1)} \\
& \quad (I' \otimes \mathcal{T}_{S,R}) \otimes (\mathcal{T}_{R,S} \otimes \Gamma \otimes \mathcal{T}_{T,U}) \\
&= \text{(associativity of } \otimes\text{, Corollary 3.3, Lemma 3.2)} \\
& \quad I' \otimes \Gamma \otimes \mathcal{T}_{T,U} \\
&= \text{(since } I \text{ is an S-invariant of } N \text{ and } \emptyset \text{ is a zero)} \\
& \quad \emptyset_{\mathbf{1}\times U}.
\end{aligned}$$

(I($\Leftarrow$)):
$$\begin{aligned}
I' \otimes \Gamma &= \text{(Corollary 5.2)} \\
& \quad I' \otimes (\mathcal{T}_{S,R} \otimes \Gamma_0 \otimes \mathcal{T}_{U,T}) \\
&= \text{(various algebraic laws)} \\
& \quad ((\mathcal{T}_{R,S} \otimes I) \otimes \Gamma_0 \otimes \mathcal{T}_{U,T}) \\
&= \text{(since } \mathcal{T}_{R,S} \otimes I \text{ is S-invariant of } N_0\text{)} \\
& \quad \emptyset_{\mathbf{1}\times U} \otimes \mathcal{T}_{U,T} \\
&= \emptyset_{\mathbf{1}\times T}.
\end{aligned}$$

Part (J) of the lemma has a symmetric proof. ■ 5.3

Corollary 5.4. *Coherence of unfolding-equivalence and invariants (part 2)*

I_0 is an S-invariant of N_0 iff $\mathcal{T}_{S,R}\otimes I_0$ is an S-invariant of N.
J_0 is a T-invariant of N_0 iff $\mathcal{T}_{T,U}\otimes J_0$ is a T-invariant of N. ■ 5.4

We illustrate these claims on our example. For instance, consider the following Q-vector I_1 (with the same Q as above):

$$I_1 : \begin{cases} Q \to \{\mu \mid \mu: \iota(Q) \to \mathbf{Z}\} \\ q_1 \mapsto \emptyset_{\{1\}} \\ q_2 \mapsto \{2,2\} \\ q_3 \mapsto \emptyset_{\{3\}} \\ q_4 \mapsto \{4,4\}, \end{cases} \qquad \text{or, in column matrix form:} \qquad \begin{array}{c|c} I_1 : & \emptyset \\ \hline q_1 & \emptyset \\ q_2 & \{2,2\} \\ q_3 & \emptyset \\ q_4 & \{4,4\} \end{array}$$

To see that I_1 is an S-invariant of N_1, we check the product $I_1'\otimes\Gamma_1$:

$$\boxed{\emptyset \,|\, \{2,2\} \,|\, \emptyset \,|\, \{4,4\}} \otimes \begin{array}{|c|c|} \hline -\{(1,x)\} & \emptyset \\ \hline \emptyset & -\{(2,y)\} \\ \hline +\{(3,x)\} & \emptyset \\ \hline \emptyset & +\{(4,y)\} \\ \hline \end{array} = \boxed{\emptyset \,|\, -\{y,y\}+\{y,y\}} = \boxed{\emptyset \,|\, \emptyset}$$

Since this product thus vanishes, I_1 is indeed an S-invariant of N_1. Note that in the evaluation of this product, the analogue of (2) must be applied. In order to recover the S-invariant of, say, N_3, which corresponds to I_1, we apply the transformation matrices mediating between N_1 and N_3 to I_1. These matrices are as follows (using the same sets R, Q, V and U as above):

$\mathcal{T}_{R,Q}$:	q_1 with $\iota(q_1)=\{1\}$	q_2 with $\iota(q_2)=\{2\}$	q_3 with $\iota(q_3)=\{3\}$	q_4 with $\iota(q_4)=\{4\}$
r_1 with $\iota(r_1)=\{1,4\}$	$\{(1,1)\}$	$\emptyset$	$\emptyset$	$\{(4,4)\}$
r_2 with $\iota(r_2)=\{2,3\}$	$\emptyset$	$\{(2,2)\}$	$\{(3,3)\}$	$\emptyset$

$\mathcal{T}_{V,U}$:	u with $\iota(u)=\{x,y\}$
v_1 with $\iota(v_1)=\{x\}$	$\{(x,x)\}$
v_2 with $\iota(v_2)=\{y\}$	$\{(y,y)\}$

To get the S-invariant I_3 of N_3 which corresponds to I_1, Lemma 5.3 tells us to apply $\mathcal{T}_{R,Q}$ from the left to I_1, i.e. to calculate $I_3 = \mathcal{T}_{R,Q}\otimes I_1$:

$$\begin{array}{|c|c|c|c|} \hline \{(1,1)\} & \emptyset & \emptyset & \{(4,4)\} \\ \hline \emptyset & \{(2,2)\} & \{(3,3)\} & \emptyset \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \emptyset \\ \hline \{2,2\} \\ \hline \emptyset \\ \hline \{4,4\} \\ \hline \end{array} = \underbrace{\begin{array}{|c|} \hline \{4,4\} \\ \hline \{2,2\} \\ \hline \end{array}}_{I_3}$$

The next calculation verifies, by computing $I_3'\otimes\Gamma_3$ and showing that this product is Zero, that I_3 is indeed an S-invariant of N_3:

$$\boxed{\{4,4\} \,|\, \{2,2\}} \otimes \begin{array}{|c|} \hline -\{(1,x)\}+\{(4,y)\} \\ \hline -\{(2,y)\}+\{(3,x)\} \\ \hline \end{array} = \boxed{\{y,y\}-\{y,y\}} = \boxed{\emptyset}$$

Finally in this section, we show that the behaviours of $N = (S, T, \omega)$ under a marking M and $N_0 = (R, U, \omega_0)$, related by transformation matrices $\mathcal{T}_{R,S}$ and $\mathcal{T}_{T,U}$ as above, are equivalent, provided the marking M_0 of N_0 corresponding to M is defined as $\mathcal{T}_{R,S} \otimes M$. Note that $M_0 = \mathcal{T}_{R,S} \otimes M \Leftrightarrow M = \mathcal{T}_{S,R} \otimes M_0$. This hinges on the next lemma.

Lemma 5.5. *$\geq$ is a congruence w.r.t. $\otimes$ (by transformation matrices)*

Let G_1, G_2 be (B, C)-matrices. Then $G_1 \geq G_2 \Rightarrow \mathcal{T}_{A,B} \otimes G_1 \geq \mathcal{T}_{A,B} \otimes G_2$.

Proof: From the semipositivity of transformation matrices. ■ 5.5

Next consider a transition t of N, a mode $m \in \iota(t)$ and the U-vector $\mathcal{T}_{U,T} \otimes \underline{(t, m)}$. The latter has an entry $\{m\}$ at $u \in U$ provided $m \in \iota(u)$, and an entry $\emptyset$ otherwise. Due to the normality of U, there is at most one entry $\{m\}$, and due to the compatibility between T and U, there is at least one entry $\{m\}$. In other words, $\mathcal{T}_{U,T} \otimes \underline{(t, m)} = \underline{(u, m)}$ where u is the unique element of U with $m \in \iota(u)$. The next results relate (given) behaviour of N to (deduced) behaviour of N_0.

Lemma 5.6. *Transitions are activated in N iff their counterparts are activated in N_0*

$$M \geq \Gamma \otimes \underline{(t, m)} \quad \text{iff} \quad (\mathcal{T}_{R,S} \otimes M) \geq \underbrace{(\mathcal{T}_{R,S} \otimes \Gamma \otimes \mathcal{T}_{T,U})}_{\Gamma_0} \otimes \underbrace{(\mathcal{T}_{U,T} \otimes \underline{(t, m)})}_{\underline{(u,m)}}.$$

Proof: Multiply from the left with $\mathcal{T}_{R,S}$ to show $(\Rightarrow)$, and with $\mathcal{T}_{S,R}$ to show $(\Leftarrow)$, using Lemma 5.5 and various algebraic laws. ■ 5.6

Similarly, it can be shown that follower markings in N are counterparts of corresponding follower markings in N_0. Moreover, there are corollaries which are analogous to Corollary 5.4, relating (given) behaviour of N_0 to (deduced) behaviour of N.

We check Lemma 5.6 on net N_1. Let M_1 be the marking of N_1 shown in Figure 1 and given above in multiset and vector forms. Intuitively, M_1 activates v_2 in mode y; we check formula (6) in order to verify this. The product $(\omega|_{S_1 \times T_1}) \otimes \underline{(v_2, y)}$ is evaluated as follows:

$$\begin{array}{|c|c|} \hline \{(1,x)\} & \emptyset \\ \hline \emptyset & \{(2,y)\} \\ \hline \emptyset & \emptyset \\ \hline \emptyset & \emptyset \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \emptyset \\ \hline \{y\} \\ \hline \end{array} = \begin{array}{|c|} \hline \emptyset \\ \hline \{2\} \\ \hline \emptyset \\ \hline \emptyset \\ \hline \end{array} \qquad \text{(compare with } M_1 : \begin{array}{|c|} \hline \{1\} \\ \hline \{2\} \\ \hline \emptyset \\ \hline \emptyset \\ \hline \end{array})$$

This product is indeed $\leq M_1$. The markings which correspond to M_1 in the nets N_2, N_3 and N_4 are depicted in Figures 2, 3 and 4, respectively. We leave it to the reader to verify that these markings are the formal correspondents of M_1 and, moreover, that they respectively activate the (unique) transitions containing y in their mode set, in mode y.

6 Lattice of unfolding-equivalent representations of a net

In this section, we show that an unfolding-equivalence class of nets forms a lattice whose bottom element is a completely unfolded net, and whose top element is a completely folded net. The previous theory does not assume that some (or all) of the involved sets are nonempty. For instance, the empty net $N_\emptyset = (\emptyset,\emptyset,\emptyset)$ is unfolding-equivalent with the singleton-place net $(\{s\},\emptyset,\emptyset)$, provided $\iota(s) = \emptyset$. If $\iota(s) \neq \emptyset$, then this is no longer true, because the two sets of places, $\emptyset$ and $\{s\}$, are then no longer compatible. In this section, we restrict our attention to the 'normal' cases; we assume that the set of places and the set of transitions of a net are nonempty and finite, and that all types of places and transitions are nonempty and finite as well. Such a net will be called *elementary* if all value sets and all mode sets are singletons, i.e., if the type of every place $s\in S$ satisfies $|\iota(s)|=1$ and the type of every transition $t\in T$ satisfies $|\iota(t)|=1$. It is called *top-level* if S and T are themselves singleton sets, i.e. if $|S|=1=|T|$.

For the remainder of this section, we assume $N = (S,T,\omega)$ and $N_0 = (R,U,\omega_0) = (S_0,T_0,\omega_0)$ to be unfolding-equivalent, i.e., related by transformation matrices as in formula (7). There are two special cases. First, if R and U are singleton sets, then this determines R and U uniquely, up to renaming, from S and T, and N_0 is a top-level net. Second, if all elements of R and U have singleton type, then this again determines R and U uniquely, up to renaming, from S and T, and N_0 is elementary.

In order to describe pure unfolding and pure folding linear-algebraically, we will consider a special class of transformation matrices. Let a transformation matrix $\mathcal{T}_{A,B}$ be called an *unfolding matrix* iff every row has precisely one non-$\emptyset$ entry, i.e., in terms of multirelations,

$$\forall a \in A\, \exists_{\mathbf{1}} b \in B\colon\ \mathcal{T}_{A,B}(a,b) \neq \emptyset_{\iota(a)\times\iota(b)}$$

(where we use the shorthand $\exists_{\mathbf{1}}$ to mean 'there exists exactly one ...'), and a *folding matrix* iff its transpose is an unfolding matrix.

Lemma 6.1. *Properties of unfolding matrices*

> A transformation matrix is a unit matrix (up to renaming) if and only if it is both an unfolding and a folding matrix. Moreover, if F and G are two unfolding matrices, then so is $F\otimes G$.

Proof: The $(\Rightarrow)$ direction of the first claim is obvious by the definition of unit matrices. For the $(\Leftarrow)$ direction, assume $\mathcal{T}_{A,B}$ to be both an unfolding and a folding matrix. Assume that there are $a\in A$ and $b\in B$ with $\iota(a)\cap\iota(b)\neq\emptyset$ and $\iota(a)\neq\iota(b)$. Because of the former, there is some $v\in\iota(a)\cap\iota(b)$. Because of the latter, there is some w with $w\in\iota(a)\backslash\iota(b)$ or $w\in\iota(b)\backslash\iota(a)$. If $w\in\iota(a)\backslash\iota(b)$, by the compatibility of A and B, there is some $b'\in B$, $b'\neq b$, with $w\in\iota(b')$, and hence, $\mathcal{T}_{A,B}$ is not an unfolding matrix since both $\mathcal{T}_{A,B}(a,b)$ (which contains (v,v)) and $\mathcal{T}_{A,B}(a,b')$ (which contains (w,w)) are different from $\emptyset$. If $w\in\iota(b)\backslash\iota(a)$, similarly, $\mathcal{T}_{A,B}$ is not a folding matrix. Hence, any two types of elements of A and B are either disjoint or equal, and hence, A and B are the same up to renaming, and $\mathcal{T}_{A,B}$ is an (A,A)- (and (B,B)-) unit matrix.

For the second claim, assume that F is an (A,B)-unfolding-matrix and G is a (B,C)-unfolding-matrix. Fix $a\in A$. Let b_0 be determined as the unique element of B with $F(a,b_0){\neq}\emptyset$, and c_0 as the unique element of C with $G(b_0,c_0){\neq}\emptyset$. From the definition of matrix multiplication and the fact that both F and G are unfolding matrices it follows that $(F{\otimes}G)(a,c){\neq}\emptyset$ if and only if $c{=}c_0$, from which the claim follows. ■ 6.1

Thus, the unfolding matrices form a subgroup of the group of transformation matrices, and so do, by symmetry, the folding matrices.

Let us call N_0 an *unfolding* of N, and N a *folding* of N_0, if the transformation matrices $\mathcal{T}_{R,S}$ and $\mathcal{T}_{U,T}$ involved in the transformation of N into N_0 by (7) are unfolding matrices. For example, in terms of our running example, N_1 is an unfolding of itself, of N_2, of N_3, and of N_4; and N_2 is an unfolding of itself and of N_4, but neither of N_1 nor of N_3.

Lemma 6.2. *'Being an unfolding of' is a partial order relation*

'Being an unfolding of' is reflexive, transitive and antisymmetric.

Proof: Reflexivity follows immediately from direction ($\Rightarrow$) of the first part of Lemma 6.1. Transitivity follows immediately from the second part of Lemma 6.1. To prove antisymmetry, assume that N_0 is an unfolding of N and N is an unfolding of N_0. It follows that the corresponding transformation matrices are the inverses, and thus the transposes, of each other; since they are both unfolding matrices, direction ($\Leftarrow$) of the first part of Lemma 6.1 implies that they are unit matrices, and hence N is the same as N_0. ■ 6.2

We now show that this partial order generates a lattice.

Proposition 6.3. *Every unfolding-equivalence class of nets is a lattice*

Let N and N_0 be unfolding-equivalent nets. Then there are a net $N_l = N{\sqcap}N_0$ (l for 'lower') and a net $N_u = N \sqcup N_0$ (u for 'upper') with the following properties: N_l is an unfolding of both N and N_0, and every unfolding of N and N_0 is an unfolding of N_l; N_u is a folding of both N and N_0, and every folding of N and N_0 is a folding of N_l.

Proof: Let the place and transition sets of N and N_0 be S, T, and R, U, respectively, as before. We have $\iota(S){=}\iota(R)$ and $\iota(T){=}\iota(U)$. We construct place sets S_l, S_u and transition sets T_l, T_u, such that

$$\iota(S_l){=}\iota(S){=}\iota(R){=}\iota(S_u) \quad \text{and} \quad \iota(T_l){=}\iota(T){=}\iota(U){=}\iota(T_u).$$

S_l and T_l will be the place set and the transition set of the greatest lower bound, and S_u and T_u will be the place set and the transition set of the smallest upper bound of N and N_0, respectively. Note that it is enough to construct these sets, since the flow relations arise through multiplication with the respective transformation matrices.

For the construction of S_l, consider the following undirected graph G_l. The vertices are the values $\iota(S)$ ($=\iota(R)$). The edges are $\langle v, w\rangle$ iff

$$\exists s{\in}S\exists r{\in}R{:}\ \{v,w\}{\subseteq}\iota(s){\cap}\iota(r).$$

Note that by the compatibility of S and R, the self-loop $\langle v, v\rangle$ is always an edge of the graph. We define S_l as the set of connected components of the graph. Note that due to the normality of S and R, such connected components are always full graphs.[1] If q is a connected component of G_l, we define $\iota(q)$ as the set of its vertices. Note that if $q \neq q'$ then $\iota(q) \cap \iota(q') = \emptyset$ (so that S_l is normal) and $\bigcup_{q\in S_l} \iota(q) = \iota(S)$ (so that S_l is compatible with both S and R). The construction of T_l is analogous.

We show that the net N_l determined by S_l and T_l is the greatest lower bound of N and N_0.

First we show that N_l is indeed a lower bound; for this, it has to be shown that N_l is an unfolding of both N and N_0. Consider the transformation matrix $\mathcal{T}_{S_l,S}$; it has to be shown that this is an unfolding matrix. (The demonstration of the same property for $\mathcal{T}_{S_l,R}$ is analogous.) Let q be a connected component of G_l. It has to be shown that, for all $s\in S$,

$$\iota(q) \cap \iota(s) = \emptyset \quad \vee \quad \iota(q) \subseteq \iota(s),$$

because it is exactly under this condition that $\mathcal{T}_{S_l,S}$ is an unfolding matrix. So suppose this is not true for some s; then there are $v \in \iota(q) \cap \iota(s)$ and $w \in \iota(q) \setminus \iota(s)$, and $\langle v, w\rangle$ cannot be an edge of G_l, contradicting the definition of q and the fact that q is a full graph.

Next we show that any other net K, which is an unfolding of both N and N_0, is also an unfolding of N_l. It has to be proved that for any two places p of K and q of N_l, $\iota(p) \cap \iota(q) = \emptyset$ or $\iota(p) \subseteq \iota(q)$. Assume the contrary, i.e., assume the existence of $v \in \iota(p) \cap \iota(q)$ and $w \in \iota(p) \setminus \iota(q)$. Then $v \in \iota(q)$ and $w \notin \iota(q)$ implies that $\langle v, w\rangle$ is not an edge of G_l. However, since K is an unfolding of N, there is a place s of N such that $\iota(p) \subseteq \iota(s)$, and since K is an unfolding of N_0, there is a place r of N_0 such that $\iota(p) \subseteq \iota(r)$. Hence, since both v and $w$are in $\iota(p)$, by the definition of G_l, $\langle v, w\rangle$ is an edge of G_l, which yields a contradiction.

The demonstration of the analogous property for transitions is analogous. This shows that any two unfolding-equivalent nets have a greatest lower bound, and thus ends the proof of the first part of the claim.

The second half of the proof actually follows by standard lattice-theoretic – but non-constructive – arguments from the first half (see, for example, Exercise 2.10.13 in [2]). In order to show how the smallest upper bound can be constructed, however, we give this construction explicitly.

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[1] To see this, consider an edge $\langle v, w\rangle$ and another edge $\langle w, x\rangle$. We claim that $\langle v, x\rangle$ is also an edge. By the definition of an edge, $\{v, w\} \subseteq \iota(s_1) \cap \iota(r_1)$ and $\{w, x\} \subseteq \iota(s_2) \cap \iota(r_2)$ for some $s_1, s_2 \in S$ and $r_1, r_2 \in R$. By the normality of S, $w \in \iota(s_1) \cap \iota(s_2)$ implies $\iota(s_1) = \iota(s_2)$, and by the normality of R, $w \in \iota(r_1) \cap \iota(r_2)$ implies $\iota(r_1) = \iota(r_2)$. Hence $\{v, x\} \subseteq \iota(s_i) \cap \iota(r_j)$, $1 \leq i, j \leq 2$, and thus, $\langle v, x\rangle$ is an edge of G_l. To see that a connected component is full, consider two vertices of some connected component. Then there is an undirected path between these two vertices, and the argument just given can be applied iteratively along this path.

For the construction of S_u, we define an undirected graph G_u. The vertices are again the values $\iota(S)$ $(=\iota(R))$. The edges are $\langle v, w\rangle$ iff

$$(\exists s{\in}S\colon \{v,w\}{\subseteq}\iota(s)) \vee (\exists r{\in}R\colon \{v,w\}{\subseteq}\iota(r)).$$

Note that because $v \in \iota(S){\cap}\iota(R)$, $\langle v, v\rangle$ is always an edge of the graph. We define S_u as the set of connected components of the graph. In this case, such connected components are not necessarily full graphs. If q is a connected component of G_u, we define $\iota(q)$ as the set of its vertices. Again, this set is normal and compatible with both S and R. The construction of T_u is analogous.

We omit the proof that the net N_u determined by S_u and T_u is indeed the smallest upper bound of N and N_0. ■ 6.3

For example, in terms ouf our running example, $N_1 = N_2 \sqcap N_3 = N_1 \sqcap N_4$ and $N_4 = N_2 \sqcup N_3$. The bottom of the lattice is always an elementary net, and the top of the lattice is always a top-level net. Note that N_1 is elementary and N_4 is top-level.

Figure 5 explains the constructions used in the proof of Proposition 6.3. Consider the nets N_2 and N_3 of our example. We wish to use the proof of Proposition 6.3 in order to construct their greatest lower and smallest upper bounds. Figure 5 shows the graphs G_l and G_u which are constructed for this purpose; all edges except the self-loops are shown. Thus, G_l has four connected components (and hence, the greatest lower bound of N_2 and N_3, which is in fact N_1, has four places) and G_u has only one connected component (and hence, the smallest upper bound of N_2 and N_3, which is in fact N_4, has only one place).

Note that the four connected components of G_l are, trivially, full graphs, while the only connected component of G_u is not a full graph.

Figure 5. The graphs G_l and G_u used for constructing the glb and the lub of N_2 and N_3

Remark 6.4. Relation to arc-weighted place/transition nets

Elementary nets in the sense defined in Section 6 and arc-weighted P/T-nets [8] are in close correspondence with each other. To transform an arc-weighted P/T-net into an elementary net, attach to every place a unique singleton value set (e.g. $\{s\}$ where s is its name) and to every transition a unique singleton mode set (e.g. $\{t\}$ where t is its name). For arcs (s,t), add a multiset containing the pair (s,t) as many times as specified by its weight, and do the same for arcs (t,s). To obtain an arc-weighted P/T-net from an elementary net, apply the converse construction, simply neglecting all value sets and all mode sets and treating all markings as multisets over $\{\emptyset\}$. For example, the arc-weighted correspondent of N_1 would be a net with four places, two transitions and four arrows. For instance, I_1 (defined above) corresponds to an ordinary S-invariant of this net, giving weight 0 to places q_1 and q_3 and weight 2 to places q_2 and q_4. ■ 6.4

7 Conclusion

The calculus we have defined in this paper has been derived by 'currying' from the usual definition of coloured nets [6, 10]. It uses a single matrix product – not different from ordinary matrix product – for all linear-algebraic purposes, which, as we have demonstrated, include folding/unfolding and other equivalent transformations. All linear-algebraic notions can be generalised directly from known place/transition-net-based linear algebra. Transformations between equivalent high-level representations of the same (elementary) net are achieved by orthogonal matrices spanning a multiplicative subgroup of the ring of matrices. Such matrices serve the same purpose as the more well-known orthogonal transformations in linear algebra, and their properties are similar to those of the sine/cosine matrices used for coordinate transformations in vector geometry (and they may also be related to [7]). When comparing our results with chapter 9 of [8], what we have achieved in addition is,

1. the incorporation of different value sets and mode sets (their exact treatment was only hinted at in [8]), which leads to a closer connection with Jensen's coloured nets (see Remark 4.1),
2. and the incorporation of orthogonal equivalence transformations.

We have finally shown that an unfolding-equivalence class of (finite) nets forms a lattice. The results of this paper have also been embedded in a more general linear-algebraic framework in [12].

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