L^p bounds for eigenfunctions and spectral projections of the Laplacian near concave boundaries

Daniel Grieser

(Ph.D. thesis, University of California, Los Angeles)

July 1992

Contents

Chapter 1

Introduction

Let Ω be a smooth, compact Riemannian manifold with boundary $\partial\Omega$, of dimension $n \geq 2$. Let Δ_Ω be the Laplace-Beltrami operator acting on functions that vanish at the boundary. For $\lambda \geq 1$ write $\chi_\lambda = \chi_{[\lambda-1,\lambda]}(\sqrt{-\Delta_\Omega})$, where χ_I is the characteristic function of the set I. Our main result is:

Theorem 1 *Let $n = 2$ and $2 \leq p \leq \infty$. If $K \subset \bar{\Omega}$ is a compact set for which all points in $K \cap \partial\Omega$ are concave then for all $f \in C_0^\infty(\Omega)$ and $\lambda \geq 1$*

$$\|\chi_\lambda f\|_{L^p(K)} \leq C_K \lambda^{\epsilon(p)} \|f\|_{L^2(\Omega)}$$

where

$$\epsilon(p) = \begin{cases} \frac{1}{2}(\frac{1}{2} - \frac{1}{p}) & \text{if } 2 \leq p \leq 6 \\ \frac{1}{2} - \frac{2}{p} & \text{if } 6 \leq p \leq \infty. \end{cases}$$

We call a point $x \in \partial\Omega$ *concave* if all geodesics through x tangent to $\partial\Omega$ are tangent to first order only and, near x, contained in $\bar{\Omega}$. See figure 1.1.

It is known ([S3]) that these estimates are true in any dimension n if $K \cap \partial\Omega = \emptyset$ with

$$\epsilon(p) = \begin{cases} \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right) & \text{if } 2 \leq p \leq p_n = 2\frac{n+1}{n-1} \\ n\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2} & \text{if } p_n \leq p \leq \infty, \end{cases}$$

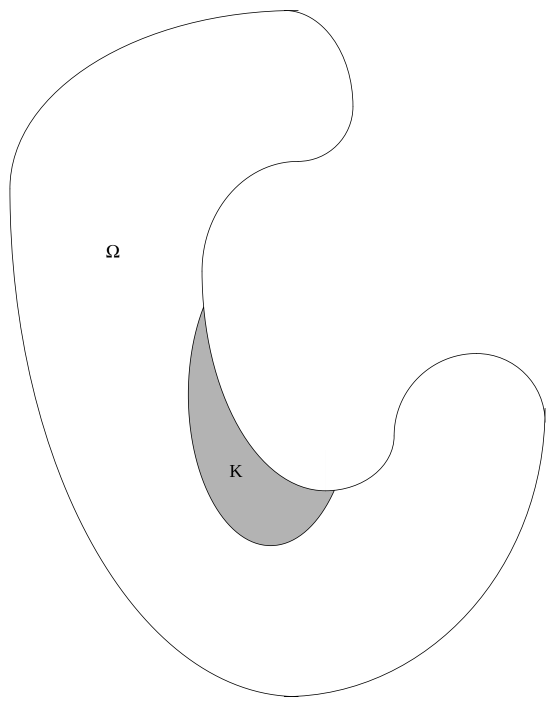

Figure 1.1: Concavity of $K \cap \partial\Omega$

and sharp if the interior of K is not empty. We *conjecture* that the same $\epsilon(p)$ works also if K is as in the theorem, for any n. We will prove this for $p=\infty$ but obtain only slightly weaker bounds for $n \geq 3, p \in (2,\infty)$.
Denote the eigenvalues of $-\Delta_\Omega$ by $\lambda_1^2 \leq \lambda_2^2 \leq \ldots$ and let $e_1, e_2, \ldots$ be a corresponding sequence of orthogonal L^2-normalized eigenfunctions. We call λ_j the *frequency* of e_j. Taking $f=e_j, \lambda=\lambda_j$ in the theorem one gets

Corollary 1.1 *With K as above, $\|e_j\|_{L^p(K)} \leq C_K \lambda_j^{\epsilon(p)}$.*

The theorem is stronger than the corollary as it gives also bounds for functions which are superpositions of eigenfunctions whose frequencies lie in a band of width one.
The restriction to convex boundary points is important:

Theorem 2 *Let $K=\Omega=\{|x|\leq 1\}$ be the unit disk in $\mathbb{R}^2$.*

a) There is a sequence of eigenfunctions (f_k) of Δ_Ω with frequencies $\mu_k \to \infty$ and a positive constant C so that

$$\|f_k\|_{L^6(\Omega)} > C\mu_k^{2/9}\|f_k\|_{L^2(\Omega)}$$

for all k. Thus, Theorem 1 is not valid here.

b) The L^∞ bound of Theorem 1 still holds, i.e.

$$\|\chi_\lambda\|_{L^2(\Omega)\to L^\infty(\Omega)} \leq C\sqrt{\lambda}.$$

Thus, the L^6 estimate is sensitive to the geometry of $\partial\Omega$ while the L^∞ estimate is not, in this case.

Why does the presence of a boundary lead to additional difficulties, and why does the geometry of the boundary matter?
As we will see below, the growth of the operator norm $\|\chi_\lambda\| = \|\chi_\lambda\|_{L^2(\Omega)\to L^p(K)}$ is closely connected with the propagation of waves in Ω. The key to the above questions lies in the peculiar behavior exhibited by waves hitting the surface of an obstacle (the boundary): For a concave obstacle (viewed from the wave) *diffraction* occurs, while for a convex obstacle one has *multiple reflections* which may degenerate into 'gliding rays', i.e. waves traveling inside the boundary surface only.

We proceed to give an outline of the proof of Theorem 1.
The relationship between χ_λ and the 'wave operator' $e^{it\sqrt{-\Delta_\Omega}}$ is fundamental:

$$\chi_\lambda = \chi_{[0,1]}(\lambda - \sqrt{-\Delta_\Omega}) = \int_{-\infty}^{\infty} \check{\chi}_{[0,1]}(t) e^{-it\lambda} e^{it\sqrt{-\Delta_\Omega}}\, dt,$$

writing $\chi_{[0,1]}$ as the Fourier transform of its inverse Fourier transform. From this it is easy to see (cf. Lemma 2.6) that

$$\exists C\ \forall \lambda > 1 \ \ \|\chi_\lambda\| \le C\lambda^{\epsilon(p)} \Longleftrightarrow \exists C\ \forall \lambda > 1 \ \ \|\rho_\lambda\| \le C\lambda^{\epsilon(p)}$$

where

$$\rho_\lambda = \int \rho(t) e^{-it\lambda} U_t\, dt,$$

ρ is any Schwartz function whose Fourier transform does not vanish on $[0,1]$, and $U_t f = \cos(t\sqrt{-\Delta_\Omega})f$ is the solution to the problem

$$\begin{aligned} (\frac{\partial^2}{\partial t^2} - \Delta)U_t f &= 0 \text{ in } \mathbb{R}_t \times \Omega_x \\ U_0 f &= f \\ (\frac{\partial}{\partial t} U_t f)_{|t=0} &= 0 \\ U_t f_{|\mathbb{R}\times\partial\Omega} &= 0. \end{aligned} \tag{1.1}$$

This has the following consequences if we choose ρ supported near 0, say in $(-\epsilon, \epsilon)$:

- The problem is *local*, i.e. the growth of $\|\chi_\lambda\|$ only depends on what Ω looks like in an ϵ-neighborhood of K. This is an immediate consequence of the finite propagation speed property of U_t, see section 2.1. Thus, if $K \subset \Omega$, $\epsilon \le \operatorname{dist}(K, \partial\Omega)$ for example, this growth is independent of the shape of $\partial\Omega$, and if Ω is a domain in $\mathbb{R}^n$, we can replace Ω by a cube containing it, not changing $\|\rho_\lambda\|$, and check the estimates for the cube by direct calculation. In fact, instead of a cube we might even take all of $\mathbb{R}^n$ (limiting case as the cube gets large), where now χ_λ is the multiplier operator $f \mapsto [\chi_{[\lambda-1,\lambda]}(|\xi|)\hat{f}(\xi)]^\vee$, and the calculation is easier as integrals are easier to handle than sums. Also, even for a nonflat metric the word 'manifold' could be avoided altogether, as no global phenomena occur. Therefore, without loss of generality one may think of domains $\Omega \subset \mathbb{R}^n$ only, with a symmetric (with respect to some measure that is smooth with respect to Lebesgue measure) variable coefficient second order elliptic operator instead of Δ.

- The growth of $\|\rho_\lambda\|$, and thus of $\|\chi_\lambda\|$, reflects non-smoothness of the map $\mathbb{R} \to \mathcal{L}(L^2(\Omega), L^p(K)), t \mapsto U_t$ near 0. In particular, U_t may be replaced by an approximation if the error is sufficiently smooth in t. Such approximations (parametrices) for the solution of the wave equation near $t = 0$, in the form of superpositions of oscillating functions, are known for the 'free' problem (no boundary condition) and for the initial boudary value problem near concave or convex boundary points.

For the proof of Theorem 1 we analyze the integral kernel of ρ_λ, using the parametrices for U_t. Recall the identities

$$\|\rho_\lambda\|_{L^2(\Omega)\to L^p(K)} = \|\rho_\lambda^*\|_{L^{p'}(K)\to L^2(\Omega)} = \|\rho_\lambda\rho_\lambda^*\|^{1/2}_{L^{p'}(K)\to L^p(K)}$$

where p' is the dual of p, $\frac{1}{p'} + \frac{1}{p} = 1$, and

$$\|A\|_{L^1(K)\to L^\infty(K)} = \sup_{x,y\in K} |A(x,y)|$$

for any operator A (we always use the same letter for operators and their integral kernels). Thus, the L^∞ bound amounts to a uniform estimate on the kernel of $A = \rho_\lambda\rho_\lambda^*$. In section 2.2 we present a lemma reducing the L^{p_n} bound to certain decay estimates of this kernel away from the diagonal $x = y$. Note that $\epsilon(p)$ is linear in p^{-1} on the intervals $[2, p_n]$ and $[p_n, \infty]$, so by linear interpolation we only need to prove the theorem for $p = 2, p_n$ and ∞, with

$$\epsilon(p_n) = p_n^{-1} \text{ and } \epsilon(\infty) = \frac{n-1}{2}.$$

In chapter 2 we present the proof for the interior case $K \cap \partial\Omega = \emptyset$. First, we consider a flat metric, taking a domain $\Omega \subset \mathbb{R}^2$ for simplicity. The L^∞ estimate is elementary, while for L^{p_n} the stationary phase method is used. Using the free parametrix for a variable coefficient wave equation, there is little difficulty in extending the results to an arbitrary metric. Essentially, it is the stability of the methods under small perturbations that makes this

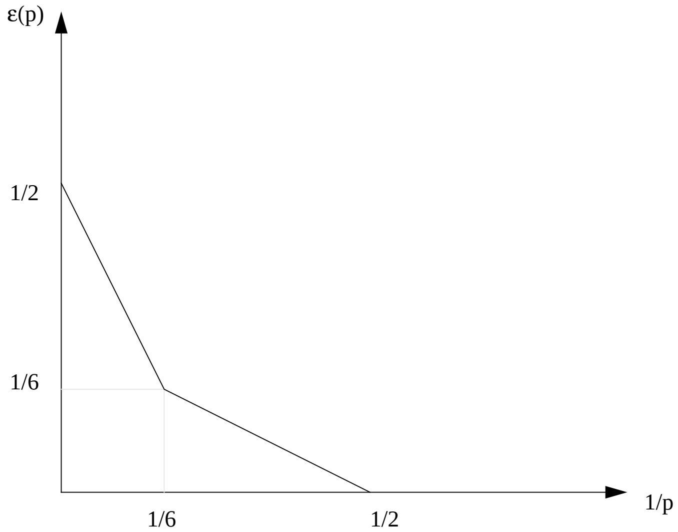

Figure 1.2: $\epsilon(p)$ for $n = 2$

possible. Some geometric explanations and a proof of sharpness for $\Omega \subset \mathbb{R}^2$ conclude chapter 2. All the ideas in chapter 2 have been known for some time, see [S3], also for further references.

What happens if $K \cap \partial\Omega \neq \emptyset$? This is the topic of chapters 3, 4 and 5. In chapter 3 we present the parametrix for the wave equation near a concave boundary point. Analogous to a light ray hitting a surface transversally, tangentially or not at all, the parametrix is the sum of 'transversal', 'grazing' and 'interior' parts. The transversal and interior parts are operators like those dealt with in chapter 2, and the estimates are easily obtained. The grazing part is a quantification of the diffraction phenomenon mentioned above. It is more difficult to construct (see [MT]) and also harder to estimate. This estimation is the core of this work and contained in chapter 4.

We prove Theorem 2 in chapter 5, using asymptotic properties of Bessel functions. A short list of open problems and appendices about Airy functions and stationary phase conclude the work.

For the sake of orientation, we remark that the weaker estimate

$$\|\chi_\lambda f\|_{L^\infty(\Omega)} \leq C\lambda^{n/2}\|f\|_{L^2(\Omega)},$$

for any smooth domain Ω, can be obtained easily from an appropriate form of Sobolev's inequality (see [H1, Thm. 17.5.3], for example).

As usual, C will denote a constant that may be different at each occurence. Generally, it may depend on Ω and other previously chosen quantities like the function ρ above, but not on λ. Investigation of the dependence of these constants on the geometry of Ω is a topic of interest, but not part of this work.

Thanks to the Professors

C. Sogge, my advisor, who had always time and tried patiently to keep my eye on what's important,

J. Ralston and G. Eskin, from whom I learned a lot about PDEs,

R. Melrose, whose 'Introduction to Microlocal Analysis' convinced me to convert to PDEs, after I had focussed on combinatorics before,

M. Aigner, my former advisor in Berlin, who supported, encouraged and challenged me in what he thought I was best at,

and to Marianne Hübner for the pictures, proofreading and moral support in times of joy and in times of despair.

Chapter 2

Basic Ideas

Here we will present special cases of Theorem 1, in order to introduce the main ideas one by one. We will prove the estimates for the interior case, $K \cap \partial\Omega = \emptyset$. For clarity, we will mostly deal with two dimensions. Generalization to n dimensions is immediate. In sections 1 and 2 we consider the case of a domain in $\mathbb{R}^2$. In section 3 we will show how the ideas in section 1 and 2, together with known parametrices for variable coefficient wave equations, yield the result in the variable coefficient case, away from the boundary (thus concluding the proof if Ω is a manifold without boundary). In section 4 we will give geometric meaning to some of the manipulations performed previously. Finally, in section 5 we show that the estimates are optimal if K contains an open set.

2.1 Localization

To illustrate the localization idea, we prove the following special case of Theorem 1:

Proposition 2.1 *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, and consider Dirichlet boundary conditions. Let $K \subset \Omega$ be a compact set in the interior of Ω. Then*

$$\|\chi_\lambda\|_{L^2(\Omega)\to L^\infty(K)} \leq C_K\sqrt{\lambda}.$$

Proof: Let $\epsilon = \text{dist}(K, \partial\Omega)$. The main point is that the solution of the wave equation with initial data supported in K will not "notice" the boundary up to time $t = \pm\epsilon$. Therefore, we can express this solution in two ways: Using the eigenfunctions e_j of Δ_Ω, and using the $\mathbb{R}^2$ "eigenfunctions"

$e^{ix\xi}$. As equality holds for all $|t| < \epsilon$, we can extract information about the e_j.

Formally, for $x \in \Omega$ and $y \in K$ write

$$\sum_j e_j(x)\overline{e_j(y)} = \delta(x-y) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{i(x-y)\xi} d\xi$$

with weak convergence.

Let $u(x,t) = (U_t \delta_y)(x)$ be the solution of the wave equation (1.1) with a point mass at y as initial data. It is well known (and not hard to prove using energy inequalities) that

$$\text{supp } u(\cdot,t) \subset \{x : \text{ there is a path from } y \text{ to } x \text{ in } \Omega \text{ of length} \le |t| \ \}. \tag{2.1}$$

So if $|t| < \epsilon$, $u(x,t)$ equals the solution of the wave equation on $\mathbb{R}^2$, that is

$$\sum_j \cos(\lambda_j t)\, e_j(x)\overline{e_j(y)} = (2\pi)^{-2} \int_{\mathbb{R}^2} \cos(t|\xi|) e^{i(x-y)\xi} d\xi \quad \text{if } |t| < \epsilon. \tag{2.2}$$

Write $\cos s = (e^{is} + e^{-is})/2$, multiply both sides by $e^{-i\lambda t}\rho(t)$, with $\rho \in C_0^\infty((-\epsilon,\epsilon))$ to be chosen later, and integrate over t to obtain

$$\rho_\lambda(x,y) := \int_{\mathbb{R}} \rho(t) e^{-i\lambda t} u(x,t)\, dt = \sum_j \frac{1}{2}[\hat{\rho}(\lambda - \lambda_j) + \hat{\rho}(\lambda + \lambda_j)] e_j(x)\overline{e_j(y)} =$$
$$(2\pi)^{-2} \int_{\mathbb{R}^2} \frac{1}{2}[\hat{\rho}(\lambda - |\xi|) + \hat{\rho}(\lambda + |\xi|)] e^{i(x-y)\xi} d\xi. \tag{2.3}$$

Clearly, $\rho_\lambda(x,y) = 0$ if $|x-y| > \epsilon$. The following simple lemma is essential:

Lemma 2.2 *If $\rho \in \mathcal{S}$ and $\hat{\rho}$ does not vanish in $[0,1]$ then, for λ sufficiently large,*

$$\|\chi_\lambda f\|_{L^2(\Omega)} \le C_\rho \|\rho_\lambda f\|_{L^2(\Omega)}.$$

Therefore, the growth of $\|\chi_\lambda\|_{L^{p'}(K)\to L^2(\Omega)}$ can be controlled using only information in a ϵ-neighborhood of K. In section 2.5 we will see that a converse estimate is also true.

Proof: $\|\chi_\lambda f\|^2_{L^2(\Omega)} = \sum_j \chi_{[0,1]}(\lambda-\lambda_j)|\langle f,e_j\rangle|^2 \leq$
$C\sum_j |\frac{1}{2}(\hat{\rho}(\lambda-\lambda_j)+\hat{\rho}(\lambda+\lambda_j))|^2\,|\langle f,e_j\rangle|^2 = C\|\rho_\lambda f\|^2_{L^2(\Omega)}$ since $\hat{\rho}(\mu)+\hat{\rho}(2\lambda-\mu)\neq 0$ for $\mu\in[0,1]$ and λ large.. ♣

It is easy to get a $\rho\in C_0^\infty((-\epsilon,\epsilon))$ as in the lemma: Take any $\rho_0\in C_0^\infty((-\epsilon,\epsilon))$ with $\hat{\rho}_0(0)=\int\rho_0\neq 0$. Then some dilation $\rho(t)=\alpha\,\rho_0(\alpha t), \alpha\geq 1$ will do.

Now write $\rho_\lambda = (\rho^+_\lambda+\rho^-_\lambda)/2$, $\rho^\pm_\lambda = \hat{\rho}(\lambda\mp\sqrt{-\Delta_\Omega})$. Then

$$\|\rho^+_\lambda(\rho^+_\lambda)^*\|_{L^1(K)\to L^\infty(K)} = \sup_{x,y\in K}|(\rho^+_\lambda(\rho^+_\lambda)^*)(x,y)| = \int_{\mathbb{R}^2}|\hat{\rho}(\lambda-|\xi|)|^2 d\xi \leq C\lambda$$

whence

$$\|\rho^+_\lambda\|_{L^2(\Omega)\to L^\infty(K)} = \|(\rho^+_\lambda)^*\|_{L^1(K)\to L^2(\Omega)} \leq C\sqrt{\lambda}$$

using Hölder inequality and duality, and ρ^-_λ is negligible because

$$\int|\hat{\rho}(\lambda+|\xi|)|^2 d\xi \leq C_N\int\frac{1}{(\lambda+|\xi|)^N}d\xi \leq C'_N\lambda^{-N+2}$$

for any N.

♣ (Proof of Proposition 2.1)

Remarks

- The essential fact is that the 'eigenfunctions' $e^{ix\xi}$ are uniformly (in ξ) bounded, and there are roughly $\lambda = \lambda^{n-1}$ of them with the relevant frequencies, with respect to Lebesgue measure in ξ. In contrast, the L^{p_n} estimate will depend essentially on cancellation that occurs when functions $x \mapsto e^{ix\xi}$ for various nearby ξ are superimposed.

- Actually, the essential feature of $\cos(t\sqrt{-\Delta_\Omega})$ is not the finite propagation speed of supports (2.1), but of singularities. This allows to generalize the method to certain nonlocal (but pseudolocal) elliptic pseudodifferential operators instead of Δ, if we are dealing with manifolds without boundary, see [S1].

- While above we thought of $e^{ix\xi}$ as 'eigenfunctions' of $\Delta_{\mathbb{R}^2}$, it will be easier in the variable coefficient case and near the boundary to use directly integral representations for the wave kernel which are not necessarily given using eigenfunctions. This will correspond more closely to $e^{ix\xi}\overline{e^{iy\xi}} = e^{i(x-y)\xi}$ viewed as standing wave, 'centered' at y.

2.2 L^p bounds

Recall that for an operator A, $\|A\|_{L^1\to L^\infty} = \sup|A(x,y)|$. There is no such simple formula relating $\|A\|_{L^{p'}\to L^p}$ to the size of the kernel $A(x,y)$, if $p \neq 1$.

The following lemma is often useful. The idea appeared for the first time in [ST] and was subsequently used to prove various L^p boundedness results (e.g. see [S3]).

Lemma 2.3 *Let $X \subset \mathbb{R}^n$ be bounded, and let $B \in C^\infty(\bar{X}\times\bar{X})$. Assume that the kernel of $A = BB^*$ satisfies:*

$$|A(x,y)| \leq \frac{\alpha}{|x_1 - y_1|^{(n-1)/2}} \text{ for all } x, y \tag{2.4}$$

and

$$\|A_{x_1x_1}\|_{L^2(H_{x_1})\to L^2(H_{x_1})} \leq \beta \tag{2.5}$$

where $H_t = \{x \in X : x_1 = t\}$ *and* $A_{x_1 y_1}$ *is the operator with integral kernel* $A_{x_1 y_1}(x', y') = A(x_1, x', y_1, y')$. *Then*

$$\|Af\|_{L^{p_n}(X)} \leq C\alpha^{2/(n+1)}\beta^{(n-1)/(n+1)}\|f\|_{L^{p'_n}(X)}$$

where C *only depends on the dimension* n.

Proof: First we show that

$$\|A_{x_1 y_1}\|_{L^{p'_n}(H_{y_1}) \to L^{p_n}(H_{x_1})} \leq C(\alpha, \beta)|x_1 - y_1|^{-(n-1)/(n+1)}. \tag{2.6}$$

where $C(\alpha, \beta) = \alpha^{2/(n+1)}\beta^{(n-1)/(n+1)}$. We use interpolation between $L^1 \to L^\infty$ and $L^2 \to L^2$ estimates:

$$\frac{1}{p_n} = \frac{1}{2} \cdot \frac{2}{p_n} + \frac{1}{\infty} \cdot (1 - \frac{2}{p_n}) = \frac{1}{2} \cdot \frac{n-1}{n+1} + \frac{1}{\infty} \cdot \frac{2}{n+1}.$$

The $L^1 \to L^\infty$ estimate is easy from (2.4):

$$\|A_{x_1 y_1}\|_{L^1(H_{y_1}) \to L^\infty(H_{x_1})} = \sup_{x', y' : x, y \in X} |A(x_1, x', y_1, y')| \leq \alpha|x_1 - y_1|^{-(n-1)/2}.$$

We now proceed to prove the L^2 estimate

$$\|A_{x_1 y_1}\|_{L^2(H_{y_1}) \to L^2(H_{x_1})} \leq \beta.$$

If $x_1 = y_1$, this is assumption (2.5). For the general case, we need to use the assumption $A = BB^*$. From $A(x, y) = \int_X B(x, z)\overline{B(y, z)}\, dz$ we see $A_{x_1 y_1} = B_{x_1} \circ B^*_{y_1}$ where $B_{x_1}(x', z) = B(x_1, x'; z)$. Now

$$\begin{aligned}
\|A_{x_1 y_1}\|^2_{L^2(H_{y_1}) \to L^2(H_{x_1})} &\leq \|B_{x_1}\|^2_{L^2(X) \to L^2(H_{x_1})} \cdot \|B^*_{y_1}\|^2_{L^2(H_{y_1}) \to L^2(X)} \\
&= \|B_{x_1} B^*_{x_1}\|_{L^2(H_{x_1}) \to L^2(H_{x_1})} \cdot \|B_{y_1} B^*_{y_1}\|_{L^2(H_{y_1}) \to L^2(H_{y_1})} \\
&= \|A_{x_1 x_1}\|_{L^2(H_{x_1}) \to L^2(H_{x_1})} \cdot \|A_{y_1 y_1}\|_{L^2(H_{y_1}) \to L^2(H_{y_1})} \leq \beta^2.
\end{aligned}$$

(2.6) is proved. The rest is 'functorialism' together with the Hardy-Littlewood-Sobolev inequality (see [S3]):

$$\|\frac{1}{x^r} * g\|_{L^p(\mathbb{R})} \leq C\|g\|_{L^q(\mathbb{R})} \text{ if } r = 1 - \frac{1}{q} + \frac{1}{p},$$

applied with $p = p_n, q = p_n'$ and $r = (n-1)/(n+1)$.

By definition,

$$(Af)(x_1, x') = \int_{\mathbb{R}} A_{x_1 y_1}[f(y_1, \cdot)](x')\, dy_1.$$

By Minkowski's inequality and (2.6), this implies

$$\begin{aligned} \|(Af)(x_1, \cdot)\|_{L^p(H_{x_1})} &\leq \int_{\mathbb{R}} \|A_{x_1 y_1}[f(y_1, \cdot)]\|_{L^p(H_{x_1})}\, dy_1 \\ &\leq C(\alpha, \beta) \int_{\mathbb{R}} |x_1 - y_1|^{-r} \|f\|_{L^{p'}(H_{y_1})}\, dy_1 \end{aligned}$$

Therefore,

$$\begin{aligned} \|Af\|_{L^p(X)} &= \|\, \|(Af)(x_1, \cdot)\|_{L^p(H_{x_1})} \|_{L^p(\mathbb{R}_{x_1})} \\ &\leq C(\alpha, \beta)\, \| \int_{\mathbb{R}} |x_1 - y_1|^{-r} \|f\|_{L^{p'}(H_{y_1})}\, dy_1 \|_{L^p(\mathbb{R}_{x_1})} \\ &\leq C_n \cdot C(\alpha, \beta)\, \|\, \|f(y_1, \cdot)\|_{L^{p'}(H_{y_1})} \|_{L^{p'}(\mathbb{R}_{y_1})} = C_n \cdot C(\alpha, \beta)\, \|f\|_{L^{p'}(X)}. \end{aligned}$$

♣

Let us show how the lemma implies the L^{p_n} estimate away from the boundary, for a domain in $\mathbb{R}^2$, i.e.

$$\|\chi_\lambda f\|_{L^6(K)} \leq C\lambda^{1/6} \|f\|_{L^2(\Omega)}.$$

As in section 2.1 this follows from

$$\||\hat{\rho}|^2(\lambda - \sqrt{-\Delta_{\mathbb{R}^2}}) f\|_{L^6(K)} \leq C\lambda^{1/3} \|f\|_{L^{6/5}(K)}.$$

We show that the 'microlocalized' kernel

$$\tilde\rho_\lambda(x,y) = (2\pi)^{-2}\int_{\mathbb{R}^2} e^{i(x-y)\xi}|\hat\rho|^2(\lambda-|\xi|)\sigma(\xi)d\xi$$

satisfies the estimates

$$|\tilde\rho_\lambda| \le \begin{cases} C\lambda(\lambda\,|\delta|)^{-1/2} & \text{if } |\delta_2|\le 2\gamma|\delta_1|, \\ C_N\lambda(\lambda\,|\delta|)^{-N} & \text{if } |\delta_2|\ge 2\gamma|\delta_1|, \text{ for any } N\ge 0. \end{cases} \tag{2.7}$$

Here σ is a smooth conical cutoff near $(1,0)$, i.e.

$$\sigma\in C^\infty(\mathbb{R}^2-0), \sigma(t\xi)=\sigma(\xi) \text{ if } t>0, \text{ and}$$

$$\sigma(\xi_1,\xi_2)=0 \text{ if } |\xi_2|\ge\gamma|\xi_1|, \text{ for a fixed } \gamma>0,$$

and we write $\delta = x-y$, $\delta_i = x_i - y_i$.

The second inequality in (2.7), combined with the L^∞ estimate $\sup|\tilde\rho_\lambda|\le C\lambda$, yields for $x_1=y_1$

$$|\tilde\rho_\lambda| \le C\lambda\,(1+\lambda|\delta_2|)^{-2}.$$

Thus, Young's inequality implies that the assumptions of the lemma are satisfied with $A(x,y)=\tilde\rho_\lambda(\frac{x}{\lambda},\frac{y}{\lambda})$ and $\alpha=\beta=\lambda$. Thus $\|A\|_{L^{6/5}\to L^6}\le C\lambda$. Rescaling shows that $\tilde\rho_\lambda$ satisfies the desired estimates, and so does ρ_λ^+ as it can be decomposed into pieces which after a rotation in the ξ variables look like $\tilde\rho_\lambda$ for some σ.

To prove (2.7), change variables $\alpha_1=|\xi|, \alpha_2=\xi_2$ to obtain

$$\tilde\rho_\lambda = (2\pi)^{-2}\iint e^{i\psi(x,y;\alpha)}\tilde\sigma(\alpha)d\alpha_2\,\hat\rho(\lambda-\alpha_1)d\alpha_1$$

where $\tilde\sigma$ is another conical cutoff function and

$$\psi(x,y;\alpha_1,\alpha_2) = \delta_1\sqrt{\alpha_1^2-\alpha_2^2}+\delta_2\alpha_2.$$

As $\hat{\rho} \in \mathcal{S}$ implies $|\int s^r \hat{\rho}(\lambda - s)ds| \leq C_{r,\rho}\lambda^r$ if $r \geq 0$, we only need to show, writing λ for α_1 and $\alpha_2 = \lambda\eta$, that

$$I_\lambda(x,y) = \lambda \int_{\mathbb{R}} e^{i\lambda\psi(x,y,1,\eta)} a(\eta) d\eta$$

satisfies the bounds (2.7) if supp $a \in (-\gamma', \gamma'), \gamma' = (1+\gamma^{-2})^{-1/2}$.

This is now a direct consequence of an investigation of the critical points of the phase ψ: We have

$$\psi'_\eta = 0 \Leftrightarrow \frac{\delta_2}{\delta_1} = \frac{\eta}{\sqrt{1-\eta^2}} = \frac{\xi_2}{\xi_1}.$$

Thus, ψ has a critical point on the support of the integrand only if $|\delta_2| \leq \gamma|\delta_1|$. More precisely,

$$|\psi'_\eta| > C|\delta| \ \text{ if } |\delta_2| \geq 2\gamma|\delta_1|. \tag{2.8}$$

Also, we have $\psi''_{\eta\eta} = -\delta_1(1-\eta^2)^{-3/2}$, so

$$|\psi''_{\eta\eta}| > C|\delta| \ \text{ if } |\delta_2| \leq 2\gamma|\delta_1|. \tag{2.9}$$

Therefore, if we write $\psi = \delta_1\tilde{\psi}$ for $|\delta_2| \leq 2\gamma|\delta_2|$ then a critical point of $\tilde{\psi}$ is uniformly nondegenerate, and stationary phase, with parameter $\lambda\delta_1$, gives the first estimate in (2.7), and (2.8) gives the second, with repeated integration by parts:

$$\begin{aligned} |I_\lambda(x,y)| = \lambda\, |\int e^{i\lambda\psi} a\, d\eta| &= \lambda\, |\int \left[(\frac{1}{i\lambda\psi'_\eta}\frac{\partial}{\partial\eta})^N e^{i\lambda\psi}\right] a\, d\eta| \\ &= \lambda\, |\int e^{i\lambda\psi}\left[(\frac{\partial}{\partial\eta}\frac{1}{i\lambda\psi'_\eta})^N a\right] d\eta| \leq C_N\lambda(\lambda|\delta|)^{-N}. \end{aligned}$$

Remark: The principal new ingredient here was the use of the stationary phase method which quantifies the cancellation of $e^{i(x-y)\xi}$ when integrating over ξ. Geometrically, nondegeneracy of the critical point of ψ is equivalent to strict convexity of the 'cospheres' $|\xi|^2 = 1$, where $|\xi|^2$ arises as the symbol of $-\Delta$. The improvement of the obtained L^6 estimate over the result of interpolation between L^2 and L^∞ estimates for χ_λ is closely connected with the improvement, also obtained by stationary phase,

$$|\int_{|\xi|=1} e^{ix\xi} d\sigma_\xi| \leq C|x|^{-1/2} \text{ as } |x| \to \infty$$

over just obvious boundedness of this integral.

2.3 Variable Coefficients

Equation (2.2) says that the wave equation in $\mathbb{R}^2$ with any initial data f can be solved by finding 'many' wavelike solutions $e^{i(x-y)\xi}\cos(t|\xi|)$, one for each direction $\xi/|\xi|$ and wavelength $|\xi|$ and 'center point' y, and superimposing them, weighted by $f(y)$. For the calculations it was convenient that these were standing waves, i.e. of the form $a_{\xi,y}(x)b_{\xi,y}(t)$. For the L^∞ estimates it was essential that the standing waves are bounded and for the L^p estimates that cancellation occurs when integrating over ξ. Here it was also important to microlocalize, i.e. to consider only values of ξ in some small cone at a time.

We will see that the well-known construction of a parametrix (approximate solution) for a variable coefficient wave equation, without boundary conditions, as oscillatory integral has the same basic properties.

In this section, we will consider a second order elliptic differential operator $P(x, D_x)$, defined in some neighborhood of a point $x_0 \in \mathbb{R}^2$. We will consider the associated wave equation

$$\begin{aligned}(\frac{\partial^2}{\partial t^2} + P(x, D_x))U_t(x, y) &= 0\\ U_0(x, y) &= \delta(x - y)\\ (\frac{\partial}{\partial t}U_t)_{|t=0}(x, y) &= 0\end{aligned} \tag{2.10}$$

for $x, y \in \omega$, some neighborhood of x_0, and $|t| < \epsilon$. To simplify the notation later on, we will also assume that the principal symbol p of $P(x, D_x)$ satisfies

$$p(x_0, \eta) = |\eta|^2. \tag{2.11}$$

This can be achieved by a linear change of coordinates.

Proposition 2.4 *For ω and ϵ sufficiently small, there exist C^∞ functions $\phi_\pm(x, y, \xi)$ (the phases) and $a_\pm(x, y, t, \xi)$ (the amplitudes), $\xi \in \mathbb{R}^2 - 0$, such that:*

a) *$\phi_\pm$ is homogeneous of degree one in ξ, $d_{x,y,\xi}\phi \neq 0$ and $a_\pm \in S^0(\omega \times \omega \times \mathbb{R}_t; \mathbb{R}^2)$ is supported near $x = y = x_0, t = 0$.*

b) *With the oscillatory integrals in*

$$K_t(x,y) = \sum_{\pm} \int_{\mathbb{R}^2} e^{i(\phi_\pm(x,y,\xi) \pm t\sqrt{p(x,\xi)})} a_\pm(x,y,t,\xi) d\xi$$

thus defined as distributions, we have

$$K_t(x,y) - U_t(x,y) \in C^\infty(\omega \times \omega \times (-\epsilon, \epsilon)).$$

c) *For each y, ξ, $\phi_\pm$ satisfy the differential equation in x:*

$$p(x, (\phi_\pm)'_x) = p(y,\xi), \text{ (eikonal equation)} \tag{2.12}$$

$$\text{and } (\phi_\pm)'_{x|x=y} = \xi, \phi_{|x=y} = 0. \tag{2.13}$$

This form of the parametrix appeared first in [H2].
The local L^∞ estimate is now immediate, just using boundedness of the integrand and smoothness in t: We only need to check

$$|\rho_\lambda(x,y)| = |\int e^{-i\lambda t} \rho(t) U_t(x,y) dt| \leq C\lambda.$$

We replace U_t by K_t, introducing an error $O(\lambda^{-\infty})$, and carry out the t-integral. We obtain

$$\rho_\lambda(x,y) \equiv \sum_{\pm} \int e^{i\phi_\pm(x,y,\xi)} \tilde{a}_\pm(x,y,\lambda \mp \sqrt{p(y,\xi)}, \xi) d\xi \mod \lambda^{-\infty}$$

where $\tilde{a}$ is the Fourier transform in t of ρa. Because P is elliptic and $\tilde{a}$ is a Schwartz function in its third argument, uniformly in $|\xi| > 1$ and x, y, we get easily

$$\int |\tilde{a}_+(x,y,\lambda - \sqrt{p(y,\xi)}, \xi)| d\xi \leq C\lambda$$

and

$$\int |\tilde{a}_-(x,y,\lambda + \sqrt{p(y,\xi)}, \xi)| d\xi \leq C_N \lambda^{-N}$$

for any N, which proves the claim.
For the L^p estimate we proceed as in section 2.2. Clearly, we only need to consider the term with a_+.

First, we split up K_t using conical cutoff functions $\sigma_j(\xi)$. Looking at one piece at a time, we may assume that σ is supported near the ξ_1 direction, in $\{|\xi_2| \leq \gamma\xi_1\}$ say. We change variables $\alpha_1 = \sqrt{p}, \alpha_2 = \xi_2$. We thus consider

$$\tilde{K}_t(x,y) = \int_{\mathbb{R}^2} e^{i(\psi(x,y,\alpha)+t\alpha_1)} b(x,y,t,\alpha) d\alpha$$

with $\operatorname{supp} b \subset \{|\alpha_2| \leq \gamma'\alpha_1\}$ for some $\gamma' < 1$, $b \in S^0$, and from (2.11), (2.13)

$$\psi = 0 \text{ if } x = y,\ \psi'_x = (\sqrt{\alpha_1^2 - \alpha_2^2}, \alpha_2) \text{ if } x = y = x_0. \tag{2.14}$$

To complete the argument as in section 2.2, we need to see that the phase ψ has the same properties with respect to stationary points as we used there. Also, the amplitude b here is not homogeneous as it was there, but the symbol estimates are all that is needed. We are thus left with showing:

> The function $\eta \mapsto \psi(x,y;1,\eta)$ has a critical point on supp b only if $\delta_2 \leq \tilde{\gamma}\delta_1$, and then it is nondegenerate. Here, $\tilde{\gamma}$ is some constant.

From (2.14) we get, using Taylor's theorem:

$$\begin{aligned}\psi'_\eta &= -\delta_1 \frac{\eta}{\sqrt{1-\eta^2}} + \delta_2 + O(|\delta|^2) + O(|\delta|\,|x - x_0|),\\ \psi''_{\eta\eta} &= -\delta_1(1-\eta^2)^{-3/2} + O(|\delta|^2) + O(|\delta|\,|x - x_0|)\end{aligned}$$

which implies the claim if x, y are sufficiently close to x_0.

2.4 A Few Words about Geometry

Here we explain the propagation of singularities for the wave kernel and the geometric meaning of microlocalization, i.e. introduction of the conical cut-offs $\sigma_j(\xi)$ in sections 2.2 and 2.3, and and how this relates to the estimates on $\rho_\lambda(x,y)$.

Formally, the operator $P = -\Delta_\Omega$ is connected with the metric g on Ω by the formula

$$p(x,\eta) = \sum_{i,j} g^{ij}\eta_i\eta_j \tag{2.15}$$

for its principal symbol[1]. As we will see below, this implies a more geometric relation between P and the metric: The solution of the wave equation for P, (2.10), propagates singularities along geodesics. More precisely, if (with notation as in Proposition 2.4)

$$K_t^\sigma(x,y) = \int e^{i\Phi(x,t,y,\xi)}\sigma(\xi)a_-\,d\xi$$

where $\Phi = \phi_- - t\sqrt{p}$, then

$$\operatorname{sing\,supp} K^\sigma = \{(x,t,y) :\ x = \gamma_t(y,\xi) \text{ for some } \xi \in \operatorname{supp}\sigma\}, \tag{2.16}$$

where $t \mapsto \gamma_t(y,\xi) \in \Omega$ is the geodesic of unit speed, starting at y, with tangent $\dot\gamma_0 = \xi^\sharp/|\xi^\sharp|_g$ at y. Here $\xi^\sharp$ is the vector with components

$$(\xi^\sharp)^i = \sum_j g^{ij}(y)\xi_j. \tag{2.17}$$

This is just the local coordinate formula for the canonical isomorphism, induced by g, of the tangent and cotangent spaces:

$$\sharp : T_y^*\Omega \to T_y\Omega.$$

[1] More precisely, $P(x,D_x) = \sum_{i,j} g^{-1/2}D_{x_i}g^{1/2}g^{ij}D_{x_j}$, with $g = \det g_{ij}$, but this just means that P is the unique operator with principal symbol (2.15) that annihilates constants and is formally selfadjoint with respect to the measure $\sqrt{g}\,dx$ induced by g on Ω.

Its inverse we denote by

$$^\flat : T_y\Omega \to T_y^*\Omega.$$

Usually, what appears naturally in calculations are cotangent vectors, i.e. vectors ξ for which $\xi^\sharp$ defined by (2.17) has a meaning as tangent or direction. Intuitively, one may often think of tangent vectors whenever we talk about cotangent vectors in the following chapters. Only in calculations it is important to keep (2.17) in mind.

Remarks:

- In this chapter, we are only interested in values of t near 0 and x, y near the interior point x_0, so we don't need to be concerned with geodesics hitting the boundary. The propagation of singularities also obeys the reflection law, see chapter 3.
- Our special choice of coordinates (see (2.11)) makes $\xi^\sharp = \xi$ at $y = x_0$, so $\dot\gamma_0$ is close to $\xi/|\xi|_g$ for y close to x_0. Therefore, if $\operatorname{supp}\sigma \cap \{|\xi| = 1\}$ is close to $(1, 0)$, then (2.16) shows that the function $t \mapsto K_t(x, y)$ will be smooth for t small and $x - y$ outside a small neighborhood of the x_1-direction, explaining the rapid decay estimate for the Fourier transform $\rho_\lambda(x, y)$ of $\rho(t)K_t$ there (second line in (2.7)).
- The singular support of the plus term in Proposition 2.4 is generated by the geodesics $\gamma_t(y, -\xi), \xi \in \operatorname{supp}\sigma$.
- One can sharpen (2.16) to the 'microlocal' statement

  $$\begin{aligned} \mathrm{WF}(K^\sigma) = \{(x, t, y; \eta, \tau, -\xi) : \xi \in \operatorname{supp}\sigma, \\ x = \gamma_t(y, \xi), \eta = [\dot\gamma_t(y, \xi)]^\flat, \tau = -\sqrt{p(y, \xi)}\}. \end{aligned} \quad (2.18)$$

  By the wave front set calculus this means that if $(y, \xi) \in \mathrm{WF}(f)$ and $\xi \in \operatorname{supp}\sigma$ then $(\gamma_t(y, \xi), [\dot\gamma_t(y, \xi)]^\flat) \in \mathrm{WF}(K_t^\sigma f)$.
- In outline, one can prove (2.18) by first showing

  $$\mathrm{WF}(K^\sigma) = \{(x, t, y; \Phi'_x, \Phi'_t, \Phi'_y) : \ \Phi'_\xi = 0 \text{ for some } \xi \in \operatorname{supp}\sigma\},$$

  by a clever integration by parts, and then using the eikonal equation (2.12) to see that the right hand side is actually generated by the bicharacteristic flow for the Hamiltonian $(p(x,\eta)-\tau^2)/2$, i.e. it is the union of curves

$$\{(x(s),t(s),x(0);\eta(s),\tau(s),\eta(0)) : s \in \mathbb{R} \text{ and near } 0\}$$

which are solutions of the system of ODE's

$$\begin{aligned} \dot{x} &= \frac{1}{2}p'_\eta & \dot{\eta} &= -\frac{1}{2}p'_x \\ \dot{t} &= -\tau & \dot{\tau} &= 0. \end{aligned}$$

From $\frac{1}{2}p'_{\eta_i}(x,\eta) = \sum_j g^{ij}(x)\eta_j$ we see (2.17), and the initial values $\eta(0)$ that actually occur are exactly the values $\xi \in \operatorname{supp}\sigma$ because of the normalization (2.13), which implies

$$\begin{aligned} \mathrm{WF}(K_0^\sigma) &= \{(x,y;\phi'_x,\phi'_y) : \phi'_\xi(x,y,\xi)=0, \xi \in \operatorname{supp}\sigma\} \\ &= \{(x,x;\xi,-\xi) : \xi \in \operatorname{supp}\sigma\}. \end{aligned}$$

2.5 Sharpness

Proposition 2.5 *If the interior of K is not empty then*

$$\limsup_{\lambda\to\infty} \lambda^{-\epsilon(p)} \|\chi_\lambda\|_{L^{p'}(K)\to L^2(\Omega)} > 0.$$

We restrict ourselves to domains in $\mathbb{R}^2$ again. Higher dimensions can be handled the same way, but for variable coefficient Laplacians there are some technical difficulties in carrying out these ideas (which are solved in [S2]). We take the lim sup in Proposition 2.5 because there can be many values λ with $\chi_\lambda = 0$.
In this section we will write $q = p'$. The following lemma contains a converse of Lemma 2.2. We write: $\|\ \| = \|\ \|_{L^q(K)\to L^2(\Omega)}$.

Lemma 2.6 *Let $\rho \in \mathcal{S}(\mathbb{R})$, and assume $\hat{\rho}$ does not vanish on $[0,1]$. Then* $\forall E\, \exists C > 0\, \forall \epsilon, 0 < \epsilon < E$

$$C^{-1} \limsup_{\lambda\to\infty} \frac{\|\chi_\lambda\|}{\lambda^\epsilon} \le \limsup_{\lambda\to\infty} \frac{\|\rho_\lambda\|}{\lambda^\epsilon} \le C \limsup_{\lambda\to\infty} \frac{\|\chi_\lambda\|}{\lambda^\epsilon}.$$

Proof: The first inequality follows directly from Lemma 2.2. To prove the second inequality, we first replace ρ_λ by $\rho_\lambda^+ = \hat\rho(\lambda - \sqrt{-\Delta_\Omega})$, using $\|\rho_\lambda^-\| = O(\lambda^{-\infty})$. Clearly, $|\hat\rho(\cdot)|^2 \le \sum_{-\infty}^{\infty} \alpha_k \chi_{[0,1]}(\cdot - k)$ for some sequence (α_k) satisfying $\alpha_k \le C_N(1+|k|)^{-N}$ for all k and any N.
Let $A = \limsup_{\lambda\to\infty} \lambda^{-\epsilon}\|\chi_\lambda\|$. For any $\delta > 0$ one can choose λ_0 so that $\|\chi_\lambda\| \le (A+\delta)\lambda^\epsilon$ for all $\lambda > \lambda_0$. For $\lambda > 2\lambda_0$ we then have

$$\|\rho_\lambda^+\|^2 \le \sum_k \alpha_k \|\chi_{\lambda-k}\|^2 =$$

$$\sum_{|k|\le\lambda/2} \alpha_k\|\chi_{\lambda-k}\|^2 + \sum_{|k|>\lambda/2} \alpha_k\|\chi_{\lambda-k}\|^2 \le (\sum_{-\infty}^{\infty}\alpha_k)\ (A+\delta)^2(\tfrac{3}{2}\lambda)^{2\epsilon} + C_N'\lambda^{-N}.$$

In the last step we used that $\|\chi_\lambda\|$ is polynomially bounded which follows easily from Sobolev's inequality as mentioned in the introduction. This implies $\limsup_{\lambda\to\infty}\lambda^{-\epsilon}\|\rho_\lambda\| \le C(A+\delta)$ for any $\delta > 0$. Letting $\delta \to 0$ gives the result. ♣
Clearly, there is at most one value of ϵ for which any of the two lim sup's is finite and nonzero. Also, λ^ϵ could be replaced by any function $h(\lambda)$ satisfying $h(2\lambda) \le C\,h(\lambda)$.

Proof of Proposition 2.5: W.l.o.g. we may assume $\{|x| < 1\} \subset K$. Choose a function $\rho \in C_0^\infty(-1/2,1/2)$ with $\hat\rho \ne 0$ on $[0,1]$. We construct families of functions f_λ, g_λ supported in $\{|x| < 1/2\}$ with

$$\|\rho_\lambda^+ f_\lambda\|_{L^2} \ge C\lambda^{\frac{2}{q}-\frac{3}{2}}\|f_\lambda\|_{L^q} \quad \text{for } 1 \le q \le 6/5 \tag{2.19}$$

$$\|\rho_\lambda^+ g_\lambda\|_{L^2} \ge C\lambda^{\frac{1}{2q}-\frac{1}{4}}\|g_\lambda\|_{L^q} \quad \text{for } 6/5 \le q \le 2. \tag{2.20}$$

Because $\operatorname{supp}\rho_\lambda f_\lambda \subset K \subset \Omega$ by the finite propagation speed of waves, the norms can be interpreted on $\mathbb{R}^2$; thus, we have transferred the problem from Ω to $\mathbb{R}^2$. By Parseval,

$$\|\rho_\lambda^+ f_\lambda\|_{L^2(\mathbb{R}^2)} = \|\hat\rho(\lambda - |\cdot|)\hat f_\lambda(\cdot)\|_{L^2(\mathbb{R}^2)} \ge \|\hat f_\lambda\|_{L^2(A)}$$

where A is the annulus $\{\lambda - 1 \le |\xi| \le \lambda\}$, and similarly for g_λ.

Choose $\psi \in C_0^\infty(\mathbb{R})$ with $\hat{\psi} \neq 0$ on $[-1,1]$, and set

$$f_\lambda(x) = \psi(\lambda x)$$
$$g_\lambda(x) = e^{i\lambda x_1}\psi(x_1)\psi(\sqrt{\lambda}x_2).$$

Then $\|f_\lambda\|_{L^q} = C\lambda^{-2/q}$ and $\|g_\lambda\|_{L^q} = C\lambda^{-1/2q}$. Also, $\hat{f}_\lambda(\xi) = \lambda^{-2}\hat{\psi}(\xi/\lambda)$. Because $\hat{\psi}(1) \neq 0$, $|\hat{f}_\lambda| \geq C\lambda^{-2}$ on A for λ big enough, so

$$\|\hat{f}_\lambda\|_{L^2(A)} \geq C\lambda^{-2}|A|^{1/2} = C'\lambda^{-3/2}$$

which gives (2.19).
To determine the L^2-norm of g, write $\hat{g}_\lambda(\xi) = \lambda^{-1/2}\hat{\psi}(\xi_1 - \lambda)\hat{\psi}(\xi_2\lambda^{-1/2})$. The point is that $[\lambda, \lambda + \frac{1}{2}] \times [-\lambda^{1/2}, \lambda^{1/2}] \subset A$. From $\hat{\psi} \neq 0$ on $[-1,1]$ we thus get

$$\|\hat{g}_\lambda\|_{L^2(A)} \geq C\lambda^{-1/2}\lambda^{1/4} = C\lambda^{-1/4},$$

and (2.20) follows.
♣

Chapter 3

Parametrix for the Initial-Boundary Value Problem

In this chapter we will present an approximate solution (parametrix) for the wave equation with Dirichlet boundary conditions, near a concave boundary point. In section 3.1 we write down the parametrix as sum of 'interior', 'transversal' and 'grazing' parts and discuss the relatively easy transversal parts (the interior parts are of the form given in chapter 2). To clarify the more difficult grazing parts, we present in section 3.2 the 'Friedlander example' where many of the calculations can be carried out explicitly, and give some geometric explanations in section 3.3. Because the kernel $K_t(x, y)$ of a grazing part is simpler when only one of the points x, y is close to the boundary, we consider only times t in some small interval I close to but not containing zero. Concavity and propagation of singularities then guarantee that for x *and* y close to $\partial\Omega$, the wave kernel is smooth and therefore negligible for our purposes.

3.1 Statement of the Parametrix, and Transversal Part

Fix a concave boundary point $x_0 \in \partial\Omega$. Let ω be the part lying in Ω of a neighborhood of x_0 in $\bar{\Omega}$. Choosing ω small enough, we may assume:

There are $t_0, \epsilon > 0$ and an open set $\omega' \subset \Omega$, $\text{dist}(\omega', \partial\Omega) > 0$, such that $\gamma_t(x,\xi) \in \omega'$ for all $x \in \omega, \xi \neq 0, |t - t_0| < 2\epsilon$, and $\gamma_t(x,\xi)$ hits the boundary at most once for $|t| < 10t_0$. See figure 3.1. $(*)$

Here $t \mapsto \gamma_t(x,\xi) \in \bar{\Omega}$ is the (possibly reflected) geodesic (of unit speed) starting at x in direction ξ, i.e. $x = \gamma_0(x,\xi), \xi = [\dot{\gamma}_0(x,\xi)]^\flat$ (see section 2.4). Reflection in the boundary obeys the usual law that the reflection angle equals the incidence angle.

For $f \in \mathcal{E}'(\Omega)$ let $Uf(x,t) = U_t f(x)$ be the solution of the wave equation (1.1), and denote by $U^{\text{free}} f$ the solution of the same problem, without boundary conditions, on some closed extension $\tilde{\Omega} \supset \Omega$, i.e. a compact Riemannian manifold of same dimension without boundary.

We are only interested in $x \in \omega$, $t \in I = (t_0 - \epsilon, t_0 + \epsilon)$. We call a continuous map $K : \mathcal{E}'(\Omega) \to \mathcal{D}'(\Omega \times \mathbb{R})$ a *microlocal parametrix at* $(y,\eta) \in T^*(\Omega)$ if, for some open cone Γ containing (y,η) and all f with $\text{WF}(f) \subset \Gamma$,

$$(Kf - Uf)_{|\bar{\omega} \times I} \in C^\infty(\bar{\omega} \times I). \tag{3.1}$$

It is essential to use $\bar{\omega}$ here because the approximation must be uniformly good as $x \to \partial\Omega$. Recall the theorem on propagation of singularities (see [H1]):

> If $\text{WF}(f) \subset \Gamma$ then Uf is smooth outside $\{(x,t) : x \in \gamma_t(\Gamma) \cup \gamma_{-t}(\Gamma)\}$.

Therefore, near any (y,η) not in $\Gamma_0 = \Gamma_0^+ \cup \Gamma_0^-$,

$$\Gamma_0^\pm = \{(y,\eta) \in T^*(\Omega) : \gamma_t(y,\eta) \in \bar{\omega} \text{ at some time } t \in \pm\bar{I}\},$$

$K \equiv 0$ is a microlocal parametrix. In particular, this holds for any $y \notin \omega'$ by $(*)$. Below we will describe microlocal parametrices near points in Γ_0. Using a cutoff near ω' and a microlocal partition of unity in $T^*\omega'$ one can then construct K such that (3.1) holds for all $f \in \mathcal{E}'(\Omega)$.

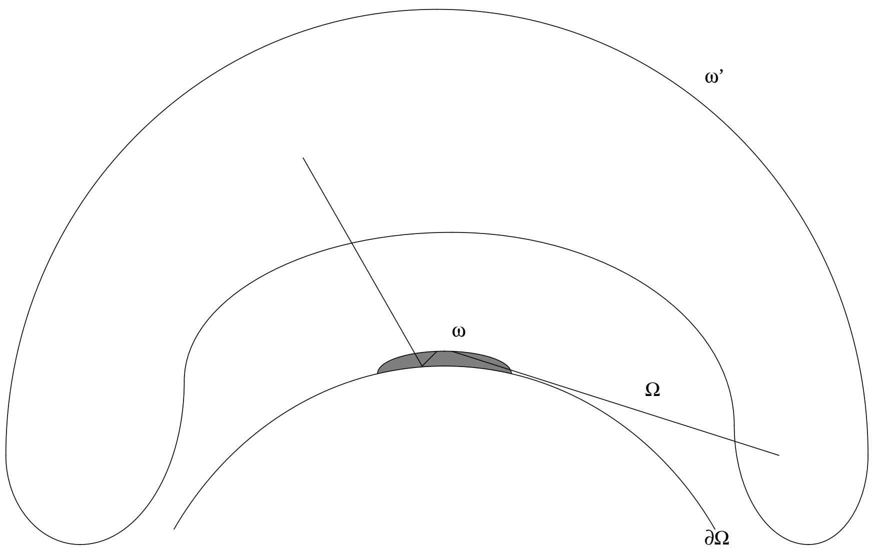

Figure 3.1: One endpoint near $\partial\Omega$, one endpoint away from $\partial\Omega$

Fix $(y, \eta) \in \Gamma_0^+$ (the case Γ_0^- is treated in an analogous manner), and let $\gamma_t = \gamma_t(y, \eta)$. Below, we write down a microlocal parametrix near (y, η). Its form depends on the nature of γ: We say that γ is *transversal, grazing* or *interior* if it hits the boundary transversally, tangentially or not at all, respectively, for some $t \in [0, 3t_0]$. Clearly, if γ is transversal or interior then so are all nearby geodesics.
In the following discussion f always denotes a distribution with wave front set close to (y, η).

Interior case: If γ is interior, we simply set $K = U^{\text{free}}$. Then $v = Kf - Uf$ satisfies $v_{|t=0} = v_{t|t=0} = 0$, solves the wave equation and has smooth boundary values for $t \in [0, 3t_0]$. Classical theorems imply that v is smooth in $\Omega \times I$.

Transversal case: Assume γ hits the boundary transversally at time $\tilde{t}$, $\gamma_{\tilde{t}} = \tilde{x} \in \partial\Omega$. Transversality means that the reflection $\bar{\xi}$ of the velocity vector $\tilde{\xi} = (\dot{\gamma}_{\tilde{t}})^\flat$ of γ at $\tilde{x}$ is distinct from $\tilde{\xi}$. That is, there is $\bar{\xi} \neq \tilde{\xi}$ in $T^*_{\tilde{x}}\tilde{\Omega}$ having the same projection onto the (co)tangent plane to the boundary, $T^*_{\tilde{x}}\partial\Omega$, see figure 3.2.

Let $\tilde{\gamma}_t = \gamma_t^{\tilde{\Omega}}(y, \eta)$ be the geodesic in $\tilde{\Omega}$ starting at (y, η) (without reflection in $\partial\Omega$) and $\bar{\gamma}_t$ its reflection in $\partial\Omega$ at $\tilde{x}$, i.e. the geodesic in $\tilde{\Omega}$ going through $(\tilde{x}, \bar{\xi})$ at time $\tilde{t}$. The extension $\tilde{\Omega}$ can be chosen (depending on (y, η)) such that both $\tilde{\gamma}$ and $\bar{\gamma}$, considered for $t \in [-3t_0, 3t_0]$, hit the boundary only at time $\tilde{t}$. Denote by $(\bar{y}, \bar{\eta}) = (\bar{\gamma}_0, (\dot{\bar{\gamma}}_0)^\flat)$ the reflection of (y, η).

The idea for our construction of the transversal parametrix is to first determine a 'reflection' $\bar{f}$ of f, with wave front set near $(\bar{y}, \bar{\eta})$, and then to write

$$K = U^{\text{free}} f - U^{\text{free}} \bar{f}. \tag{3.2}$$

Thus, $\bar{f}$ should be determined such that $U^{\text{free}}\bar{f}$ has the same boundary values as $U^{\text{free}} f$.
To construct $\bar{f}$, it is useful to write, with $Q = \sqrt{-\Delta_{\tilde{\Omega}}}$,

$$U_t^{\text{free}} = \cos(tQ) = \frac{1}{2}(e^{-itQ} + e^{itQ}) = \frac{1}{2}(U_t^1 + U_t^2).$$

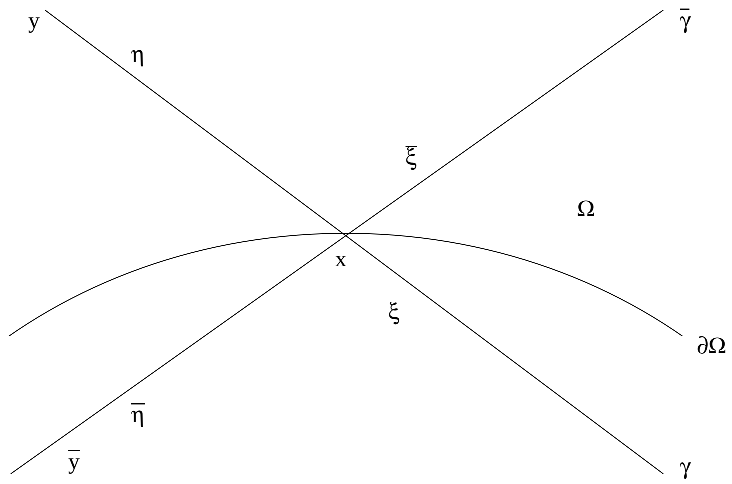

Figure 3.2: Geodesics in transversal case

$\frac{1}{2}U^1$ and $\frac{1}{2}U^2$ are, modulo C^∞, the minus and plus terms in Proposition 2.4, respectively. They have the advantage to propagate singularities only in one direction:

$$\text{sing supp}\, U_t^{\frac{3}{2}\pm\frac{1}{2}} f \subset \{\gamma_t^{\tilde{\Omega}}(x, \mp\xi) : (x, \xi) \in \text{WF}(f)\},$$

see section 2.4.

Now we consider the boundary value problem

$$(\frac{\partial^2}{\partial t^2} - \Delta_{\tilde{\Omega}})u = 0,\ u_{|\partial\Omega\times\mathbb{R}} = g \tag{3.3}$$

where $g = (U^1 f)_{|\partial\Omega\times\mathbb{R}}$. It is well-known that, in the transversal case, this behaves like a hyperbolic initial value problem. That is, there are solutions $u_\pm(x,t)$, unique modulo C^∞, for $t \in J = [0, 3t_0]$ and $x \in \tilde{\Omega}$, with sing supp $u_-(t,\cdot)$ near $\tilde{\gamma}_t$ and sing supp $u_+(t,\cdot)$ near $\bar{\gamma}_t$, and the operators $B^\pm$, $u_\pm = B^\pm g$, are Fourier integral operators (here g varies over boundary values of functions f with WF(f) conically close to (y,η)). In fact, one can take $B^- g = U^1 f$, and if we set

$$\bar{f} = (B^+ g)_{|t=0},$$

then $B^+ g = U^1 \bar{f}$ also. Now define K by (3.2). We claim that this is a microlocal parametrix near (y,η). Set $v = (Kf - Uf)_{|\Omega\times J}$. Clearly, $v(x,t)$ satisfies the wave equation. Because WF$(\bar{f})$ is near $(\bar{y},\bar{\eta})$ and $\bar{y} \in \tilde{\Omega} - \bar{\Omega}$, we also have $v_{|t=0}, v_{t|t=0} \in C^\infty(\Omega)$. Finally, $U^{\text{free}} f \equiv U^1 f$ and $U^{\text{free}} \bar{f} \equiv U^1 \bar{f} \mod C^\infty$ near $\partial\Omega \times J$, so v has smooth boundary values for $t \in J$, and again we can conclude that $v \in C^\infty(\Omega \times J)$.

Grazing case: In the interior and transversal cases, K could be constructed as Fourier integral operator, with smooth canonical relation. That this cannot work here is easy to see: If $\gamma_t(y,\eta)$ is tangent to $\partial\Omega$ at $\tilde{t}$ say the 'propagation of singularities map' $(y,\eta) \mapsto \gamma_{2\tilde{t}}(y,\eta)$ is not smooth, but only $C^{1/2}$, see figure 3.3. A parametrix in this case was constructed by Melrose and Taylor ([MT]; see also the nice presentation in [ZW]). We state the result formally and give some explanations in the next two sections.

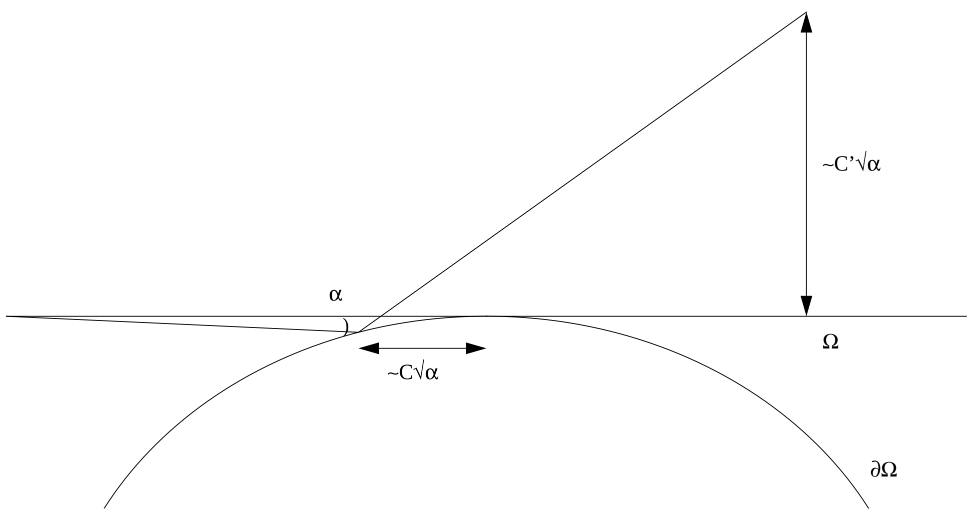

Figure 3.3: Geodesics in grazing case

In the proposition below, we summarize the discussion above.

We will use coordinates $(t, x) = (t, x_2, \dots, x_{n+1})$ on $\mathbb{R}_t \times \omega$, with $\partial\Omega = \{x_{n+1} = 0\}$. Even if we are dealing with a domain Ω in $\mathbb{R}^n$ (i.e. a flat metric), this straightening out of the boundary will make Δ a variable coefficient operator $P = P(x, D_x)$. We choose our coordinates such that

$$P(x_0, D_x) = D_x^2.$$

Throughout, we use the notation

$$x = (x_2, \dots, x_n, x_{n+1}) = (x', x_{n+1}) = (x'', x_n, x_{n+1}) \text{ and}$$
$$\xi = (\xi_1, \dots, \xi_n) = (\xi_1, \xi') = (\xi_1, \xi'', \xi_n).$$

Proposition 3.1 *For ω sufficiently small, the solution operator U for the initial boundary value problem (1.1), followed by restriction to $\omega_x \times I_t$, can be written as a finite sum*

$$\sum_l K^{(l)} \circ P^{(l)} + R$$

where R is smoothing, uniformly as $x \to \partial\Omega$, i.e.

$$R_t(x, y) \in C^\infty(\bar{\omega}_x \times \Omega_y \times I_t),$$

$P^{(l)}$ is a microlocal cutoff near a point $(y^{(l)}, \eta^{(l)}) \in T^*\omega'$, i.e. a pseudodifferential operator in $\Psi^0_{1,0}(\omega')$ whose symbol is supported near $(y^{(l)}, \mathbb{R}_+\eta^{(l)})$, and each $K^{(l)}$ is either (with extensions $\tilde{\Omega} = \tilde{\Omega}^{(l)}$ as above)

interior, *then $K^{(l)}$ is just the free solution U^{free} and given as in Proposition (2.4), or*

transversal, *then*

$$K^{(l)} = U^{free} - U^{free} \circ V^{(l)}$$

where $V^{(l)}$ is a Fourier integral operator of order 0, elliptic near $(\overline{y^{(l)}}, \overline{\eta^{(l)}}; y^{(l)}, \eta^{(l)})$, or

grazing, *then* $K^{(l)} = M \circ G$, *with (everything depending on* l*)*

$$M_t f(x) = \int_{\mathbb{R}^n} \left[g\left(A_-(\zeta) - A_+(\zeta)\frac{A_-}{A_+}(\zeta_0)\right) + h\left(A'_-(\zeta) - A'_+(\zeta)\frac{A_-}{A_+}(\zeta_0)\right)\right] e^{i(\theta + t\xi_1)} \hat{f}(\xi)\, d\xi \quad (3.4)$$

where $g(x,\xi) \in S^0, h(x,\xi) \in S^{-1/3}$ *have conical support near* $x = x_0, \xi = \bar{\xi} = (1, 0, \dots, 0)$, *and* G *is a Fourier integral operator of order* $1/6$ *elliptic near* $((\tilde{t}, 0), \bar{\xi}; y, \eta)$, *with* $\tilde{t}$ *defined by* $\gamma_{\tilde{t}}(y^{(l)}, \eta^{(l)}) \in \partial\Omega$. *The phase functions* $\zeta(x,\xi), \theta(x,\xi)$ *are smooth near* $(x_0, \bar{\xi})$ *in* $\bar{\Omega} \times \mathbb{R}^n$ *and homogeneous in* ξ *of degree* $2/3$ *and* 1 *respectively. They satisfy the eikonal equations*

$$\begin{gathered} p(x, \theta'_x) - \zeta p(x, \zeta'_x) = \xi_1^2, \\ p(x, \theta'_x, \zeta'_x) = 0, \end{gathered} \quad (3.5)$$

in $\{\zeta \leq 0\}$, *where* $p(x,\xi)$ *is the principal symbol of* P *and* $p(x,\xi,\eta)$ *denotes the bilinear form associated with* p. *The matrix* $\theta''_{x'\xi'}$ *is non-singular, and we have the normalization*

$$\begin{gathered} \zeta_0 = \zeta_{|x \in \partial\Omega} = -\xi_n \xi_1^{-1/3}, \text{ and} \\ \zeta'_{x_{n+1}} < 0 \text{ on } \partial\Omega. \end{gathered}$$

$A_\pm$ are Airy functions defined in Appendix A.
$\theta'_x(x_0, \bar{\xi})$ is tangential to the boundary, and by a rotation in the x'-coordinates we can make

$$\theta'_{x_i}(x_0, \bar{\xi}) = \delta_{in}. \quad (3.6)$$

To complete the discussion of the transversal case, we need to show that the operator $V : f \mapsto \bar{f}$ is an elliptic Fourier integral operator of order 0. We can write $V = R_0 \circ B^+ \circ R_\partial \circ U^1$, where R_0, R_∂ denote restriction to $\{t = 0\}$ and the boundary, respectively. All the operators on the right hand side are Fourier integral operators in the relevant domains, and transversality of γ easily implies the transversality condition on their canonical relations that guarantees that their composition is also a Fourier integral operator. $B^+g = U^1\bar{f}, B^-g = U^1 f$ imply that $V^{-1} = R_0 \circ B_- \circ R_\partial \circ U^1$, and this shows ellipticity and $\operatorname{ord} V = \operatorname{ord} V^{-1} = 0$.

Remarks:

- The form given for the transversal parametrix is a generalization of an elementary idea how to find solutions of the wave equation in $\mathbb{R}^n$ that vanish on the hyperplane $x_{n+1} = 0$: take any solution $u(x,t)$ of the wave equation and use $u(x', x_{n+1}, t) - u(x', -x_{n+1}, t)$.
- The transversal parametrix can also be constructed without reference to an extension $\tilde{\Omega}$, but the form given here is very intuitive and also makes the estimates for Theorem 1 very easy.
- The construction of the transversal part works for any geometry of the boundary; concavity was never used.

3.2 The Friedlander Example (Model)

In ([FL]), the following boundary value problem is considered:

$$\left[\frac{\partial^2}{\partial t^2} - \frac{1}{1+x_{n+1}}\left(\frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_{n+1}^2}\right)\right] u(t,x) = 0 \text{ for } x \in \Omega = \mathbb{R}^n_{x_{n+1}>0} \tag{3.7}$$

$$u(t,x',0) = f(t,x') \in \mathcal{E}'(t \geq 0)$$

$$u(t,x) = 0 \text{ if } t < 0 \tag{3.8}$$

Here $x = (x_2, \dots, x_{n+1}) = (x', x_{n+1})$ as before. This equation describes wave propagation in a layered medium, i.e. the propagation speed $1/\sqrt{1+x_{n+1}}$ depends only on one coordinate, here the distance to the boundary $\partial\Omega = \{x_{n+1} = 0\}$. Because this speed is highest at the boundary, waves or rays approaching it get refracted away from the x_{n+1}-direction, so $\partial\Omega$ is concave with respect to the metric $g_{ij} = (1+x_{n+1})\delta_{ij}$ that has these rays as geodesics, see figure 3.4. For $n \geq 3$, the operator $P = -\frac{1}{1+x_{n+1}}\Delta_x$ is not the Laplace-Beltrami operator of g, but this is not important here as only the principal part matters for us.

The virtue of this example is that most computations can be carried out explicitly (in particular, the Airy function appears naturally). R. Melrose proved ([M2]) that in some geometric sense, any boundary value problem near a concave boundary is equivalent to it, and that this equivalence can be used to construct solutions. Therefore, it serves as a model for our general situation.

To solve (3.7), take the Fourier transform in (t, x'), whose dual variables we call (τ, μ), to obtain

$$
\begin{aligned}
(-\tau^2(1 + x_{n+1}) + \mu^2 - \frac{\partial^2}{\partial x_{n+1}^2})\hat{u}(\tau, \mu, x_{n+1}) = 0 \text{ if } x_{n+1} > 0 \\
\hat{u}(\tau, \mu, 0) = \hat{f}(\tau, \mu)
\end{aligned}
$$

$$\hat{u} \text{ has an analytic continuation to } \operatorname{Im} \tau < 0.$$

Up to a change of coordinates, this is Airy's equation $A''(z) = zA(z)$, and we obtain for $\tau > 0$

$$\hat{u}(\tau, \mu) = \frac{A_+(-\tau^{-4/3}(\tau^2 - |\mu|^2 + x_{n+1}\tau^2))}{A_+(-\tau^{-4/3}(\tau^2 - |\mu|^2))} \hat{f}(\tau, \mu).$$

The choice of Airy function is dictated by the 'forward' condition (3.8). In order to put the solution in a form corresponding to the parametrix in section 3.1, we set $\xi_1 = \tau, \xi'' = (\mu_2, \dots, \mu_{n-1}), \xi_n = (\tau^2 - |\mu|^2)/\tau$ and get

$$u(t, x) \equiv (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{A_+(\zeta)}{A_+(\zeta_0)} e^{i(\theta(x,\xi)+t\xi_1)} g(\xi) \hat{f}(\xi_1, \xi'', \sqrt{\xi_1^2 - \xi_1\xi_n - |\xi''|^2})\, d\xi \tag{3.9}$$

modulo $C^\infty(\mathbb{R}_t \times \bar{\Omega}_x)$ where

$$
\begin{aligned}
\zeta &= -(\xi_n + x_{n+1}\xi_1)/\xi_1^{1/3}, \ \zeta_0 = -\xi_n/\xi_1^{1/3}, \\
\theta &= x''\xi'' + x_n\sqrt{\xi_1^2 - \xi_1\xi_n - |\xi''|^2}, \text{ and} \\
g(\xi) &= \frac{\xi_1}{2\sqrt{\xi_1^2 - \xi_1\xi_n - |\xi''|^2}} \rho(\xi),
\end{aligned}
$$

with a conical cutoff function ρ which equals 1 for $\xi_1 > 3|\xi'|$ and vanishes for $\xi_1 \leq 2|\xi'|$, and we assume that $\hat{f}(\xi_1, \xi'', \sqrt{\xi_1^2 - \xi_1\xi_n - |\xi''|^2})$ is rapidly decaying for $\xi_1 \leq 3|\xi'|$. To get the C^∞ error for u, we use the fact that the boundary value problem with smooth data has a smooth solution. The restriction on $\hat{f}$ represents no loss of generality since every f can be decomposed into parts which after a rotation in the (t, x')-variables have the required property.

Now in order to solve the initial boundary value problem for the operator P, we write as in the transversal case in section (3.1)

$$K = U^{\text{free}} - B^+ \circ R_\partial \circ U^{\text{free}}, \tag{3.10}$$

where B^+ is the operator in (3.9) and R_∂ denotes restriction to the boundary $\mathbb{R}_t \times \mathbb{R}^{n-1}_{x'}$. In principle, one could use the parametrix for U^{free} from section 2.3 here, but the cancellations in the composition become easier to handle when we write it in a different way:

It is clear from the construction of ζ and θ that $Ai(\zeta)e^{i(\theta+t\xi_1)}$ satisfies the wave equation (3.7). Thus, if we can find, for $k \in \mathcal{E}'(\mathbb{R}^n_{x_{n+1}>0})$, a distribution Gk for which

$$\int Ai(\zeta)e^{i\theta} g(\xi) (Gk)\hat{\ }(\xi)\, d\xi \equiv k \tag{3.11}$$

then

$$U_t^{\text{free}} k(x) \equiv \int Ai(\zeta)e^{i\theta + t\xi_1} g(\xi)(Gk)\hat{\ }(\xi)\, d\xi.$$

Using this and formula (7.2) from Appendix A in (3.10) gives the parametrix (3.4), with $h = 0$.

We will only outline the idea why Gk exists if k is supported away from the boundary, more than one unit with above choice of $\operatorname{supp}\rho$. For details, also in the general case, see ([ZW]).

The assumptions imply that $x_{n+1} + \xi_n/\xi_1 > C > 0$ on $(\operatorname{supp} k) \times (\operatorname{supp}\rho)$. So we can write $Ai(\zeta) = \sum_\pm \Psi_\pm(\zeta) e^{\mp i\frac{2}{3}(-\zeta)^{3/2}}$, and as ξ varies, the oscillation directions $(\phi^\pm)'_x$ of $e^{i\phi^\pm}$, $\phi^\pm = \mp\frac{2}{3}(-\zeta)^{3/2} + \theta$, sweep out a full neighborhood of some points $\eta_\pm$ (just check that $\phi''_{x\xi}$ is nonsingular). So a function k as above whose oscillation directions are concentrated near η_+ or η_-, i.e. whose Fourier transform is rapidly decaying far away from these directions, is representable as a superposition as in (3.11). One can check directly that η_+, η_- point away from respectively towards the boundary, so we are mainly interested in functions k microlocally supported near η_-. In a more formal manner, the above argument says that G can be obtained as the inverse of the elliptic Fourier integral operator with kernel $\int e^{i(\phi^- - z\xi)}\Psi_-(\zeta)g(\xi)\,d\xi$. Its order is 1/6 because $\Psi_-(\zeta) \in S^{-1/6}_{1,0}$ in the region considered.

3.3 Some Geometric Aspects of the Grazing Part

First, we will give interpretations of ξ, ζ and θ. Then we will give geometric meaning to the various parts of the grazing parametrix.

Let $K = K^{(l)}$ be one of the grazing parts in Proposition 3.1, and let (y^0, η^0) be close to $(y^{(l)}, \eta^{(l)})$. Let $\gamma = \gamma(y^0, \eta^0)$ and $\tilde{\gamma} = \gamma^{\tilde{\Omega}}(y^0, \eta^0)$, so γ may be reflected, but $\tilde{\gamma}$ doesn't notice the boundary. To γ there corresponds a unique value ξ^0 of the integration parameter ξ with the following property: Whenever we introduce a conical cutoff $\sigma(\xi)$ with $\sigma(\xi^0) \neq 0$ in the integral defining M_t, then the obtained operator M_t^σ will 'notice' a singularity at (y^0, η^0), i.e. for all distributions f with $(y^0, \eta^0) \in \mathrm{WF}(f)$, $M_t^\sigma G f$ is singular at $\gamma_t(y^0, \eta^0)$.
This implies that at (y^0, ξ^0), $e^{i\phi^-}$ oscillates in direction η^0, i.e. ξ^0 can be obtained from

$$(\phi^-)'_x(y^0, \xi^0) = \eta^0,$$

where ϕ^- is the total phase as in the last paragraph of the previous section.

The geometric relation between ξ^0 and γ is not as simple as in section 2.4, where we had $\xi^0 = \eta^0$. The following facts about ξ, ζ and θ are consequences of $(\dot{\tilde{\gamma}}_t)^\flat = (\phi^-)'_x(\tilde{\gamma}_t, \xi^0)$ and the properties stated in Proposition 3.1:

- ξ_1 is the frequency of oscillation of $e^{i\phi^\pm}$. In fact, one readily verifies that the eikonal equations (3.5) are equivalent to the system $p(x, (\phi^\pm)'_x) = \xi_1^2$. Now recall $|\eta|_g = \sqrt{p(y, \eta)}$ for $\eta \in T_y^*(\Omega)$.

- $\tilde{\gamma}$ is tangential to the surface $S_{\xi^0} = \{x : \zeta(x, \xi^0) = 0\}$ which is roughly parallel to $\partial\Omega$ at a distance approximately proportional to $|\xi_n^0/\xi_1^0|$. ξ_n^0 is positive or negative if γ hits the boundary transversally or not at all, respectively, and $\xi_n^0 = 0$ characterizes grazing γ. In the model, $S_{\xi^0} = \{x_{n+1} = -\xi_n^0/\xi_1^0\}$.

- The direction of $\tilde{\gamma}$ at the point of contact x^0 with S_{ξ^0} is $[\theta'_x(x^0,\xi^0)]^\flat$; thus, the normalizations (3.6) and $g_{ij}(x_0)=\delta_{ij}$ mean that we are considering geodesics roughly in the x_n-direction, which shall be referred to as 'the ray direction' in chapter 4.

- From the eikonal equations it then follows easily that, at $(x_0,\bar{\xi})$, $\theta''_{x_n\xi_j}\neq 0$ iff $j=n$ (see also Lemma 4.3 in section 4.3), and so the nondegeneracy of $\theta''_{x'\xi'}$ is equivalent to nondegeneracy of $\theta''_{x''\xi''}$, which just means that $(\xi^0)''$ parametrizes diffeomorphically the direction of the projection of $\tilde{\gamma}$ onto S_{ξ^0}. In the model, this direction is precisely $(\xi^0)''$.

We proceed to interpret various parts of M_t. Fix cutoff functions $\chi,\phi\in C^\infty(\mathbb{R})$ with

$$\chi(s)=\begin{cases}0 & \text{if } s<1\\ 1 & \text{if } s>2\end{cases}$$
$$\phi=1-\chi.$$

Let us say that an expression like $(\chi A_-)(\zeta)$ *generates* a line l (always a part of a geodesic) if

$$\text{sing supp}\left[\int [g(\chi A_-)(\zeta)+h(\chi A'_-)(\zeta)]\, e^{i(\theta+t\xi_1)}(Gf)\hat{}\,(\xi)\,d\xi\right]\supset l$$

for every f with $(y^0,\eta^0)\in \mathrm{WF}(f)$. Thus for example, saying that singularities are propagated according to the reflection law is equivalent to saying that $A_-(\zeta)-A_+(\zeta)\frac{A_-}{A_+}(\zeta_0)$ generates γ, by Proposition 3.1. We also have (see figure 3.4):

- $Ai(\zeta)$ generates $\tilde{\gamma}$. The splitting $Ai=-\omega A_- -\omega^2 A_+$ corresponds to a splitting of $\tilde{\gamma}$ as follows:

- $(\phi A_-)(\zeta)$ generates the 'incoming part' of $\tilde{\gamma}$, i.e. the line $\{\tilde{\gamma}_t : t\le t_0\}$ where t_0 is the time when $\tilde{\gamma}$ hits the surface S_{ξ^0} tangentially.

- $(\phi A_+)(\zeta)$ generates the 'outgoing part' of $\tilde{\gamma}$, i.e. the line $\{\tilde{\gamma}_t : t\ge t_0\}$.

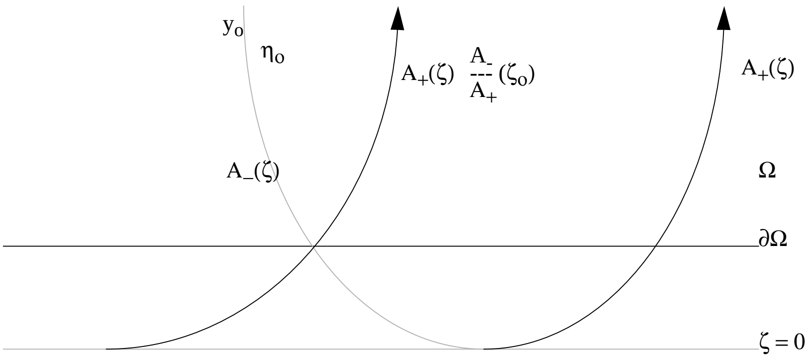

Figure 3.4: Meaning of terms in the grazing parametrix

- $(\phi A_+)(\zeta)\frac{A_-}{A_+}(\zeta_0)$ generates (an extension of) the reflected part of γ, i.e. $\{\gamma_t : t \geq \tilde{t}\}$ if γ hits the boundary at $t = \tilde{t}$, plus the backward extension of this as geodesic in $\tilde{\Omega}$ to the point where it becomes outgoing. If γ doesn't hit the boundary then the same line is generated as by $(\phi A_+)(\zeta)$.

- The part $\chi(\zeta)[A_-(\zeta)-A_+(\zeta)\frac{A_-}{A_+}(\zeta_0)] = -\omega^2\chi(\zeta)[Ai(\zeta)-A_+(\zeta)\frac{Ai}{A_+}(\zeta_0)]$ defines a convergent integral only for $x \in \bar{\Omega}$ and generates nothing there, since it is has exponential behavior, decreasing in Ω and increasing outside.

Chapter 4

Estimates near the boundary

Here we show how the methods in chapter 2 together with the parametrix in chapter 3 yield Theorem 1.

Because this parametrix is valid only for t near some point $t_0 \neq 0$, we take ρ supported in $(t_0 - \epsilon, t_0 + \epsilon)$. By Lemma 2.2, we may then replace χ_λ by $\int e^{-it\lambda}\rho(t)U_t\,dt$, and therefore we need to show

$$\|\int e^{-it\lambda}\rho(t)K_tPf\,dt\|_{L^p(\omega)} \leq C\lambda^{\epsilon(p)}\|f\|_{L^2(\Omega)}$$

for every part $K \circ P = K^{(l)} \circ P^{(l)}$ of the parametrix.

As P is bounded on $L^2(\Omega)$, the analysis in section (2.3) finishes the interior parts.

4.1 Transversal Parts

Here $K \circ P = (U^{\text{free}} - U^{\text{free}} \circ V) \circ P$.
$U^{\text{free}} \circ P$ is treated like an interior part, and so is $U^{\text{free}} \circ V \circ P$:

$$\|\int e^{-it\lambda}\rho(t)U_t^{\text{free}}VPf\,dt\|_{L^p(\omega)} \leq C\lambda^{\epsilon(p)}\|VPf\|_{L^2(\tilde{\Omega})} \leq C'\lambda^{\epsilon(p)}\|f\|_{L^2(\Omega)}$$

because elliptic Fourier integral operators of order 0 are bounded on L^2.

4.2 Grazing Parts: Reductions

Here $K = M \circ G$. Because G is elliptic near $((t_0, 0), \bar{\xi}; y, \eta)$ and of order $1/6$, $D_{x_1}^{-1/6} \circ G \circ P$ is bounded on L^2, so we can argue as in section 4.1 and replace $G \circ P$ by $D_{x_1}^{1/6}$. It will be useful to further split up

$$T_\lambda = \int e^{-i\lambda t} \rho(t) M_t \circ D_{x_1}^{1/6} \, dt$$

into pieces $T_\lambda = \sum_{j=1}^{8} T_\lambda^{(j)}$, corresponding to different summands in (3.4) and to regions of oscillatory respectively exponential behavior of the Airy functions:
Choose cutoff functions $\chi, \phi \in C^\infty(\mathbb{R})$ with

$$\chi(s) = \begin{cases} 0 & \text{if } s < 1 \\ 1 & \text{if } s > 2 \end{cases}$$
$$\phi = 1 - \chi.$$

The $T_\lambda^{(j)}$ then correspond to the following parts in M_t:

$$T_\lambda^{(1)} : \phi(\zeta) A_-(\zeta)$$
$$T_\lambda^{(2)} : \phi(\zeta) A_+(\zeta) \frac{A_-}{A_+}(\zeta_0)$$
$$T_\lambda^{(3)} : \chi(\zeta) Ai(\zeta)$$
$$T_\lambda^{(4)} : \chi(\zeta) A_+(\zeta) \frac{Ai}{A_+}(\zeta_0)$$

and $T_\lambda^{(5)}, \dots, T_\lambda^{(8)}$ analogous for the $A'_\pm$ terms. Note that we use Ai where positive ζ matter, $A_\pm$ otherwise (recall (7.2) in appendix A). For an interpretation of the various terms, see section 3.3. Also, we define similarly

$$T_\lambda^{(1+)} : \phi(\zeta) A_+(\zeta)$$
$$T_\lambda^{\text{free}} : Ai(\zeta).$$

$T_\lambda^{\mathrm{free}}$ is just a reparametrization of the parametrix for the free wave equation, see chapter 3.

We will use Lemma 2.3 for the operators $T_\lambda^{(j)} T_\lambda^{(j)*}$. Carrying out the t integral we get

$$T_\lambda^{(j)}(x,z) = \int \hat\rho(\lambda - \xi_1)\, L^{(j)}(x,\xi) e^{-iz\xi}\, d\xi$$

$$\text{and thus } T_\lambda^{(j)} T_\lambda^{(j)*}(x,y) = \int_{\mathbb{R}^n} |\hat\rho(\lambda - \xi_1)|^2\, L^{(j)}(x,\xi)\overline{L^{(j)}(y,\xi)}\, d\xi$$

where, for example, $L^{(1)}(x,\xi) = g(x,\xi)(\phi A_-)[\zeta(x,\xi)]\, \xi_1^{1/6} e^{i\theta(x,\xi)}$. Thus, setting

$$I_\lambda^{(i,j)}(x,y) = \int_{\mathbb{R}^{n-1}} L^{(i)}(x,\lambda,\xi')\overline{L^{(j)}(y,\lambda,\xi')}\, d\xi',$$

we need to show

$$\|I_\lambda^{(j,j)}\|_{L^{p'}\to L^p} \le C\lambda^{2\epsilon(p)},\ j = 1,\dots,8.$$

We will only consider the terms with $j = 1,\dots,4$ because the others can be treated the same way.

Note that we are spared consideration of mixed terms because we need only upper bounds:

$$\|T\|_{L^2\to L^p} \le \sum_1^8 \|T^{(j)}\|_{L^2\to L^p} = \sum_1^8 \|T^{(j)}T^{(j)*}\|_{L^{p'}\to L^p}^{1/2}.$$

For example, $I^{(1,2)}$ would be much harder to analyze directly. In $I^{(j,j)}$, quotients of Airy functions only occur for the 'elliptic' terms $j = 4$ or 8, and these are easy to analyze, see subsection 4.3.2. For $j = 2$ or 6, the quotient $A_-/A_+(\zeta_0)$ cancels out because of $|A_-| = |A_+|$, so $I^{(2,2)} = I^{(1+,1+)}$. Geometrically, this means that if x and y lie on a reflected geodesic γ, on the same side of the reflection point, then going from y along γ through the reflection point and then back to x has the same effect as going from y to x directly, without ever noticing the reflection.

We will, however, consider the mixed term $I^{(1,1+)}$ because for it the L^2 estimate in Lemma 2.3 can be proved using Young's inequality, while for $I^{(1,1)}, I^{(2,2)}$ it cannot. A functional analytic argument then yields the L^2 estimate for these operators.

Remark on notation: Here and from now on x and y both denote variables in ω, i.e. close to the boundary. In chapter 3, y was restrained to ω', away from the boundary.

4.3 Grazing Parts: Estimates on the Kernels

Proposition 4.1 *Let $I = I^{(j,j)}, j = 1, \ldots, 4, free$ or $I = I^{(1,1+)}$. For every $\tilde{\epsilon} > 0$ one can choose ω sufficiently small and supp g, supp h sufficiently close to the ξ_1 direction so that for all $x, y \in \omega$*

$$\lambda^{-(n-1)}|I_\lambda(x,y)| \leq C \text{ (for the } L^\infty \text{ estimate)} \tag{4.1}$$

and for the L^p-estimate, with any $N \geq 1$,

$$\lambda^{-(n-1)}|I_\lambda(x,y)| \leq \begin{cases} C|\lambda\delta|^{-(n-1)/2} & \text{if } |\delta_n| > (1-\tilde{\epsilon})|\delta| \quad \text{(4.2ray)} \\ C_N \lambda^{-1/3} |\lambda^{2/3}\delta|^{-N} & \text{if } |\delta_{n+1}| > \tilde{\epsilon}|\delta|. \quad \text{(4.2perp)} \\ C_N|\lambda\delta|^{-N} & \text{if } |\delta''| > \tilde{\epsilon}|\delta| \quad \text{(4.2else)} \end{cases} \tag{4.2}$$

Furthermore, for $I^{(free,free)}$ the sharper estimate (4.2else) is true whenever $|\delta_n| \leq (1-\tilde{\epsilon})|\delta|$, and for $I = I^{(1,1+)}$, $I^{(3,3)}$ and $I^{(4,4)}$, one has

$$\lambda^{-(n-1)}|I_\lambda(x,y)| \leq C\lambda^{-1/3}. \tag{4.3}$$

Here $\delta = x - y$ and $x = (x_2, x_3, \ldots, x_{n+1}) = (x'', x_n, x_{n+1})$.

(4.2ray) describes the behavior of $|I(x,y)|$ if $x - y$ is close to the ray direction (cf. the third remark in section 3.3), (4.2perp) if $x - y$ is close to perpendicular to the boundary, and (4.2else) the remaining cases. Of course, the estimates obtained here for $I^{(\text{free,free})}$ are the same as those proved in chapter 2. The weakness of (4.2perp) is the reason why we can prove Theorem 1 only if $n = 2$, see section 4.4. Of course, (4.2else) is void then, but it is of independent interest. The reason for the different behavior near the x_{n+1} direction is that, close to the boundary, the length scale of oscillations of the integrand in (3.1) in this direction is $\lambda^{-2/3}$ while in the other directions it is λ^{-1}.

The proof of (4.1) only uses the size of the $L^{(j)}$. For (4.2) oscillations are important. We will get (4.2ray) from a stationary phase analysis and (4.2perp) and (4.2else) using integration by parts only.

In the expression for $I(x,y)$ we change variables $\xi' = \lambda\eta'$ and write $\zeta = \zeta(x; 1, \eta')$ and $\tilde{\zeta} = \zeta(y; 1, \eta')$, and similarly for θ. Then we have, for example,

$$\lambda^{-(n-1)} I_\lambda^{(1,1+)}(x,y) =$$
$$\lambda^{1/3} \int_{\mathbb{R}^{n-1}} (\phi A_-)(\lambda^{2/3}\zeta)\, \overline{(\phi A_+)(\lambda^{2/3}\tilde{\zeta})}\, e^{i\lambda(\theta-\tilde{\theta})} g(x,\lambda,\lambda\eta')\overline{g(y,\lambda,\lambda\eta')}\, d\eta',$$

and the integrand is nonzero only on a small set of η' near 0, independent of λ.

4.3.1 Proof of the L^∞ estimate

Although the uniform bound (4.1) will be a byproduct of the analysis for the L^p-estimate, we give a simpler, independent proof here.

Let us consider $I_\lambda^{(1,1+)}$, for example. By Cauchy-Schwarz it suffices to show

$$\lambda^{1/3} \int_{|\eta'|<1} |(\phi A_-(\lambda^{2/3}\zeta)|^2 \, d\eta' < C. \tag{4.4}$$

Now $|\phi A_-|(t) \leq Ct^{-1/4}$, so the left hand side is majorized by $C \int_{|\eta'|<1} |\zeta|^{-1/2} \, d\eta'$. For $x \in \partial\Omega$, $\zeta = -\eta_n$ and so $|\zeta'_{\eta_n}| = 1$. Thus, for x near the boundary, $|\zeta'_{\eta_n}| > 1/2$. This implies $\int_{|\eta_n|<1} |\zeta|^{-1/2} \, d\eta_n < C$, and (4.4) follows.
The other cases are treated the same way; for $I^{(4,4)}$ notice

$$|A_+(\lambda^{2/3}\zeta) \frac{Ai}{A_+}(\lambda^{2/3}\zeta_0)| = |\frac{A_+(\lambda^{2/3}\zeta)}{A_+(\lambda^{2/3}\zeta_0)} Ai(\lambda^{2/3}\zeta_0)| \leq |Ai(\lambda^{2/3}\zeta_0)|$$

because $|A_+|$ is increasing and $\zeta \leq \zeta_0$ in Ω.

4.3.2 Proof of the decay estimates (4.2)

We will deal with $I^{(4,4)}$ in subsection 4.3.2. For the other integrals, we first get rid of the Airy functions. To this end, we could replace them by their asymptotic expansions. But this introduces non-smooth phases. The following lemma from [ZW] lets us use smooth phase functions throughout:

Lemma 4.2

$$(\phi A_\pm)(\lambda^{2/3}\zeta) \equiv \lambda^{1/3} \int_{\mathbb{R}} e^{i\lambda(T^3/3+T\zeta)} \Psi_\pm(\lambda^{1/3}T)\psi(T) \, dT \quad mod \ \lambda^{-\infty}$$

with $\psi \in C_0^\infty(\mathbb{R}), \Psi_\pm \in S^0(\mathbb{R}), \Psi'_\pm \in \mathcal{S}(\mathbb{R})$, *and* $\Psi_\pm(t)$ *rapidly decaying as* $t \to \mp\infty$.

Proof: See [ZW, Lemma 4.1]. One simply replaces $A_\pm(\zeta)$ by its asymptotics $\Psi_\pm(\zeta)e^{\mp i\frac{2}{3}(-\zeta)^{3/2}}$ and uses stationary phase. ♣
Any ψ works that equals one in $|T| < r$ where r can be chosen small if ω is small and $\operatorname{supp} g, \operatorname{supp} h$ are close to the ξ_1 direction.

For unity of presentation, we also write the (Schwartz) function χAi as in the lemma:

$$(\chi Ai)(\lambda^{2/3}\zeta) \equiv \lambda^{1/3}\int_{\mathbb{R}} e^{i\lambda(T^3/3+T\zeta)}\alpha_0(\lambda^{1/3}T)\psi(T)\,dT \qquad \mod \lambda^{-\infty}$$

with $\alpha_0 \in \mathcal{S}$ (simply take $\alpha_0(t) = (2\pi)^{-1}e^{-it^3/3}(\chi Ai)\hat{}(t)$).

Thus we have

$$\lambda^{-(n-1)}I_\lambda^{(i,j)} \equiv \lambda\int e^{i\lambda\Phi(x,y,T,S,\eta')}\alpha(\lambda^{1/3}S)\beta(\lambda^{1/3}T)a(x,y,T,S,\eta',\lambda)\,dTdSd\eta'. \tag{4.5}$$

Here

$$\Phi = \frac{1}{3}(T^3 - S^3) + T\zeta - S\tilde{\zeta} + \theta - \tilde{\theta} \tag{4.6}$$

and a is supported near $x = y = x_0, S = T = 0, \eta' = 0$, and satisfies

$$|D^\alpha_{S,T,\eta'}a| \le C_\alpha.$$

α and β are given in table 4.1. Note that $\alpha\beta \in \mathcal{S}$ for $I^{(1,1+)}$ and $I^{(3,3)}$.

For the free parametrix, written differently, we proved the estimates already in chapter 2, so we expect the main difficulty to come from the λ-dependence of α and β.

Our strategy in analyzing (4.5) will be as follows:
We carry out the T, η_n integral first. The stationary point is uniformly nondegenerate, so we get a factor λ^{-1}. For δ near the ray the remaining phase will have Hessian (with respect to S, η'') of order $|\delta|$, and stationary phase yields (4.2ray). Away from the ray there is no stationary point, and integration by parts yields the other estimates. The various nondegeneracy claims are usually easy to check directly at $x = y = x_0, \eta' = 0, S = T = 0$, and therefore hold nearby also, by smoothness of the phase. Thus, they are true for $x, y \in \omega$ and S, T, η' in the supports of $\psi(S), \psi(T), g(x,\lambda,\lambda\eta')$ and $h(x,\lambda,\lambda\eta')$ if ω and these supports are chosen small enough.

In the next subsection, we investigate the phase. Then we finish the proof of Proposition 4.1 in subsection 4.3.2, except for the elliptic term $I^{(4,4)}$, which is treated in subsection 4.3.2.

(i,j)	α	β
(free,free)	1	1
$(1,1)$	$\overline{\Psi_-}$	Ψ_-
$(2,2)$	$\overline{\Psi_+}$	Ψ_+
$(3,3)$	$\overline{\alpha_0}$	α_0
$(1,1+)$	$\overline{\Psi_+}$	Ψ_-

Table 4.1: Symbols

Analysis of the phase

It is useful to keep in mind the model, where

$$\zeta = -\eta_n - x_{n+1},\ \theta = x_n\sqrt{1-\eta_n-|\eta''|^2} + x''\eta''.$$

The stationary points of Φ with respect to T, η_n are determined by

$$\begin{aligned} \Phi'_T &= T^2 + \zeta = 0 \\ \Phi'_{\eta_n} &= T\zeta'_{\eta_n} - S\tilde{\zeta}'_{\eta_n} + (\theta'_{\eta_n} - \tilde{\theta}'_{\eta_n}) = 0. \end{aligned} \tag{4.7}$$

If $x = y \in \partial\Omega$ then $\zeta = -\eta_n, \theta = \tilde{\theta}$ and the Hessian $\mathrm{Hess}_{T,\eta_n}\Phi = \begin{pmatrix} 2T & -1 \\ -1 & 0 \end{pmatrix}$ is nondegenerate. Hence this is true for x, y near x_0, and the implicit function theorem gives functions $\bar{\eta}_n(S,\eta'',x,y), \bar{T}(S,\eta'',x,y)$ as unique solutions of (4.7), with

$$\begin{aligned} \bar{T} &= S \text{ if } x = y, \\ \bar{\eta}_n &= S^2 \text{ if } x = y \in \partial\Omega. \end{aligned} \tag{4.8}$$

Let $\phi(x,y,S,\eta'') = \Phi(x,y,\bar{T},S,\eta'',\bar{\eta}_n)$ be the phase at the stationary point.
From (4.8) and (4.6) we see

$$\begin{aligned} \phi_{|x=y} &= 0, \\ \phi'_{x|x=y} &= S\bar{\zeta}'_x + \bar{\theta}'_x, \end{aligned}$$

with $\bar{\zeta}'_x, \bar{\theta}'_x$ equal to ζ'_x, θ'_x evaluated at $\eta_n = \bar{\eta}_n, x = y$. Therefore,

$$\phi = (x-y)\cdot(S\bar{\zeta}'_x + \bar{\theta}'_x + O(|x-y|)). \tag{4.9}$$

We need to investigate closer the dependence of ϕ'_x on S and η''. But first we collect some information about θ:

Lemma 4.3 *$\theta'_{x_{n+1}} = 0$ at $x = x_0$, and for $x = x_0$, $\eta' = 0$*

$$\theta''_{x_n \eta''} = 0,\ \theta''_{x_n \eta_n} \neq 0 \tag{4.10}$$

$$\theta''_{x''\eta''},\ (\theta'_{x_n})''_{\eta''\eta''} \text{ are nondegenerate.} \tag{4.11}$$

Proof: Recall the eikonal equations, which read under the normalizations at $x = x_0$:

$$|\theta'_x|^2 + \eta_n |\zeta'_x| = 1, \tag{4.12}$$

$$\theta'_x \cdot \zeta'_x = 0. \tag{4.13}$$

As $\zeta \equiv \eta_n$ on $\partial\Omega$, we have $\zeta'_{x_i} \neq 0$ iff $i = n+1$, so (4.13) gives $\theta'_{x_{n+1}} = 0$. Differentiating (4.12) with respect to (η'', η_n) gives (4.10), using $\theta'_{x_i} = \delta_{in}$ at $x = x_0, \eta' = 0$. Because $\theta''_{x'\eta'}$ is nonsingular, (4.10) implies that $\theta''_{x''\eta''}$ is nonsingular. Finally, differentiating (4.12) twice in η'' gives

$$0 = \sum_{k=2}^{n} (|\theta'_{x_k}|^2)''_{\eta''\eta''} = 2\sum_{k=2}^{n} \theta'_{x_k} (\theta'_{x_k})''_{\eta''\eta''} + 2\sum_{k=2}^{n} \theta''_{x_k\eta''} \cdot \theta''_{x_k\eta''}$$

and therefore

$$(\theta'_{x_n})''_{\eta''\eta''} = -(\theta''_{x''\eta''})^t \theta''_{x''\eta''}$$

which proves the second part of (4.11).
♣

We return to ϕ. In the model, $\phi'_x = (\eta'', \sqrt{1 - S^2 - |\eta''|^2}, -S)$ at $x = y = x_0$.

Lemma 4.4 *At $x = y = x_0$, $(\eta'', S) = 0$ we have*

a) $(\phi'_{x_n})''_{\eta''\eta''}$ nondegenerate, $(\phi'_{x_n})''_{SS} \neq 0$.

b) $\phi''_{x_i S} \neq 0$ iff $i = n+1$.

c) $\phi''_{x''\eta''}$ nondegenerate and $\phi''_{x_i\eta''} = 0$ for $i = n$ and $i = n+1$.

Proof: At $x = y = x_0$, $\phi'_x = S\zeta'_x(x_0; \eta'', S^2) + \theta'_x(x_0; \eta'', S^2)$. So $\phi'_x = \theta'_x$ and $(\phi'_{x_n})''_{SS} = 2\theta''_{x_n\eta_n}$ at $S = 0$. Therefore, a) and c) are direct consequences of Lemma 4.3. Also, $\phi''_{x_iS} = \zeta'_{x_i}$ gives b). ♣

Let us illustrate with an example how we will use the lemma:
Say $|\delta_{n+1}| > \tilde{\epsilon}|\delta|$. We show that $\phi'_S = \delta_{n+1}\psi(x, y, S, \eta'')$ with ψ smooth in η'', S and $|\psi| > C > 0$ near $x = y = x_0, (\eta'', S) = 0$.

Clearly,

$$\psi = \begin{cases} \phi''_{x_{n+1}S} \text{ if } x = y, \text{ and else} \\ \phi''_{x_{n+1}S|x=y} + \frac{\delta''}{\delta_{n+1}}\phi''_{x''S|x=y} + \frac{\delta_n}{\delta_{n+1}}\phi''_{x_nS|x=y} + O(\frac{|\delta|^2}{\delta_{n+1}}). \end{cases}$$

By b) in Lemma (4.4), $\psi \neq 0$ if $x = y = x_0, (\eta'', S) = 0$. Then $|\delta/\delta_{n+1}| < \tilde{\epsilon}^{-1}$, and continuity of the ϕ-derivatives implies the claim.

Main Terms

Let

$$J = J_\lambda(S, \eta'', x, y) = \lambda \int e^{i\lambda\Phi}\, \beta(\lambda^{1/3}T) a\, dT d\eta_n,$$

with a as in (4.5). By stationary phase,

$$J = e^{i\lambda\phi}\, b(x, y, S, \eta'', \lambda)$$

where

$$|D^\alpha_{S,\eta''} b| \leq \begin{cases} C_\alpha & \text{if } \beta \equiv 1 \\ C_\alpha \lambda^{|\alpha|/3} & \text{else.} \end{cases}$$

We now have

$$\lambda^{-(n-1)} I^{(i,j)}_\lambda \equiv \int e^{i\lambda\phi(x,y,S,\eta'')} \alpha(\lambda^{1/3}S)\, b\, dS d\eta''.$$

We first prove (4.2ray). By Lemma 4.4, $\phi = \delta_n\tilde{\phi}$ with $\text{Hess}_{S,\eta''}\tilde{\phi}$ nondegenerate if $|\delta_n| > (1 - \tilde{\epsilon})|\delta|$. Stationary phase, with parameter $\lambda|\delta_n|$, now implies the estimate near the ray for $I^{(\text{free,free})}$.

Because the stationary phase method requires that differentiation of the amplitude brings out a factor of at most $(\lambda|\delta_n|)^{1/2}$ (see Appendix B), it is not directly applicable to the other cases. Therefore, we need to keep track of the λ-dependence of b.

Using Taylor's formula, write

$$\beta(\lambda^{1/3}T) = \sum_{j=0}^{k-1} \lambda^{j/3}(T-\bar{T})^j \, \beta^{(j)}(\lambda^{1/3}\bar{T}) + \lambda^{k/3}(T-\bar{T})^k h(\lambda^{1/3}T, \lambda^{1/3}\bar{T})$$

with h smooth and bounded, and k to be determined.

Putting this into the expression for J, we gain a factor of $\lambda^{j/3-\lceil j/2\rceil}$ in the jth term because $(T-\bar{T})^j$ vanishes to order j at the stationary point. Thus, if k is big enough, the term involving h will be less than some constant, and we only need to show:

$$|\int e^{i\lambda\delta_n\tilde{\phi}} \, \alpha(\lambda^{1/3}S) \, \gamma(\lambda^{1/3}\bar{T}(S,\eta'',x,y)) \, c(S,\eta'',x,y,\lambda) \, dSd\eta''| \leq C(\lambda|\delta_n|)^{-(n-1)/2}$$
$$\text{if } \alpha', \gamma' \in \mathcal{S} \text{ and } |D^{\alpha}_{S,\eta''}c| \leq C_\alpha.$$

If $n=2$, η'' is not there. Choose $\rho \in C_0^\infty(\mathbb{R})$ and introduce a factor $[1-\rho(\sqrt{\lambda|\delta_n|}(S-\bar{S}))]$ under the integral, where $\bar{S}=\bar{S}(x,y)$ is the stationary point; this introduces an error of $(\lambda|\delta_n|)^{-1/2}$, and a single integration by parts shows that the remaining integral is of the same order, because the L^1 norm of the derivative of the amplitude is uniformly bounded.

If $n>2$, we first note, using (4.8),

$$|D_{\eta''}(\gamma(\lambda^{1/3}\bar{T}))| \leq C\lambda^{1/3}|\delta_n| \leq C(\lambda|\delta_n|)^{1/3}$$

and similarly for higher derivatives, so that we can use stationary phase in the η'' variables. An argument as above takes care of the S integration, and we are done.

We now turn to the estimates (4.2perp) and (4.2else) away from the ray. Again, this is easy for $I^{(\text{free,free})}$: By Lemma 4.4 we have $|D_{S,\eta''}\phi| > C|(\delta'', \delta_{n+1})|$, thus simple integration by parts in

$$\lambda^{-(n-1)} I^{(\text{free,free})} \equiv \int e^{i\lambda\phi} \, b \, dSd\eta''$$

gives the result. In the other cases, we have the $\lambda^{1/3}$-dependence of α and β, so we will lose a factor $\lambda^{1/3}$ with each integration by parts. We show that this actually happens only when integrating with respect to S, not with η''. Because the η''-gradient of the phase can become zero when δ is close to perpendicular to $\partial\Omega$ we get only the weaker result (4.2perp) there.

First, we argue as before, replacing $\beta(\lambda^{1/3}T)$ by its Taylor expansion around $\beta(\lambda^{1/3}\bar{T})$, to see that we only need to consider

$$\int e^{i\lambda\phi}\alpha(\lambda^{1/3}S)\gamma(\lambda^{1/3}\bar{T})c(S,\eta'',x,y,\lambda)\,dSd\eta'' \tag{4.14}$$

with α, γ and c as before.

If $|\delta''| > \tilde{\epsilon}|\delta|$ then Lemma 4.4 implies $\phi'_{\eta''} = \delta''\cdot A(S,\eta'',x,y)$ for some uniformly nondegenerate matrix A, so

$$|\int e^{i\lambda\phi}\alpha(\lambda^{1/3}S)\gamma(\lambda^{1/3}\bar{T})c\,dSd\eta''| =$$

$$\lambda^{-N}|\int e^{i\lambda\phi}\alpha(\lambda^{1/3}S)(D_{\eta''}\frac{\phi'_{\eta''}}{|\phi'_{\eta''}|^2})^N\gamma(\lambda^{1/3}\bar{T})c\,dSd\eta''| \le C_N|\lambda\delta''|^{-\frac{2}{3}N}$$

since $|D_{\eta''}\gamma(\lambda^{1/3}\bar{T})| \le \lambda^{1/3}|\delta| \le |\lambda\delta|^{1/3}$.

If $|\delta_{n+1}| > \tilde{\epsilon}|\delta|$ then Lemma 4.4 shows similarly $\phi'_S = \delta_{n+1}\psi$ with $|\psi| > C > 0$, and integration by parts gives (4.2perp) since any S-derivative falling on α or γ brings out a factor of $\lambda^{1/3}$ but also makes the S-integral of order $\lambda^{-1/3}$ since $\alpha', \gamma' \in \mathcal{S}$.

Finally, for $I^{(3,3)}$ the improvement (4.3) follows from $\alpha \in \mathcal{S}$. For $I^{(1,1+)}$, $\alpha(\cdot)\gamma(\cdot+s) \in \mathcal{S}$ uniformly for $|s| < C$. Therefore, if $|\delta| \le \lambda^{-1/3}$ then

$$\int |\alpha(\lambda^{1/3}S)\gamma(\lambda^{1/3}\bar{T})c|\,dSd\eta'' \le C\int |\alpha(\lambda^{1/3}S)\gamma(\lambda^{1/3}[S+O(|\delta|)])|dS \le C\lambda^{-1/3}$$

and if $|\delta| \ge \lambda^{-1/3}$ then (4.2perp) for $N=1$ shows $|I| \le C\lambda^{-2/3}$.

The Elliptic Term $I^{(4,4)}$

Write

$$L = \lambda^{-(n-1)} I^{(4,4)} = \lambda^{1/3} \int u(-\lambda^{2/3}\eta_n, \lambda^{2/3}x_{n+1}\rho)\overline{u(-\lambda^{2/3}\eta_n, \lambda^{2/3}y_{n+1}\tilde{\rho})} e^{i\lambda(\theta-\tilde{\theta})} g\tilde{g}\, d\eta'' d\eta_n$$

with

$$u(s,t) = (\chi A_+)(s-t)\frac{Ai}{A_+}(s)$$

and $\zeta(x;1,\eta') = -\eta_n - x_{n+1}\rho(x,\eta')$, $\rho > 0$.

Lemma 4.5 *$u \in \mathcal{S}(\mathbb{R}_s \times \mathbb{R}_{t\geq 0})$ and*

$$|D^\alpha_{\eta''} u(s, \lambda^{2/3}x_{n+1}\rho)| \leq C_\alpha.$$

Proof: $s \geq t$ on $\operatorname{supp} u$ and $|u(s,t)| \leq |Ai(s)|$ if $t \geq 0$ (because $|A_+|$ is increasing) imply $u \in \mathcal{S}$ there. Also, $\rho > C > 0$ gives

$$|D_{\eta''} u(s, \lambda^{2/3}x_{n+1}\rho)| = |\lambda^{2/3}x_{n+1}\rho'_{\eta''} u'_t(s, \lambda^{2/3}x_{n+1}\rho)| \leq C|\frac{\lambda^{2/3}\ x_{n+1}\rho'_{\eta''}}{\lambda^{2/3}\ x_{n+1}\rho}| \leq C'$$

and similarly for higher derivatives. ♣

Now if $|\delta_n| > (1-\tilde{\epsilon})|\delta|$ then we get from Lemma 4.3 $\theta - \tilde{\theta} = \delta_n \hat{\phi}$ with $\hat{\phi}''_{\eta''\eta''}$ uniformly nondegenerate and $|\hat{\phi}'_{\eta_n}| > C > 0$. By Lemma 4.5, we can use stationary phase in η'' and obtain

$$|L| \leq C\lambda^{1/3}\lambda^{-2/3}|\lambda\delta|^{-(n-2)/2}.$$

If we then integrate by parts once in η_n we get

$$|L| \leq C\lambda^{-1/3}|\lambda^{1/3}\delta|^{-1}|\lambda\delta|^{-(n-2)/2},$$

and these two estimates easily imply

$$|L| \leq C|\lambda\delta|^{-(n-1)/2}.$$

If $|\delta''| > \tilde{\epsilon}|\delta|$, we integrate by parts in η'' as for the main terms, and get

$$|L| \leq C\lambda^{1/3}\lambda^{-2/3}|\lambda\delta|^{-N}.$$

Finally, if $|\delta_{n+1}| > \tilde{\epsilon}|\delta|$, we simply write

$$\begin{aligned}|L| &\leq C_N\lambda^{1/3}\int(1+\lambda^{2/3}|\eta_n|)^{-2}(1+\lambda^{2/3}x_{n+1}\rho)^{-N}(1+\lambda^{2/3}y_{n+1}\tilde{\rho})^{-N}\,d\eta''d\eta_n\\ &\leq C_N\lambda^{1/3}\lambda^{-2/3}|\lambda^{2/3}\delta_{n+1}|^{-N},\end{aligned}$$

using $|\delta_{n+1}| \leq \max(x_{n+1}, y_{n+1})$.

4.4 Grazing Parts: Completion of the Proof

Here we show how the estimates on the various kernels in Proposition 4.1 imply, for $n = 2$,

$$\|I_\lambda^{(j,j)}\|_{L^{p'}(\omega)\to L^p(\omega)} \leq C\lambda^{2\epsilon(p)},\ j = 1,\dots,4,\ p = p_2 = 6,$$

with the help of Lemma 2.3. The role of x_1 in the lemma is played by $x_n = x_2$ in the present setup. Thus, by the same scaling argument as in section 2.2, we need to prove for $I = I_\lambda^{(j,j)}$

$$\lambda^{-1}|I(x,y)| \leq C|\lambda\delta_2|^{-1/2} \tag{4.15}$$

$$\|I\|_{L^2(H_{x_2})\to L^2(H_{x_2})} \leq C \tag{4.16}$$

where $H_s = \{x \in \omega : x_2 = s\}$, and in (4.16) we write $I = I_{x_2x_2}$ for simplicity.

Near the ray, (4.15) is just (4.2ray). Away from it, we use (4.2perp) with $N = 1$ and (4.1):

$$\lambda^{-1}|I| \leq C\min(\frac{1}{|\lambda\delta|}, 1) \leq C\sqrt{\frac{1}{|\lambda\delta|}} = C|\lambda\delta|^{-1/2}.$$

The L^2-bound (4.16), where $\delta_2 = 0$, can be proved using Young's inequality in the 'better' cases $I^{(\text{free},\text{free})}, I^{(3,3)}, I^{(4,4)}$ and $I^{(1,1+)}$: For $I^{(\text{free},\text{free})}$, (4.1) and (4.2else) give

$$|I| \leq C\frac{\lambda}{(1+\lambda|\delta_3|)^2}, \text{ so } \int |I|\, dx_3, \int |I|\, dy_3 \leq C.$$

For $I^{(3,3)}, I^{(4,4)}$ and $I^{(1,1+)}$, (4.3) and (4.2perp) give

$$|I| \leq C\frac{\lambda^{2/3}}{(1+\lambda^{2/3}|\delta|)^2}, \text{ so } \int |I|\, dx_3, \int |I|\, dy_3 \leq C.$$

To obtain (4.16) for $I^{(1,1)}$ and $I^{(2,2)}$, we write

$$Ai - \chi Ai = -\omega\phi A_- - \omega^2\phi A_+$$

and infer, using the positivity of the operators $I^{(1,1)}$ and $I^{(2,2)} = I^{(1+,1+)}$ on $L^2(H_{x_2})$ and writing $\|\ \| = \|\ \|_{L^2(H_{x_2})\to L^2(H_{x_2})}$,

$$\max(\|I^{(1,1)}\|, \|I^{(2,2)}\|) \leq \|I^{(1,1)} + I^{(2,2)}\| =$$
$$\|I^{(\text{free},\text{free})} + I^{(3,3)} - I^{(3,\text{free})} - I^{(\text{free},3)} - \omega^2 I^{(1,1+)} - \omega^2 I^{(1+,1)}\| \leq$$
$$2(\|I^{(\text{free},\text{free})}\| + \|I^{(3,3)}\| + \|I^{(1,1+)}\|) \leq C.$$

Chapter 5

The Case of the Disk

Here we prove Theorem 2. A complete system of (nonnormalized) eigenfunctions on the unit disk is given by

$$e_{nm}(r,\phi) = e^{in\phi} J_n(\lambda_{nm} r), n \in \mathbb{Z}, \gtrdot = \nVdash, \nvDash, \nVdash, \dots$$

where r, ϕ are polar coordinates, J_n is the nth Bessel function and λ_{nm} the mth positive zero of J_n. λ_{nm} is the frequency of e_{nm}. J_n can be defined as

$$J_n(t) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(t \sin\theta + n\theta)}\, d\theta \tag{5.1}$$

and solves the differential equation

$$t^2 J_n'' + t J_n' + (t^2 - n^2) J_n = 0$$

which results from separation of variables when the eigenvalue equation for Δ is written in polar coordinates.

In section 2.2 we proved

$$\|e_{nm}\|_{L^6(r \le R)} \le C_R \lambda_{nm}^{1/6} \|e_{nm}\|_{L^2(\Omega)}$$

for each $R < 1$, so the functions in Theorem 2a) must be big near the boundary.

Also, because the treatment of the transversal parts in chapters 3 and 4 is independent of the geometry of the boundary, we expect that only eigenfunctions make trouble with gradients almost tangential to the boundary. For the e_{nm} this means that the ratio of angular frequency n to radial frequency (which depends monotonically on λ_{nm} and therefore on m) should be big and thus m small. In fact, after collecting some facts on the asymptotic behavior of Bessel functions we will see that $f_n = e_{n1}$ works for a). In contrast, the oscillation of e_{0m} is purely radial, and the estimates of Theorem 1 are easily verified directly.

In order to make sufficiently precise estimates we need to know the behavior of J_n near λ_{n1}. Since $\lambda_{n1} \sim n$ as $n \to \infty$, see Lemma 5.2 below, this is contained in the first part of the following Lemma:

Lemma 5.1

$$\begin{aligned} J_n(nz) &= n^{-1/3} g(z)\, Ai(n^{2/3}\zeta(z)) + O(n^{-4/3}(1+n^{2/3}\zeta)^{-1/4} g(z)), \\ J_n'(nz) &= n^{-2/3} \tilde{g}(z)\, Ai'(n^{2/3}\zeta(z)) + O(n^{-4/3}(1+n^{2/3}\zeta)^{1/4} \tilde{g}(z)), \end{aligned} \tag{A}$$

uniformly in $z > 0$ with functions $g, \tilde{g}, \zeta \in C^\omega(0,\infty)$, g and $\tilde{g}$ nonvanishing and $\zeta(1) = 0$, $\zeta' < 0$ on $(0,\infty)$.

$$\begin{aligned} J_n(nz) &= \frac{1}{\sqrt{2\pi n}\,(z^2-1)^{1/4}} \sin(\tfrac{2}{3}n(-\zeta)^{3/2} + \tfrac{\pi}{4}) + O(n^{-3/2} z^{-1/2}), \\ J_n'(nz) &= \frac{1}{\sqrt{2\pi n}} \frac{(z^2-1)^{1/4}}{z} \cos(\tfrac{2}{3}n(-\zeta)^{3/2} + \tfrac{\pi}{4}) + O(n^{-3/2} z^{-1/2}), \end{aligned} \tag{B}$$

uniformly in $z > C$ for any $C > 1$.

In fact, g, $\tilde{g}$ and ζ can be computed explicitly as

$$\frac{2}{3}|\zeta|^{3/2} = \begin{cases} \sqrt{z^2-1} - \arctan\sqrt{z^2-1} & \text{if } z \geq 1 \\ \ln \dfrac{1+\sqrt{1-z^2}}{z} - \sqrt{1-z^2} & \text{if } z \leq 1 \end{cases}$$

$$g(z) = \left(\frac{4\zeta}{1-z^2}\right)^{1/4}, \quad \tilde{g} = \zeta' g,$$

and then (B) is just a special case of (A), using the Airy function asymptotics (see appendix A). But (B) can easily be obtained directly by stationary phase from (5.1). The O errors can be improved by using lower order terms. For proofs and full asymptotic expansions see [OL].

$\zeta(z)$ is related to the functions $\zeta(x,\xi), \theta(x,\xi)$ in chapter 3 as follows: Set $x_2 = \phi, x_3 = r-1$ and $\alpha(z) = z^{-2/3}\zeta(z)$. Then $\alpha(1) = 0, \alpha'(1) \neq 0$, so α has an inverse β near 1. Then

$$\zeta(x;1,\eta) = \alpha((x_3+1)\beta(\eta))\ (x_3+1)^{2/3} \text{ and}$$
$$\theta(x;1,\eta) = x_2/\beta(\eta).$$

The parameter η corresponds to λ, n via $\eta = \alpha(\lambda/n)$.

The following lemma states some implications of Lemma 5.1 which we will need. They could be stated in sharper form.
If α, β are functions of λ, n, m, we write

$$\alpha \approx \beta \text{ for } c\alpha \leq \beta \leq C\alpha$$

with some positive constants c and C, for sufficiently large λ. As always, C may denote a different constant at each occurence.

Lemma 5.2 *For any $\epsilon > 0$,*

$$\int_0^1 J_n^2(\lambda r)\, rdr \approx \begin{cases} \lambda^{-1} & \text{if } \frac{\lambda}{n} \geq 1+\epsilon, \\ \lambda^{-1}(-\zeta(\frac{\lambda}{n}))^{1/2} & \text{if } 1+\epsilon\lambda^{-2/3} \leq \frac{\lambda}{n} \leq 2. \end{cases} \tag{a}$$

$$J_n(nz) \leq \begin{cases} C(nz)^{-1/2} & \text{if } z \geq 1+\epsilon, \\ Cn^{-1/3}\,(1-n^{2/3}\zeta)^{-1/4} & \text{if } z \geq 1, \\ Cn^{-1/3}\,e^{-n\zeta^{3/2}} & \text{if } 1/2 \leq z \leq 1. \end{cases} \tag{b}$$

$$\lambda_{nm} = n + \alpha_m n^{1/3} + O(n^{-2/3}) \text{ as } n \to \infty \text{ with } m \text{ fixed, } \alpha_1 > 0. \tag{c}$$

$$|\gamma m^{2/3} + n^{2/3}\zeta(\frac{\lambda_{nm}}{n})| \leq C(m^{-1/3} + n^{-1}) \text{ for some constant } \gamma. \tag{d}$$

The zeroes of J_n and J_{n+1} interlace, i.e. $\lambda_{nm} < \lambda_{n+1,m} < \lambda_{n,m+1}$, (e)

and $\lambda_{n+1,m} - \lambda_{nm} > 1$.

Sketch of proof (for details see [OL]):
(a) From Bessel's equation we see $\frac{1}{2}\frac{d}{dt}\left[t^2(J_n')^2(t)+(t^2-n^2)J_n^2(t)\right]=tJ_n^2(t)$, so

$$\int_0^\lambda J_n^2(t)\,tdt=\frac{\lambda^2}{2}\left[(J_n')^2(\lambda)+(1-(\tfrac{n}{\lambda})^2)J_n^2(\lambda)\right].$$

Now use Lemma 5.1 with $z=\lambda/n$. Note that if $J_n(\lambda)=0$ then we get

$$\int_0^1 J_n^2(t)\,tdt=\frac{1}{2}(J_n')^2(\lambda)=\frac{1}{2}n^{-4/3}\tilde{g}^2(Ai'(n^{2/3}\zeta))^2+O(n^{-2}) \tag{5.2}$$

for $z=\frac{\lambda}{n}\leq C$.
(b) is clear from Lemma 5.1 and the Airy function estimates in Appendix A.
(c),(d): If a_m is the mth zero of the Airy function then (A) in Lemma 5.1 implies

$$a_m=n^{2/3}\zeta(\frac{\lambda_{nm}}{n})+O(n^{-1}).$$

This gives the asymptotics for λ_{nm}. The Airy function asymptotics gives $a_m=-Cm^{2/3}+O(m^{-1/3})$, and this shows (d).
(e) The interlacing is a well-known elementary fact, see [WA] for example. The gap statement is harder. One can show $\frac{d^2\lambda_{nm}}{dn^2}<0$ (e.g. using [WA, equation (3) in 15.6]). Also, (c) implies $\frac{d\lambda_{nm}}{dn}\to 1$ as $n\to\infty$.
♣
Proof of Theorem 2: a) We show that

$$\frac{\|e_{n1}\|_6}{\|e_{n1}\|_2}\geq C\lambda_{n1}^{2/9}.$$

This reflects the fact that e_{n1} is 'essentially' supported in a strip of width $n^{-2/3}$ near the boundary, so the quotient is approximately $(n^{-2/3})^{\frac{1}{6}-\frac{1}{2}}=n^{2/9}$. Note that $\lambda_{n1}\sim n$ by (c).
(a) shows $\|e_{n1}\|_2\approx n^{-2/3}$, and (A), (c) and $Ai(0)\neq 0$ give

$$J_n(n+tn^{1/3})\approx n^{-1/3} \qquad \text{for } 0\leq t\leq \alpha_1/2,$$

from which we conclude $\|e_{n1}\|_6 \geq Cn^{-1/3}(n^{-2/3})^{1/6}$, and the claim follows.
b) We need to show

$$\sum_{\substack{n,m \\ \lambda-1 \leq \lambda_{nm} \leq \lambda}} \frac{J_n^2(\lambda_{nm} r)}{\|J_n(\lambda_{nm} \cdot)\|_2^2} \leq C\lambda.$$

Write Q_{nm} for one summand. Because of Proposition 2.1, we may assume $r > 1/2$. By (e), there is at most one summand for each m. For $m = 1, 2, \ldots$ let $n(m)$ be the index n for which λ_{nm} is closest to $\lambda - 1/2$. Clearly, the claim follows from

$$\sum_{m: n(m)>0} Q_{nm} \leq C\lambda. \tag{5.3}$$

n is a decreasing function of m, even strictly decreasing as long as $n > 0$, by the interlacing property. By (c), $n(1) \leq \lambda$, and therefore $n(\lambda + 1) = 0$, so the sum goes at most up to $m = \lambda$. On the other hand, (d) for $n = 1$ and the asymptotics $\zeta(z) \sim -Cz^{2/3}$ as $z \to \infty$ show $n(2\epsilon\lambda) > 0$ for some $\epsilon > 0$ independent of λ, and strict monotonicity implies

$$n(m) > 2\epsilon\lambda - m.$$

Now consider the terms in (5.3) with $m > \epsilon\lambda$. Then we have $n \leq \lambda - \epsilon\lambda$, so $\frac{\lambda}{n} > \frac{1}{1-\epsilon} > 1$, and (a) and (b) give

$$\sum_{\substack{m > \epsilon\lambda \\ n(m)>0}} Q_{nm} \leq C\lambda \sum_{\epsilon\lambda}^{\lambda} \frac{1}{\lambda r} \leq C\lambda.$$

This part of the sum corresponds to the transversal part in chapter 3.

We now turn to the harder part with $m \leq \epsilon\lambda$ and $1 < \frac{\lambda}{n} \leq \frac{1}{\epsilon}$. Clearly, (d) remains true if λ_{nm} is replaced by λ, since $|\lambda_{nm} - \lambda| < \lambda_{n,m+1} - \lambda_{n,m-1}$. By (a), (c) and (d) we now have

$$\|J_n(\lambda_{nm} \cdot)\|_2^2 \approx \lambda^{-1} |\zeta(\frac{\lambda_{nm}}{n})|^{1/2} \approx \lambda^{-4/3} m^{1/3}.$$

So all that remains is to show

$$I = \sum_{\substack{m<\epsilon\lambda \\ \zeta>0}} m^{-1/3} e^{-n\zeta^{3/2}} \le C\lambda^{1/3},$$
$$II = \sum_{\substack{m<\epsilon\lambda \\ \zeta<0}} m^{-1/3} (1-n^{2/3}\zeta)^{-1/2} \le C\lambda^{1/3}$$

where $\zeta = \zeta(\frac{\lambda}{n}r)$ and $n = n(m)$ as always. We will even see $I \le C$.

One way to proceed would be to define $n(m)$ as function of real m, replace sums by integrals and use ζ as new variable. But let us stay with integers m here. In any case, we need to estimate the change of ζ with m: Let $m > \bar{m}$, $\bar{n} = n(\bar{m})$, so $n < \bar{n}$. With $\zeta_0 = \zeta(\frac{\lambda}{n})$, and $\bar{\zeta}, \bar{\zeta}_0$ defined using $\bar{n}$, we then have, using $\zeta' < 0$, $r \in [1/2, 1]$ and $\bar{n} \approx \lambda$,

$$\begin{aligned}\bar{\zeta} - \zeta &\approx r(\frac{\lambda}{n} - \frac{\lambda}{\bar{n}}) \approx \bar{\zeta}_0 - \zeta_0 \approx \lambda^{-2/3}(\bar{n}^{2/3}\bar{\zeta}_0 - \bar{n}^{2/3}\zeta_0) \\ &\ge \lambda^{-2/3}(\bar{n}^{2/3}\bar{\zeta}_0 - n^{2/3}\zeta_0) = \gamma\lambda^{-2/3}(m^{2/3} - \bar{m}^{2/3} + O(\bar{m}^{-1/3})),\end{aligned}$$

and this implies

$$\bar{\zeta} - \zeta \ge C\frac{m-\bar{m}}{m^{1/3}\lambda^{2/3}} \tag{5.4}$$

for some $C > 0$ if $m - \bar{m} \ge C_0$.
We now analyze the sums I and II.
ζ is decreasing in m. Let $M = \max\{m : \zeta > 0\}$. Applying (5.4) repeatedly gives for $m + C_0 \le M$

$$\zeta \ge C\lambda^{-2/3}(\frac{1}{(m+C_0)^{1/3}} + \frac{1}{(m+2C_0)^{1/3}} + \cdots) \ge C\lambda^{-2/3}(M^{2/3} - m^{2/3})$$

where the sum goes at most up to M. If one uses this in the expression for I and estimates the sum by an integral, one gets

$$I \le C\int_1^M m^{-1/3} e^{-C'(M^{2/3}-m^{2/3})^{3/2}}\, dm,$$

and a change of variable $z = M^{2/3} - m^{2/3}$ shows $I \leq C$.

With II we proceed similarly: For $m > M$,

$$\zeta < -C\lambda^{-2/3}(m^{2/3} - M^{2/3}),$$

so

$$II \leq C \int_M^\lambda \frac{1}{m^{1/3}\,(m^{2/3} - M^{2/3})^{1/2}}\,dm,$$

and changing variables $z = m^{2/3} - M^{2/3}$ shows

$$II \leq C \int_0^{\lambda^{2/3}} z^{-1/2}\,dz = 2C\lambda^{1/3}.$$

This finishes the proof of part b).

♣

Remark: The method of estimating the L^2-norm of J_n using the identity (5.2) involving its derivative might seem artificial. We could have proceeded directly and used the Airy function asymptotics only. The reason why we chose our method is the following:

The method leads to a sum over squares of terms like

$$\frac{Ai(n^{2/3}\zeta)}{Ai'(n^{2/3}\zeta_0)}$$

where $n^{2/3}\zeta_0$ ranges over the zeroes of Ai. As M. Williams showed in [WI], the parametrix for the wave kernel near a convex boundary can always be written in terms of such a sum, using the residue theorem on its usual representation as complex line integral. This strongly suggests that the estimates above essentially suffice to prove the L^∞ bound near a convex boundary point of an arbitrary domain.

Chapter 6

Remarks and Problems

- The foremost problem is to generalize Theorem 1 to higher dimensions. While $n = 3$ can probably be handled by a better argument to obtain the L^2 estimate needed in Lemma 2.3, avoiding Young's inequality, the case $n \geq 4$ is less clear as the decay of the kernels on a $\lambda^{-2/3}$ scale in directions close to perpendicular to the boundary seems to make Lemma 2.3 too weak then.

- Next is the convex case because here a parametrix for the wave equation is known. Is the example in chapter 5 worst possible?

- Sogge used the interior estimates proved in chapter 2 to deduce certain results about convergence of Bochner-Riesz means on Ω, see [S3]. What are the implications of the different behavior of $\|\chi_\lambda\|_{L^2 \to L^p}$ on the disk for these means?

- It would be very interesting to decide if the L^∞ estimate $\|\chi_\lambda f\|_{L^\infty(\Omega)} \leq C\lambda^{(n-1)/2}\|f\|_{L^2(\Omega)}$ is true for any geometry of the boundary. One would need to find a method of proving such estimates without using parametrices for the wave equation.

- The asymptotics of the L^p-norms of eigenfunctions seems unclear. Corollary 1.1 gives an upper bound. But unlike for the spectral projections, this is not sharp in all cases: On the cube, $\|e_j\|_{L^p}/\|e_j\|_{L^2}$ is uniformly (in j) bounded, for the standard basis (e_j) of eigenfunctions. Is it true that this quotiont must be unbounded if there is any curvature, either in the metric or in the boundary?

Chapter 7

Appendix A: Airy functions

Ai is the unique (up to constant factor) bounded solution of

$$A''(z) = zA(z).$$

Set $A_-(z) = Ai(\omega z), A_+(z) = Ai(\omega^2 z)$, where $\omega = e^{2\pi i/3}$. $A_\pm$ are also solutions of this equation. Ai and $A_\pm$ are called Airy functions. They are linearly related by

$$Ai + \omega A_- + \omega^2 A_+ \equiv 0. \tag{7.1}$$

To see this, just evaluate this expression and its derivative at 0. The Airy functions have oscillatory behavior for negative argument and exponential behavior for positive argument. More precisely,

$$Ai(z) = \Psi(z)e^{-\frac{2}{3}z^{3/2}}$$

where Ψ is holomorphic and has the asymptotic expansion, uniformly in $\{z \in \mathbb{C} : |\arg F| \le (\nVdash - \delta)\pi\}$ for any $\delta > 0$,

$$\Psi(z) \sim z^{-1/4} \sum_0^\infty a_j z^{-\frac{3}{2}j}.$$

In particular,

$$\begin{aligned} A_\pm(z) &= \Psi_\pm(z)e^{\mp\frac{2}{3}i(-z)^{3/2}} && \text{if } z < 0, \\ A_\pm(z) &= \Psi_\pm(z)e^{\frac{2}{3}z^{3/2}} && \text{if } z > 0, \\ Ai(z) &= \Psi(z)e^{-\frac{2}{3}z^{3/2}} && \text{if } z > 0 \end{aligned}$$

with $\Psi_\pm(z) = \Psi(\omega^\mp z)$. Also, from this and (7.1),

$$Ai(z) = c(-z)^{-1/4}\sin(\tfrac{2}{3}(-z)^{3/2} + \tfrac{\pi}{4}) + O((-z)^{-3/2}) \text{ for } z \to -\infty.$$

Here are a few more useful facts: $A_- = \overline{A_+}$ for real z, and $|A_\pm|$ is monotonically increasing. In the context of the grazing ray parametrix it is useful that (7.1) implies

$$A_-(\zeta) - A_+(\zeta)\frac{A_-}{A_+}(\zeta_0) = -\omega^2\left(Ai(\zeta) - A_+(\zeta)\frac{Ai}{A_+}(\zeta_0)\right). \tag{7.2}$$

Chapter 8

Appendix B: Stationary Phase

Here I state the stationary phase lemma in the form I need it. The proof is standard and therefore only sketched.

Let $\phi(x,y)$ be a smooth function of $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, defined near (x_0, y_0), and suppose $\phi(x_0, \cdot)$ has a nondegenerate critical point at y_0, i.e.

$$\phi'_y(x_0, y_0) = 0, \phi''_{yy} \text{ nondegenerate.}$$

By the implicit function theorem there is a smooth function $y(x)$ near x_0, with $y(x_0) = y_0$, such that $\phi(x, \cdot)$ has, for y near y_0, a unique critical point at $y = y(x)$. By continuity of ϕ''_{yy} it is nondegenerate.

For each $\lambda > 1$ let $a(x, y, \lambda)$ be a smooth function of x, y with support sufficiently close to (x_0, y_0) and independent of λ.

Assume

$$|D^\alpha_{x,y} a(x, y, \lambda)| \leq C_\alpha \lambda^{r|\alpha|} \tag{8.1}$$

for all multi-indices α, for some $r \leq 1/2$.

Lemma 8.1 *Let*

$$I^\gamma_\lambda(x) = \int e^{i\lambda\phi(x,y)} (y - y(x))^\gamma a(x, y, \lambda)\, dy$$

for some multiindex γ. Then

$$I_\lambda^\gamma(x) = \lambda^{-m/2-\lceil|\gamma|/2\rceil} e^{i\lambda\phi(x,y(x))} b(x,\lambda)$$

where b satisfies the same estimates as a:

$$|D_x^\alpha b(x,\lambda)| \leq C_\alpha \lambda^{r|\alpha|}.$$

Here $\lceil s\rceil$ is the smallest integer greater than or equal to s. Thus the nondegenerate stationary point gives a decay, in powers of $\lambda \to \infty$, of $-\frac{1}{2}$(number of integration variables), and each order of vanishing of the amplitude at the stationary point gives an additional gain of $-\frac{1}{2}$, with odd orders slightly better.

For $r = 0$ a proof can be found in [S3], for example. The main idea (for $\alpha = 0$ say) is to cut off the integral smoothly at a distance $\lambda^{-1/2}$ from the stationary point, introducing an error of the order of the desired estimate, and repeatedly integrating by parts in the remaining integral. As the λ-dependence of the cutoff function is as in (8.1), with $r = \frac{1}{2}$, the resulting estimate is not weakened by allowing to have a stronger λ-dependence of a as in (8.1), as long as $r \leq \frac{1}{2}$.

Bibliography

[FL] F. G. Friedlander, *The wave front set of the solution of a simple initial-boundary value problem with glancing rays,* Math. Proc. Camb. Phil. Soc. **79**(1976), 145-159.

[H1] L. Hörmander, *The analysis of linear partial differential operators Vols. III and IV*, Springer Verlag, Berlin, 1985.

[H2] L. Hörmander, *The spectral function of an elliptic operator,* Acta Math. **121**(1968), 193-218.

[M1] R. B. Melrose, *Microlocal parametrices for diffractive boundary value problems,* Duke Math. J. **42**(1975), 605-635.

[M2] R. B. Melrose, *Equivalence of glancing hypersurfaces,* Invent. Math. **37**(1976), 165-191.

[M3] R. B. Melrose, *Weyl's conjecture for manifolds with concave boundary,* Proc. Symp. Pure Math. **36**(1980), 257-274.

[MT] R. B. Melrose and M. E. Taylor, *Boundary problems for the wave equation with grazing and gliding rays*, manuscript.

[OL] F. Olver, *The asymptotic expansion of Bessel functions of large order,* Phil. Trans. Roy. Soc. London Ser. A **247**(1954), 328-368.

[SS] A. Seeger and C. D. Sogge, *Bounds for eigenfunctions of differential operators,* Indiana Math. J. **38**(1989), 669-682.

[S1] C. D. Sogge, *Concerning the L^p norm of spectral clusters for second order elliptic operators on compact manifolds*, J. Func. Anal. **77**(1988), 123-134.

[S2] C. D. Sogge, *Remarks on L^2 restriction theorems for Riemannian manifolds,* Analysis at Urbana 1, Camb. Univ. Press, Cambridge, 1989, 416-422.

[S3] C. D. Sogge, *Fourier integrals in classical analysis*, Camb. Univ. Press, to appear.

[ST] R. Strichartz, *A priori estimates for the wave equation and some applications,* J. Func. Anal. **5**(1970), 218-235.

[WA] G. N. Watson, *Theory of Bessel functions*, 2nd edition, Camb. Univ. Press, Cambridge, 1948

[WI] M. Williams, *On extending solutions to Dirichlet problems across the boundary as solutions,* Duke Math. J. **52**(3)(1985), 547-563.

[ZW] M. Zworski, *High frequency scattering by a convex obstacle,* Duke Math. J. **61**(2)(1990), 545-634.