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A NATURAL DIFFERENTIAL OPERATOR ON CONIC SPACES

Daniel Grieser

Institut für Mathematik
Carl von Ossietzky Universität Oldenburg
26111 Oldenburg, Germany

Abstract. We introduce the notion of a conic space, as a natural structure on a manifold with boundary, and define a natural first order differential operator, ${}^c d_\partial$, acting on boundary values of conic one-forms. Conic structures arise, for example, from resolutions of manifolds with conic singularities, embedded in a smooth ambient space. We show that pull-backs of smooth ambient one-forms to the resolution are ${}^c d_\partial$-closed, and that this is the only local condition on one-forms that is invariantly defined on conic spaces. The operator ${}^c d_\partial$ extends to conic Riemannian metrics, and ${}^c d_\partial$-closed conic metrics have important geometric properties like the existence of an exponential map at the boundary.

1. **Introduction.** Let X be a manifold with boundary. The conic tangent bundle, denoted ${}^cTX$ and defined below, is a natural vector bundle over X of rank $\dim X$. It is isomorphic to the 'usual' tangent bundle TX, though not canonically. We call the pair $(X, {}^cTX)$ a conic space. This notion is motivated by the desire to have a coordinate invariant framework for calculations in the geometry and analysis on manifolds with conic singularities: If M is a manifold with conic singularities, embedded in a smooth manifold Z, and M_{res} is the resolution of M obtained by blowing up the conic points (for precise definitions see Section 4) then the fibres of ${}^cTM_{\mathrm{res}}$ correspond naturally

- to the tangent spaces of M at smooth points of M and
- to limits of tangent spaces of M, from various directions, at the singular points of M,

see Corollary 1. That is, ${}^cTM_{\mathrm{res}}$ is nothing but the Nash bundle of M. Therefore, in this case ${}^cTX$ has a very intuitive geometric meaning.

Differential geometric objects such as differential forms or Riemannian metrics have natural analogues on a conic space, obtained by replacing the usual tangent bundle TX with ${}^cTX$. We call these conic forms, conic metrics etc. In the case where $X = M_{\mathrm{res}}$, smooth forms or metrics on the ambient space Z, when restricted to the smooth part of M and lifted to the interior of X, extend smoothly to corresponding conic objects on X. They also satisfy an infinite number of compatibility conditions at ∂X, stemming from the differential structure of the ambient space Z at the conic singularities of M. These conditions are relations between derivatives of various orders of the coefficients of the object under consideration, in any local coordinate system.

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The question arises if any of these conditions can be formulated invariantly in the category of conic spaces. We show that this is the case precisely for one of these conditions. This is a first order condition, expressible as ${}^c d_\partial \omega = 0$, where ${}^c d_\partial$ is a natural first order differential operator on any conic space X, derived from the exterior derivative and acting at the boundary ∂X. The operator ${}^c d_\partial$ is defined on conic one-forms but extends to higher forms, metrics etc. It can be viewed as a directional derivative in directions tangent to the boundary, and as such is scalar valued. See Section 3 for the definition of ${}^c d_\partial$ and proofs of its properties stated above, and Theorem 4.1 for the statement on uniqueness.

It is clear that the condition on a conic Riemannian metric g to be ${}^c d_\partial$-closed, i.e. to satisfy ${}^c d_\partial g = 0$, should have geometric implications. We formulate two such implications in Section 5: The existence of a certain normal form for g near the boundary, and the existence of an exponential map based at the boundary, see Theorem 5.2. Here we use a theorem of Melrose and Wunsch [3] which implies the equivalence of these two conditions.

Besides ${}^c TX$ there is another natural structure on a manifold with boundary, the b-tangent bundle ${}^b TX$ introduced by Melrose [4]. There are many similarities between conic and b-geometry. In particular, Melrose introduced an 'exactness' condition on b-metrics which is equivalent to the existence of a certain normal form near the boundary. Exactness is also a first order condition at the boundary. However, there are some important differences between the b- and the conic case: In the conic case, the closedness condition is equivalent to an exactness condition, see Remark 2, while the exactness condition in the b-case is global on the boundary and has no local equivalent. Another difference is that in the conic case the closedness condition for metrics is induced from a corresponding condition on one-forms. There appears to be no corresponding condition on one-forms in the b-case.

2. Definition of ${}^c TX$.

We will introduce all objects invariantly and also express them in local coordinates. Local coordinates $r, \phi_1, \cdots, \phi_m$ near a boundary point of X are called *boundary coordinates* if r is a boundary defining function (so $\partial X = r^{-1}(0)$, $r \geq 0$ on X and $dr \neq 0$ at ∂X). Then $\phi_1, \cdots, \phi_m$ are, when restricted to ∂X, local coordinates on ∂X.

It is easiest to define the dual bundle ${}^c T^* X$ first and then to dualize. Let $i : \partial X \to X$ be the inclusion and

$$\mathcal{M} = \{\alpha \in \Gamma(T^* X) : i^* \alpha = 0\}$$

be the space of one-forms on X vanishing on vectors tangent to the boundary. The space $\mathcal{M}$ is clearly a $C^\infty(X)$-module. Also, it is locally free. This is clear near any interior point of X (since one just gets all one-forms there), and in a neighborhood U of a boundary point one has, with respect to any boundary coordinate system,

$$\mathcal{M}_{|U} = \{a\, dr + \sum_{i=1}^m c_i\, d\phi_i : \ a, c_i \in C^\infty(U),\ c_{i|U\cap\partial X} = 0\ \forall i\},$$

and since $c_i \in C^\infty(U)$, $c_{i|U\cap\partial X} = 0 \Leftrightarrow c_i = r b_i$ for some $b_i \in C^\infty(U)$, the module $\mathcal{M}_{|U}$ is spanned freely by $dr, r d\phi_1, \cdots, r d\phi_m$ over $C^\infty(U)$. Since locally free $C^\infty(X)$-modules correspond to vector bundles, we may define:

Definition 2.1. Let X be a manifold with boundary. The conic cotangent bundle, ${}^c T^* X$, is the vector bundle whose space of sections is given by $\mathcal{M}$, defined above. The conic tangent bundle, ${}^c TX$, is the dual of ${}^c T^* X$.

An explicit model for ${}^cT^*_qX$, for $q \in X$, is ${}^cT^*_qX = \mathcal{M}/\mathfrak{m}_q\mathcal{M}$, $q \in X$, where $\mathfrak{m}_q \subset C^\infty(X)$ is the maximal ideal of functions vanishing at q.

If $r, \phi_1, \cdots, \phi_m$ are local boundary coordinates then $dr, rd\phi_1, \cdots, rd\phi_m$ is a local basis of (the space of sections of) ${}^cT^*X$; we write

$$\partial_r, \frac{1}{r}\partial_{\phi_1}, \cdots, \frac{1}{r}\partial_{\phi_m}$$

for the dual local basis of (the space of sections of) ${}^cTX$.

The precise meaning and a justification for this notation becomes apparent from a careful examination of the relationship of ${}^cTX$ and TX:

Lemma 2.2 (and Definition)**.** *Interpreting a section of ${}^cT^*X$ as a one-form on X defines a natural bundle homomorphism*

$$\iota : {}^cT^*X \to T^*X,$$

which is an isomorphism over the interior of X but over $q \in \partial X$ has one-dimensional range

$$L^*_q := \operatorname{ran} \iota_q \subset T^*_qX$$

equal to the conormal bundle of ∂X. In any boundary coordinate system, one has $L^*_q = \operatorname{span}\{dr\}$.

The dual homomorphism $\iota' : TX \to {}^cTX$ identifies smooth vector fields on X with certain sections of ${}^cTX$. It is an isomorphism over the interior of X but at any boundary point q has one-dimensional range

$$L_q = \operatorname{span}\{\partial_r\} \subset {}^cT_qX$$

(with respect to any boundary coordinate system) and is naturally dual to L^*_q. We have $L_q \cong T_qX/T_q\partial X$ naturally.

We call L the canonical line bundle *of X.*

We will see later that in a blow-up situation L corresponds to the canonical line bundle over projective space.

Proof. Clearly, ι is a well defined bundle homomorphism, and an isomorphism over the interior. In boundary coordinates, it sends $a\,dr + \sum_{i=1}^m b_i\, rd\phi_i \in {}^cT_qX$ ($a, b_i \in \mathbb{R}$ here) to itself, interpreted as element of T_qX. At the boundary, i.e. $r = 0$, one has

$$\iota\left(a\,dr + \sum_{i=1}^m b_i\, rd\phi_i\right) = a\,dr \quad \text{at } \partial X. \tag{1}$$

This proves the claims for L^*_q. The rest is clear by definition of the dual map, with L_q the orthogonal space to $\ker \iota_q = \operatorname{span}\{rd\phi_1, \cdots, rd\phi_m\}$ with respect to the natural pairing of ${}^cT_qX$ and ${}^cT^*_qX$. The last statement follows from the fact that $\ker \iota'_q = T_q\partial X$ and general nonsense. □

Concretely, the last statement of the lemma means: A smooth vector field on X, $a\,\partial_r + \sum_{i=1}^m c_i\,\partial_{\phi_i}$, can be written $a\,\partial_r + \sum_{i=1}^m (rc_i)\,\frac{1}{r}\partial_{\phi_i}$. Thus, interpreted as section of ${}^cTX$, it reduces to $a\,\partial_r$ at any boundary point (since $rc_i = 0$ for $r = 0$).

3. **The boundary differential for ${}^cTX$.**

3.1. **Definition of ${}^c d_\partial$.** In addition to the line bundle L there is more structure canonically associated with the conic tangent bundle: A differential operator along the boundary. In order to define it we first compute the exterior derivative of a one-form which is a section of ${}^cT^*X$.

$$d\left(a\,dr + \sum_i b_i\, r d\phi_i\right) = \sum_i a_{\phi_i}\, d\phi_i \wedge dr + \sum_i (b_i r)_r\, dr \wedge d\phi_i + \sum_{i,j} (b_i r)_{\phi_j}\, d\phi_j \wedge d\phi_i$$
$$= \sum_i (a_{\phi_i} - b_i)\, d\phi_i \wedge dr + O(r) \tag{2}$$

where a_{ϕ_i} denotes the partial derivative and $O(r)$ denotes r times a smooth section of $\bigwedge^2 T^*X$. We are interested in the leading term on the right, i.e. in the restriction to the boundary. We see that it is already determined by the restrictions of a and the b_i to ∂X, and that it only contains terms of the form $d\phi_i \wedge dr$, so the following definition makes sense.

Definition 3.1. The *conic boundary differential* is the operator

$${}^c d_\partial : \Gamma({}^cT^*_{\partial X}X) \to \Gamma(T^*\partial X \otimes L^*)$$

defined by extending an element of $\Gamma({}^cT^*_{\partial X}X)$ to the interior of X, applying the exterior derivative d and restricting to the boundary.

Here $T^*\partial X \otimes L^* \subset \bigwedge^2 T^*_{\partial X}X$ via $\nu \otimes \nu' \mapsto \nu \wedge \nu'$. Whenever convenient, we also consider ${}^c d_\partial$ as acting on $\Gamma({}^cT^*X)$; then it is just d followed by restriction to the boundary.

We may also view ${}^c d_\partial$ as defining a *directional derivative along the boundary.* For any vector field $v \in \Gamma(T\partial X)$ this is given by the operator

$$\iota_v \circ {}^c d_\partial : \Gamma({}^cT^*_{\partial X}X) \to \Gamma(L^*)$$

where ι_v is interior product, i.e. $(\iota_v \omega)(w) = \omega(v, w)$. From (2), in coordinates,

$${}^c d_\partial \left(a\,dr + \sum_i b_i\, r d\phi_i\right) = \sum_i (a_{\phi_i} - b_i)\, d\phi_i \otimes dr. \tag{3}$$

Note that when we fix a boundary defining function r we can turn ${}^c d_\partial$ into an operator $\Gamma({}^cT^*_{\partial X}X) \to \Gamma(T^*\partial X)$, by contracting with ∂_r (the dual basis of L for the bass dr of L^*). This just means leaving out dr on the right in (3). The directional derivative is then simply scalar-valued.

From (3) or from the invariant definition we have the product rule

$${}^c d_\partial (f\alpha) = f\, {}^c d_\partial \alpha + df \otimes \alpha \tag{4}$$

for $f \in C^\infty(\partial X)$ and $\alpha \in \Gamma({}^cT^*_{\partial X}X)$. Strictly speaking, on the right we should write $\iota(\alpha)$ where ι is the operator in (1) which from a conic one-form only retains its dr-part, hence takes values in L^*. To simplify notation we leave out ι here and below.

${}^c d_\partial$ extends to higher tensors via the Leibniz rule. We only consider symmetric 2-tensors here.

Definition 3.2. The conic boundary differential extends to conic symmetric 2-tensors as

$${}^c d_\partial : \Gamma(S^2\, {}^cT^*_{\partial X}X) \to \Gamma(T^*\partial X \otimes L^* \otimes L^*)$$

by setting

$$^c d_\partial(\alpha \otimes \beta) = {}^c d_\partial(\alpha) \otimes \beta + {}^c d_\partial(\beta) \otimes \alpha.$$

Again α, β on the right should be understood as $\iota(\alpha)$, $\iota(\beta) \in \Gamma(L^*)$. This is well-defined (i.e. for example plugging in $f\alpha \otimes \beta$ gives the same result as $\alpha \otimes f\beta$, for a function f on ∂X) as can easily be checked using (4).

In boundary coordinates we calculate from (3)

$$^c d_\partial \left(A dr^2 + 2 \sum_i B_i dr\, r d\phi_i + \sum_{i,j} C_{ij} r d\phi_i\, r d\phi_j \right) = \\ = \sum_i (A_{\phi_i} - 2B_i)\, d\phi_i \otimes dr \otimes dr. \quad (5)$$

Again, by fixing a boundary defining function the boundary differential turns into a real-valued directional derivative along the boundary.

Remark 1. There is a variant of $^c d_\partial$: Fixing a boundary defining function r consider the operator $^c(rd)_\partial : \Gamma({}^cT^*_{\partial X}X) \to \Gamma(\bigwedge^2 {}^cT^*_{\partial X}X)$. This is defined by restricting the image $rd\omega$, for $\omega \in \Gamma({}^cT^*X)$, as a *conic* 2-form to the boundary, as opposed to $^c d_\partial$, where we restricted $d\omega$ as a 'regular' 2-form. This has the effect that the last term on the right side of (2), which disappears in $^c d_\partial \omega$, is still present in $^c(rd)_\partial \omega$, as $\sum_{i<j} [(b_i)_{\phi_j} - (b_j)_{\phi_i}]\, r d\phi_j \wedge r d\phi_i$.

In the sequel we are mainly interested in forms satisfying $^c d_\partial \omega = 0$. This is equivalent to $^c(rd)_\partial \omega = 0$ since $a_{\phi_i} = b_i\, \forall i$ implies $(b_i)_{\phi_j} = (b_j)_{\phi_i}\, \forall i, j$. So in this regard $^c d_\partial$ and $^c(rd)_\partial$ are equivalent.

However, $^c(rd)_\partial$ is slightly less natural than $^c d_\partial$ since it involves the choice of a boundary defining function. Also, it is not clear how to extend $^c(rd)_\partial$ to symmetric two-tensors.

Remark 2. For conic 1-forms, $^c d_\partial$-closedness is equivalent to an exactness condition: In local coordinates, $\omega = a\, dr + \sum_i b_i\, r d\phi_i$, we have from (3) that $^c d_\partial \omega = 0 \iff a_{\phi_i} = b_i\, \forall i \iff \omega = d(ra) + O_{^cT^*X}(r)$. Here a is defined globally at the boundary – after choice of a boundary defining function –, as inner product of ω with ∂_r. In other words, the sequence $rC^\infty(X) \xrightarrow{d_\partial} \Gamma({}^cT^*_{\partial X}X) \xrightarrow{^c d_\partial} \Gamma(T^*\partial X \otimes L^*)$ is exact at the middle term, where d_∂ denotes d followed by restriction. Here $rC^\infty(X)$ could be replaced by $rC^\infty(\partial X)$ or more invariantly by $\Gamma(L^*)$, and then one obtains a short exact sequence, but there is no apparent use of this.

3.2. **Naturality.** Let X, Y be manifolds with boundary. We denote by r_X, L^*_X etc. a boundary defining function for X, the line bundle L^* for X etc., and similarly for Y. A natural class of maps between X, Y are the *interior b-maps*. These are smooth maps $f : X \to Y$ satisfying

$$f^* r_Y = r_X^e\, h$$

for some $e \in \mathbb{N}$ and $h \in C^\infty(X)$, $h > 0$. Intuitively, this means that as $q \to \partial X$ to order one, the image $f(q)$ approaches ∂Y to order e. In particular, $f(\partial X) \subset \partial Y$. Recall that an *immersion* is a smooth map f whose differential $f_{*q} : T_qX \to T_{f(q)}Y$ is injective for each $q \in X$. An interior b-map $f : X \to Y$ which is an immersion must have $e = 1$ and is called an *immersed p-submanifold* of Y.

Proposition 1. *The conic boundary differential is natural with respect to interior b-maps. More precisely, let $f : X \to Y$ be an interior b-map of manifolds with boundary. Then the pull-back $f^* : \Gamma(T^*Y) \to \Gamma(T^*X)$ induces maps*

$$f^* : \Gamma({}^cT^*Y) \to \Gamma({}^cT^*X), \quad f^* : \Gamma(L_Y^*) \to \Gamma(L_X^*),$$

and $f^* \circ {}^cd_{\partial Y} = {}^cd_{\partial X} \circ f^*$.

Furthermore, if $f : X \to Y$ is an immersed p-submanifold then $f_{*q} : {}^cT_qX \to {}^cT_{f(q)}Y$ is injective for all $q \in X$ and maps L_{Xq} to $L_{Y f(q)}$.

Proof. If $\alpha \in \Gamma({}^cT^*Y)$ and $q \in \partial X$, $v \in T_q\partial X$ then $(f^*\alpha)_q(v) = \alpha_{f(q)}(df(v)) = 0$ since $f : \partial X \to \partial Y$, so $f^*\alpha \in \Gamma({}^cT^*X)$. The second statement is obvious, and the naturality follows from naturality of d, i.e. $f^* \circ d_Y = d_X \circ f^*$, and the definition of ${}^cd_\partial$.

Now let f be also an immersion. We prove the dual of the last statements. If r_Y is a boundary defining function for Y then $r_X = f^*r_Y$ is one for X, which proves $f^* : L_Y^* \to L_X^*$; if $\alpha \in {}^cT_q^*X$ is arbitrary then $\alpha = a\, dr_X + r_X\beta$ for some $a \in \mathbb{R}$ and $\beta \in T_q^*X$. Since $f^* : T^*_{f(q)}Y \to T_q^*X$ is surjective, $\beta = f^*\beta'$ for some $\beta' \in T^*_{f(q)}Y$ and thus $\alpha = f^*(a\, dr_Y + r_Y\beta')$, so α is in the image of $f^* : {}^cT^*_{f(q)}Y \to {}^cT_q^*X$ which was to be shown. □

In the sequel we will only consider embedded rather than immersed p-submanifolds for simplicity.

4. **The case of a resolved conic singularity.** We recall the notion of blow-up of a manifold Z in a point $p \in Z$. This is a manifold with boundary, denoted $[Z, p]$, together with a smooth map $\beta : [Z, p] \to Z$ which maps $\partial[Z, p]$ to p and is a diffeomorphism from the interior of $[Z, p]$ to $Z \setminus \{p\}$. Since all that follows is local near $\partial[Z, p]$ one may w.l.o.g. think of $Z = \mathbb{R}^n$, $p = 0$, and then $[Z, p] = \mathbb{R}_+ \times S^m$, where $m = n - 1$ and $S^m \subset \mathbb{R}^n$ is the unit sphere, and $\beta(r, \omega) = r\omega$. See [4] or [2] for a more in-depth discussion.

A subset $M \subset Z$ is called a *submanifold with conic singularity at p* if $M \setminus \{p\}$ is a submanifold and the strict transform $M_{\text{res}} := \overline{\beta^{-1}(M \setminus \{p\})}$ is a p-submanifold of $[Z, p]$. That is, M_{res} is a submanifold with boundary of $[Z, p]$ which meets $\partial[Z, p]$ transversally, and $\partial M_{\text{res}} \subset \partial[Z, p]$. Then the map

$$\beta_M : M_{\text{res}} \overset{i_M}{\to} [Z, p] \overset{\beta}{\to} Z, \tag{6}$$

with $i_M : M_{\text{res}} \overset{i_M}{\to} [Z, p]$ the inclusion, is called the *(embedded) resolution* of M.

4.1. **Conic tangent bundle of a resolved conic singularity.** We first give an interpretation of the conic tangent bundle on $[Z, p]$.

Proposition 2. *Let Z be a smooth manifold, $p \in Z$, $\tilde{Z} = [Z, p]$ and $\beta : \tilde{Z} \to Z$ the blow-down map. Then ${}^cT\tilde{Z}$, ${}^cT^*\tilde{Z}$ are canonically isomorphic to β^*TZ, β^*TZ^*, respectively.*

More precisely, the isomorphism $\beta_* \circ (\iota')^{-1} : {}^cT(\tilde{Z} \setminus \partial\tilde{Z}) \to T(Z \setminus \{p\})$, where ι' is defined in Lemma 2.2, extends to a smooth map ${}^cT\tilde{Z} \to TZ$ which, for each $q \in \partial\tilde{Z}$, is an isomorphism ${}^cT_q\tilde{Z} \to TZ_p$.

Proof. It suffices to prove the statement for the cotangent bundle. This says: Let $q \in \partial\tilde{Z} = \beta^{-1}(p)$. Then a local basis of sections of T_p^*Z pulls back, under β, to a local basis of sections of ${}^cT_q^*\tilde{Z}$.

In suitable boundary coordinates $r, \phi_1, \dots, \phi_{n-1}$ based at q and $x_1, \dots, x_n$ based at p, the blow-down map β has 'components'

$$\beta^* x_1 = r\phi_1, \dots, \beta^* x_{n-1} = r\phi_{n-1},\ \beta^* x_n = r \tag{7}$$

(homogeneous or projective coordinates). Then

$$\beta^* dx_1 = r d\phi_1 + \phi_1 dr, \dots, \beta^* x_{n-1} = r d\phi_{n-1} + \phi_{n-1} dr,\ \beta^* dx_n = dr,$$

and the right hand sides of this clearly form a local basis of ${}^cT^*_q\tilde{Z}$. □

Note that the line bundle $L \subset {}^cT_{\partial\tilde{Z}}\tilde{Z}$ is just the canonical line bundle in the 'usual' sense: Each $q \in \partial\tilde{Z}$ corresponds to a direction at p, hence determines a line in T_pZ, and then L_q is just the corresponding line in ${}^cT_q\tilde{Z}$ under the isomorphism given by the Proposition.

From the proposition we get an interpretation of the conic tangent bundle on the resolution of a manifold with conic singularities.

Corollary 1. *Let Z be a smooth manifold and $M \subset Z$ be a submanifold with conic singularity at p. Let $\beta_M : M_{res} \to Z$ be the resolution of M. Then*

$$(\beta_M)_*({}^cT_q M_{res}) = \begin{cases} T_{f(q)}M & \text{if } q \notin \partial M_{res}, \text{i.e. } f(q) \neq p, \\ \lim_{p' \to_q p} T_{p'}M & \text{if } q \in \partial M_{res}. \end{cases}$$

Here $p' \to_q p$ means that $p' \neq p$ approaches p in the direction q (recall that points of $\partial M_{\mathrm{res}}$ correspond to directions of approach to p). The latter limit is taken in the Grassmannian of TZ. Part of the statement is that the limit exists. A bundle over a resolution M_{res} of an embedded singular space M together with a map $(\beta_M)_*$ having the properties above is called a *Nash bundle* for M. Thus, ${}^cTM_{\mathrm{res}}$ is a Nash bundle for M.

Proof. $p' \to_q p$ is equivalent to $q' \to q$ in M_{res}, where $q' = \beta^{-1}(p')$. Also, ${}^cTM_{\mathrm{res}}$ my be considered as subspace of ${}^cT[Z,p]$ by Proposition 1. Therefore, the statement follows from the continuity in Proposition 2. □

4.2. **Pull-backs of ambient objects are ${}^cd_\partial$-closed.** The following is the main motivation for defining ${}^cd_\partial$:

Proposition 3. *Let $M \subset Z$ be a submanifold with conic singularity and resolution $\beta_M : M_{res} \to Z$. Then for each smooth one-form α on Z*

$$\beta_M^*\alpha \in \Gamma({}^cT^*M_{res}) \quad \text{and} \quad {}^cd_\partial(\beta_M^*\alpha) \equiv 0.$$

Similarly, for any smooth 2-tensor g on Z we have $\beta_M^* g \in \Gamma(S^2\, {}^cT^*X)$ and ${}^cd_\partial(\beta_M^* g) \equiv 0$.

The pull-back $\beta_M^*\alpha$ or $\beta_M^* g$ may be thought of as α or g written in polar coordinates.

Proof. By Proposition 2 we have $\beta^*\alpha \in \Gamma({}^cT^*[Z,p])$ where $\beta : [Z,p] \to Z$ is the blow-down map. Then restriction to M_{res} maps this to $\Gamma({}^cT^*M_{\mathrm{res}})$ by naturality, Proposition 1, and this proves the first statement.

Again by naturality the second statement follows if we show ${}^cd_\partial(\beta^*\alpha) = 0$ on $[Z,p]$. This is just naturality of d together with the fact that $\beta_*(v) = 0$ for $v \in T\partial[Z,p]$: By definition, ${}^cd_\partial\beta^*\alpha$ equals $d\beta^*\alpha$ restricted to the boundary. Now $d\beta^* = \beta^* d$, and $\iota_v\beta^* = \beta^*\iota_{\beta_* v}$ where ι_v is inner product (plugging in a vector in a form).

So for all v tangent to $\partial[Z,p]$, $\iota_v {}^c d_\partial \beta^*\alpha = \iota_v \beta^* d\alpha = \beta^* \iota_{\beta_* v} d\alpha = 0$ because of $\beta_* v = 0$. This proves the claim.

The last statement follows from building a two-tensor as sum of tensor products of one-forms. □

4.3. **Uniqueness of ${}^c d_\partial$.** In addition to ${}^c d_\partial(\beta_M^*\alpha) = 0$, a pull-back one-form satisfies infinitely many more conditions, involving higher order derivatives. However, we now show that, among these, the condition ${}^c d_\partial(\beta_M^*\alpha) = 0$ is the only condition which is invariantly defined on conic spaces. Since we are dealing with local conditions, the language of sheaves is appropriate. Thus, let Ω^1_Z be the sheaf of one-forms on a manifold Z and ${}^c\Omega^1_X$ be the sheaf of conic one-forms on a conic space X, so ${}^c\Omega^1_X(U) = \Gamma({}^cTU)$ for any open $U \subset X$.

Theorem 4.1. *Assume that for each manifold with boundary X we are given a sheaf $\mathcal{C}_X \subset {}^c\Omega^1_X$ satisfying the following conditions:*

a) $\mathcal{C}_X \subset \ker {}^c d_\partial$

b) If $\beta_M : M_{res} \to Z$ is the resolution of a space $M \subset Z$ with conic singularities then

$$\beta_M^* \Omega^1_Z \subset \mathcal{C}_{M_{res}}$$

c) Invariance under local diffeomorphisms: If X, Y are manifolds with boundary, $U \subset X$ and $V \subset Y$ open and $F : U \to V$ is a diffeomorphism, then

$$\mathcal{C}_X(U) = F^*\mathcal{C}_Y(V).$$

Then $\mathcal{C}_X = \ker {}^c d_\partial$ for all X.

Proof. Let $\omega \in \ker {}^c d_\partial \subset \Gamma({}^cT^*X)$ and $q \in X$. We need to show that $\omega_{|U} \in \mathcal{C}_X(U)$ for some neighborhood U of q. Since X is locally diffeomorphic to an open subset of a space $[Z,p]$, we may assume $X = [Z,p]$ because of c). Clearly, ${}^c\Omega^1 = \Omega^1$ over the interior of X, so we may asume $q \in \partial X$. If $r, \phi_1, \dots, \phi_m$ are projective coordinates on $X = [Z,p]$ near q as in (7) we know from condition b) that the following forms are in $\mathcal{C}_X$:

$$dr,\ d(r\phi_i),\ (r\phi_i)dr,\ rd(r\phi_i) \quad (i = 1, \dots, m)$$

hence also $r^2 d\phi_i = rd(r\phi_i) - (r\phi_i)dr$. By c) this remains true if one replaces r by any boundary defining function (bdf) ρ. We will show that ω is a sum of finitely many terms of this form.

Write $\omega = a\,dr + \sum_i b_i\, r\, d\phi_i$ with a, b_i smooth. Since ${}^c d_\partial \omega = 0$ we know that $a_{\phi_i} = b_i\, \forall i$ at $r = 0$, so $b_i - a_{\phi_i} = rc_i$ with c_i smooth. Write $\omega = d(ar) + \sum_i (b_i - a_{\phi_i})\, r d\phi_i = d(ar) + \sum_i c_i\, r^2 d\phi_i$. Choose $C \in \mathbb{R}$ big enough so that $a + C > 0$. Then $\rho = (a + C)r$ is a bdf, hence $d(ar) = d\rho - C dr \in \mathcal{C}_X$. Choose $C' \in \mathbb{R}$ big enough so that $c_i + C' > 0$, then $\rho' = \sqrt{c_i + C'} r$ is a bdf and $c_i r^2 d\phi_i = (\rho')^2 d\phi_i - C' r^2 d\phi_i \in \mathcal{C}_X$, and we are done. □

5. Normal forms and exponential map for conic metrics.

Definition 5.1. Let X be a manifold with boundary. A *conic metric* on X is a positive definite symmetric 2-tensor on ${}^cTX$. Equivalently, it is a Riemannian metric on the interior of X which in any boundary coordinate system $r, \phi_1, \dots, \phi_m$ can be written

$$g = A\, dr^2 + 2dr \sum_{i=1}^m B_i\, r d\phi_i + \sum_{i=1}^m \sum_{j=1}^m C_{ij}\, r d\phi_i\, r d\phi_j \tag{8}$$

with $A, B_i, C_{ij} \in C^\infty(X)$ and the quadratic form

$$(\zeta, \theta_1, \dots, \theta_m) \mapsto A(q)\,\zeta^2 + 2\zeta \sum_{i=1}^m B_i(q)\theta_i + \sum_{i=1}^m \sum_{j=1}^m C_{ij}(q)\theta_i\theta_j$$

positive definite for each $q \in X$.

As usual, the point is that the smoothness and positive definiteness hold up to (that is, including) the boundary.

It is clear from Proposition 3 that smooth metrics on the ambient space Z of a manifold with conic singularity M, when written in polar coordinates (i.e. pulled back to M_{res}), are conic. In addition, they are ${}^c d_\partial$-closed, and also satisfy infinitely many more conditions at the boundary. We now show that closedness alone implies important geometric properties.

Theorem 5.2. *Let g be a conic metric on a manifold with boundary X. The following conditions are equivalent:*

a) g is ${}^c d_\partial$-closed, that is ${}^c d_\partial g = 0$.

b) (Normal form 1) There are local boundary coordinates in which g has the form (8) with

$$A = 1 + O(r^2),\ \ B_i = O(r)\ \forall i.$$

c) (Exponential map based at the boundary) There is $\varepsilon > 0$, a neighborhood U of ∂X and a diffeomorphism $\Phi : \partial X \times [0, \varepsilon) \to U$ so that, for each $q \in \partial X$,

$$t \mapsto \Phi(q, t) \text{ is the unique unit speed geodesic with } \Phi(q, 0) = q$$

d) (Normal form 2) There are local boundary coordinates in which g has the form (8) with

$$A = 1,\ \ B_i = 0\ \forall i \quad \text{in a neighborhood of } \partial X.$$

Proof. a)⇒b): We first show that, for any conic metric, the r-coordinate can be chosen so that even $A \equiv 1$. Geometrically this just means taking r to be distance to the boundary. To find r analytically, assume $g = Ad\rho^2 + \dots$ in some coordinates ρ, ϕ_i (dots mean terms involving $d\phi_i$). If $r = \rho h$ then $dr = (\rho h)_\rho d\rho + \dots$, so if we choose the function h so that $(\rho h)_\rho = \sqrt{A}$ then we get $g = dr^2 + \dots$. For this we simply set $h(\rho, \phi) = \frac{1}{\rho}\int_0^\rho \sqrt{A(\sigma, \phi)}\, d\sigma = \int_0^1 \sqrt{A(\rho s, \phi)}\, ds$. Next, (5) shows that a) is equivalent to

$$A_{\phi_i} - 2B_i = 0 \quad \text{at } \partial X, \quad i = 1, \dots, m \tag{9}$$

in any local coordinate representation (8), so $A = 1$ implies $B_i = 0$ at the boundary.

b)⇒c): This was proved by Melrose and Wunsch in [3].

c)⇒d): Choose any coordinates in ∂X and let r be the coordinate on $[0, \varepsilon)$. Using Φ this gives coordinates on $U \subset X$. Then $A = 1$ by unit speed, and $B_i = 0$ by the Gauss Lemma (level sets of r are orthogonal to the geodesics $\Phi(p, \cdot)$).

d)⇒a) is clear from (9). □

If $X = M_{\mathrm{res}}$ is the resolution of a space $M \subset Z$ with conic singularity at p, the map Φ in c) indeed gives an exponential map based at p: Since points $q \in \partial M_{\mathrm{res}}$ correspond to rays tangent to M at p, we get a unique geodesic in M starting at p for every tangent direction, and smooth dependence of the geodesic on the direction, as well as uniqueness in the sense that for all points p' in some neighborhood of p there is a unique geodesic from p' to p inside the neighborhood. The situation is remarkably more complicated for cuspidal singularities, see [1].

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E-mail address: `grieser@mathematik.uni-oldenburg.de`