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HOOK-FREE COLORINGS AND A PROBLEM OF HANSON

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Hanson posed the following problem: What is the minimum number $\chi(n)$ of colors needed to color all subsets of an n-set such that there is no monochromatic triple A, B, C with $A\cup B=C$? It is known that $\chi(n)\leqq\lceil(n+1)/2\rceil$, while Erdős and Shelah proved $\chi(n)\geqq\lceil(n+1)/4\rceil$. Their proof suggests the following notion: Let C be any finite plane point-configuration. The hook-free coloring number $\chi(C)$ is the smallest number of colors needed for C such that no monochromatic hooks arise, i.e. if (c_x, c_y) are the coordinates of point $c\in C$, then there are no 3 distinct points $a, b, c\in C$ with $a_x=b_x<c_x$, $b_y=c_y<a_y$. In this paper $\chi(R_{m,n})$ is determined exactly for an $m\times n$-rectangle, and asymptotically for the triangular staircase. As a corollary one obtains $\chi(n)\geqq 0.293n$.

1. Introduction

The motivation for this paper is the following problem which Professor Erdős attributes to Hanson (see [2]): Color all subsets of $X=\{1, ..., n\}$ such that any color class is unionfree, i.e. there are no distinct subsets A, B, C with $A\cup B=C$. What is the minimum number $\chi(n)$ of colors needed?

The following coloring is well-known and probably due to Hanson himself: Put all 1-subsets, 3-subsets, 7-subsets, ... into color class 1, all 2-subsets, 5-subsets, 11-subsets, ... into class 2, and, in general, all $2i$-subsets, $(4i+1)$-subsets, $(8i+3)$-subsets, ... into color class $i+1$, $1\leqq i\leqq n/2$. This procedure obviously provides a coloring as required and uses $\lceil(n+1)/2\rceil$ colors. Thus

$$\chi(n)\leqq\left\lceil\frac{n+1}{2}\right\rceil.$$

What about lower bounds? In [2], Erdős and Shelah showed that $\chi(n)\geqq\lceil(n+1)/4\rceil$. Their beautiful argument (which prompted the present paper) runs as follows. Let us just consider the subfamily $\mathcal{J}\subseteqq 2^n$ of all intervals $[i,j]$, $1\leqq i\leqq n/2<j\leqq n$. We associate to $\mathcal{J}$ the complete bipartite graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$, with $[i,j]$ corresponding to the edge joining i and j. We arrange the numbers $1, ..., \lfloor n/2\rfloor$ on the left from top to bottom, and $\lfloor n/2\rfloor+1, ..., n$ on the right from bottom to top (see figure 1 for $n=11$).

Suppose $\mathcal{J}$ is colored union-free. Consider numbers i, k, l, j with $1\leqq i<k\leqq n/2<l<j\leqq n$. Then $[i,j]=[i,l]\cup[k,j]$, and hence these three intervals cannot

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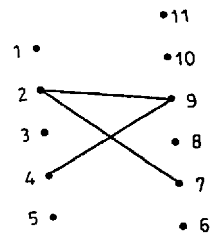

Fig. 1

all receive the same color. For the complete bipartite graph this means that three edges forming a "hook" cannot all be colored the same, where by a hook we mean a path of length 2, going from left to right upward, then to left and finally to right downward (see figure 1 with $i=2$, $k=4$, $l=7$, $j=9$). Observe now that any cycle *must* contain such a configuration, implying that any color class in the graph is a forest. Hence a color class cannot contain more than $n-1$ edges which means that the number of colors is at least

$$\frac{\left\lfloor \frac{n^2}{4} \right\rfloor}{n-1} \geqq \frac{n+1}{4}.$$

Let us recast and slightly generalize Erdős' argument. It is convenient to look at our problem in the dual form, i.e. we want to color the sets *intersection-free.* Whenever $A \cap B = C$ for three distinct sets then A, B, C cannot all receive the same color. By exchanging the color of each set with that of its complement, it is clear that the two problems are equivalent.

Consider the restricted problem of coloring the set $\mathscr{I}$ of all intervals $[i, j]$, $1 \leqq i \leqq j \leqq n$, intersection-free. Thus $|\mathscr{I}| = \binom{n+1}{2}$. To $\mathscr{I}$ we associate a staircase SC_n of length n, numbering the rows and columns $1, \ldots, n$, and letting the interval $[i, j]$ correspond to the pair (i, j). Figure 2 shows SC_5.

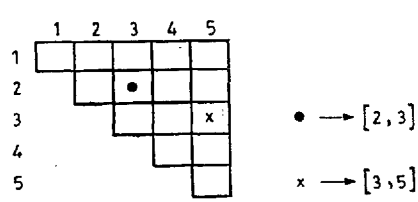

Fig. 2

Consider now an intersection-free coloring of $\mathscr{I}$. We claim that this corresponds precisely to the absence of monochromatic *hooks* in SC_n as defined in the abstract. Indeed, $[i, j] = [k, l] \cap [s, t]$ for three distinct intervals implies $i = \max(k, s)$, $j = \min(l, t)$. If, say, $i = k$ then $j = t$, and we obtain the hook as in figure 3.

Conversely, any such hook corresponds precisely to an intersection relation. Thus we conclude that the problem of coloring $\mathscr{I}$ intersection-free is exactly the same as coloring the staircase "hook-free".

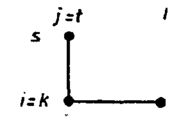

Fig. 3

It is therefore reasonable to ask for the *hook-free coloring number* $\chi(\mathscr{C})$ in the sense just described for any plane point-configuration $\mathscr{C}$. In Section 2 we obtain the precise result for a rectangular array $R_{m,n}$ of size m by n, and apply this to the staircase SC_n in Section 3.

2. Hook-free colorings of rectangles

Let $R(m, n)$ be a rectangular array of size m by n. By $\chi(m, n)$ we denote the minimum number of colors needed to color the points of $R(m, n)$ such that no monochromatic hooks arise.

Theorem 1. $\chi(1, 1)=1$ *and for* m, n *with* $m+n\geqq 3$ *we have*

$$\chi(m, n) = \left\lceil \frac{mn-1}{m+n-2} \right\rceil. \tag{1}$$

Proof. The result $\chi(1, 1)=1$ is obvious, so let us assume $m+n\geqq 3$. We first establish that $\chi(m, n)\geqq \lceil (mn-1)/(m+n-2)\rceil$. Consider any hook-free coloring g of $R(m, n)$ and suppose it needs t colors. Since, obviously, $\chi(m, n)\leqq \min(m, n)$, we may assume $t\leqq n$. Put a circle around a point $p=(i, j)$ iff there is another point q in the same row to the right of p which received the same color as p, i.e. $q=(i, l)$ with $l>j$ and $g(p)=g(q)$.

We note two facts:

a) In any row there are at least $n-t$ circles, since at most one point of each color remains uncircled.
b) In any column different from the last there are no two circles of the same color, since otherwise we would plainly obtain a monochromatic hook, considering our circle rule. Hence there are at most t circles in any column different from the last. Furthermore, if there are precisely t circles in a column, then again by the circle rule, one of these circles must appear in the first row.

Now let K be the total number of circles in rows 2 to m. By a) and b) we obtain

$$(m-1)(n-t) \leqq K \leqq (t-1)(n-1).$$

Rearranging the two sides of this inequality yields precisely $t\geqq (mn-1)/(m+n-2)$, and thus $\chi(m, n)\geqq \lceil (mn-1)/(m+n-2)\rceil$.

To prove the upper bound we define the following coloring f. We number the rows $0, 1, \ldots, m-1$ and the columns $0, 1, \ldots, n-1$, where again we assume $m+n \geqq 3$, and set $r = \lceil (mn-1)/(m+n-2) \rceil$. Now we define f by

$$f(i,j) = \left\lceil \frac{(n-1)i+(m-1)j}{m+n-2} \right\rceil \bmod r \tag{2}$$

for $0 \leqq i \leqq m-1$, $0 \leqq j \leqq n-1$.

We have to show that f is hook-free. Suppose there is a monochromatic hook (i,j), (k,j), (k,l) with $i<k$, $j<l$. By the definition of f we have

$$\left\lceil \frac{(n-1)i+(m-1)j}{m+n-2} \right\rceil \equiv \left\lceil \frac{(n-1)k+(m-1)j}{m+n-2} \right\rceil \equiv \left\lceil \frac{(n-1)k+(m-1)l}{m+n-2} \right\rceil \bmod r. \tag{3}$$

Since $i<k$, $j<l$ it is clear that the first term in (3) is less than or equal to the second which is less than or equal to the third. It is now an easy matter to deduce from the congruence conditions in (3) that, in fact, they are all equal. Hence we obtain from $i<k$, $j<l$

$$\left\lceil \frac{(n-1)i+(m-1)j}{m+n-2} \right\rceil = \left\lceil \frac{(n-1)k+(m-1)l}{m+n-2} \right\rceil \geqq \left\lceil \frac{(n-1)(i+1)+(m-1)(j+1)}{m+n-2} \right\rceil =$$

$$= \left\lceil \frac{(n-1)i+(m-1)j+m+n-2}{m+n-2} \right\rceil =$$

$$= \left\lceil \frac{(n-1)i+(m-1)j}{m+n-2} \right\rceil + 1,$$

a contradiction. ∎

Observe that the coloring as described in the proof is completely determined by the coloring of the first row and first column, since by (2) $f(i+1, j+1) \equiv f(i,j)+1 \pmod r$.

Corollary. *For a square $R(n, n)$ the hook-free chromatic number is*

$$\chi(n, n) = \left\lceil \frac{n+1}{2} \right\rceil. \tag{4}$$

3. Hook-free coloring of staircases

Let us return to the staircase SC_n introduced in Section 1. We denote by $\chi(SC_n)$ the minimum number of colors required in any hook-free coloring.

Theorem 2. *For the staircase SC_n $(n \geqq 2)$ we have*

$$\left\lceil n - \frac{1}{2} - \sqrt{\frac{n^2}{2} - \frac{3n}{2} + \frac{5}{4}} \right\rceil \leqq \chi(SC_n) \leqq \left\lceil \frac{n+1}{3} \right\rceil. \tag{5}$$

Proof. For the lower bound we use the same argument as in the proof of Theorem 1. Suppose a proper coloring uses t colors, where we may assume $t \leqq n-1$ since, obviously, $\chi(SC_n) \leqq n-1$. Circle the points as before, and let K be the total number of circles in rows 2 to n. Any row r_i ($i \geqq 2$) contains at least $\max(n+1-i-t, 0)$ circles. Any column c_j ($j \leqq n-1$) contains at most $\min(j, t)$ circles, and if it contains t circles then one of them must appear in row 1. Hence

$$(6) \qquad (1+2+\ldots+(n-1-t)) \leqq K \leqq (1+2+\ldots+(t-2))+(t-1)(n-t).$$

Rearranging the inequality (6) yields

$$(7) \qquad t^2-(2n-1)\,t+\frac{n^2+n}{2}-1 \leqq 0.$$

It follows that t must be at least as large as the smaller root of the quadratic equation in (7), and this root is precisely the lower bound in (5).

To prove the upper bound, consider the case $n \equiv 0 \pmod 3$ first. We superimpose on SC_n a square Q of side length $(2n)/3$ touching the point in the upper right-hand corner. See figure 4 for $n=9$. By (4), Q can be colored in a hook-free manner with $n/3+1=\lceil (n+1)/3 \rceil$ colors. Take the restriction Q' of this coloring to the part that is inside SC_n. The small upper staircase U and the lower staircase L are both of length $n/3$. Hence by taking a color for each of the rows of U and a color for each of the columns in L, we may extend the coloring of Q' to all of SC_n.

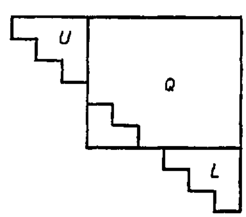

Fig. 4

In the case $n \equiv 1 \pmod 3$ we superimpose a square of length $(2n-2)/3$ and when $n \equiv 2 \pmod 3$ we take one of size $(2n-1)/3$. In each case, $\lceil (n+1)/3 \rceil$ colors suffice, and the proof is complete. ∎

Since the lower bound argument furnished the exact result for rectangular arrays, it seems probable that the same holds for SC_n, at least asymptotically. A rather complicated argument can be used to show that

$$(8) \qquad \chi(SC_n) \leqq \left\lceil n-\frac{1}{2}-\sqrt{\frac{n^2}{2}-n+\frac{1}{4}} \right\rceil + 8,$$

thus establishing

$$(SC_n) \sim \left(1-\frac{1}{\sqrt{2}}\right) n \sim 0.293n \quad (n \to \infty).$$

We just give the idea of this coloring. We choose a number t such that $n-1/2-\sqrt{(n^2/2)-n+1/4}\leqq t<3n/10$ (n large enough). The coloring is done in two steps: First we mark all positions of the staircase which may receive colors that also appear to the right in the same row (the circled points in our proof of Theorem 2). Secondly, we color all positions one by one observing the marked positions in step 1 and not creating monochromatic hooks. At the end, not all marked positions are also circled in the sense above, which accounts for the fact that we cannot achieve the lower bound (5) exactly, but only asymptotically.

The lower bound (5) gives the correct value of $\chi(SC_n)$ for n up to 21, and it is quite plausible that it is exact for all n.

By our remarks in Section 1 we have $\chi(SC_n)\leqq\chi(n)$, and thus

$$\frac{\chi(n)}{n}\geqq 0.293 \quad (n\to\infty), \tag{9}$$

a result which has also been obtained by Kleitman and West (personal communication).

It was suggested by Hanson (see [2]) that a careful analysis of the argument by Erdős and Shelah presented in Section 1 would, in fact, yield $\chi(n)\geqq n/3$ $(n\to\infty)$, but as (8) shows, this is not the case. It has come to our knowledge, however, that Kleitman bettered (9) by proving $\chi(n)/n\geqq\log 2/2\sim 0.349$ $(n\to\infty)$, using the Erdős—Ko—Radó theorem [1]. Hanson himself conjectured that the trivial coloring mentioned in Section 1 cannot be improved asymptotically, i.e. $\chi(n)/n\sim 1/2$; but this remains open.

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